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NON-COMMUTATIVITY OF CONDITION SPECTRUM

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Abstract For a general complex unital Banach algebra $\mathcal{A}$, the spectrum always commutes :

$$\text{for all } a, b \in \mathcal{A}, \quad \sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

In this paper, we prove that the above commutative property is not true if we replace the usual spectrum by condition spectrum. Further we study the similar question for more general spectrum called Ransford spectrum.

1 Introduction

For a complex unital Banach algebra $\mathcal{A}$, the spectrum of an element $a \in \mathcal{A}$ is given by

$$\sigma(a) := \{\lambda \in \mathbb{C} : (\lambda - a) \notin \mathcal{G}(\mathcal{A})\},$$

where $\mathcal{G}(\mathcal{A})$ is the group of invertible elements in $\mathcal{A}$. In fact, it is well known that the spectrum $\sigma(a)$ is a non-empty compact set [1]. It is to be observed that in the definition of $\sigma(a)$ above, $(\lambda - a)$ means $(\lambda.1 - a)$, where 1 is the multiplicative unit of $\mathcal{A}$.

The following is a well known property [4] of the spectrum that, for all $a, b \in \mathcal{A}$,

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}. \tag{1.1}$$

This property (1.1) is called the commutativity of spectrum. The proof of (1.1) comes from a simple agebraic fact that if $(1 - ab)^{-1} = u$, then $(1 - ba)^{-1} = 1 + bua$. There are various notions of spectra. One would like to see the property (1.1) for those spectra. For example, in [5], a counter-example was furnished, which demonstrates that the exponential spectrum [3] does not commute. In this paper, we would like to consider the same for condition spectrum [6], which is a special Ransford spectrum in the context of operator algebras. In fact, in Section 3, we produce examples to show the non-commutativity of condition spectrum in Examples 3.5, 3.6, 3.9, 3.12 and 3.13 in the algebra of bounded linear operators on Banach space ℓ^p, $1 \leq p < \infty$. Moreover in Section 4, in Example 4.3, we have also shown non-commutativity of the specific Ransford spectrum with the corresponding Ransford set $\Omega := \mathcal{A} \setminus \{0\}$ in operator algebra.

2 Preliminaries

Let us start with the definitions of Ransford set and Ransford spectrum.

Definition 2.1 (Ransford Set). [8, 9] Let X be a complex Normed Linear space and let $1 \in X \setminus \{0\}$ be fixed. A set $\Omega(\subseteq X)$ is a Ransford set if the following holds

(i) Ω is open,

(ii) $0 \notin \Omega$,

(iii) $1 \in \Omega$,

(iv) $\lambda \in \mathbb{C} \setminus \{0\}, a \in \Omega \Rightarrow \lambda a \in \Omega$.

Example 2.2. The following are some examples of Ransford sets [8, 9].

(i) $\mathcal{A} \setminus \{0\}$, for a complex Banach algebra $\mathcal{A}$ containing a non-zero element 1,

(ii) $\mathcal{G}(\mathcal{A})$, the set of invertible elements in a Banach algebra $\mathcal{A}$,

(iii) $Exp(\mathcal{A})$, the set of exponential elements [3, 5] for any complex unital Banach algebra $\mathcal{A}$, given by

$$Exp(\mathcal{A}) = \{e^{a_1} \cdots e^{a_n} : a_1, \ldots, a_n \in \mathcal{A}, n \geq 1\},$$

(iv) $\Omega_\epsilon := \{a \in \mathcal{G}(\mathcal{A}) : \|a\|\|a^{-1}\| < \frac{1}{\epsilon}\}$, where $0 < \epsilon < 1$.

Definition 2.3 (Ransford Spectrum). [9] The Ransford spectrum of an element $x \in \mathcal{A}$ is given by

$$\sigma_\Omega(x) = \{\lambda \in \mathbb{C} : (x - \lambda 1) \notin \Omega\}.$$

Some basic properties of Ransford spectrum are given in [8, 9]. It is to be observed that $\sigma_\Omega(x)$ is compact set in $\mathbb{C}$ [8].

The usual spectrum and exponential spectrum are also Ransford spectra with the corresponding Ransford sets $\mathcal{G}(\mathcal{A})$ and $Exp(\mathcal{A})$ respectively. One of them satifies the commutative property while the other does not. So it is natural to ask whether the Ransford spectra other than these spectra commute or not, i.e., whether for all $a, b \in \mathcal{A}$,

$$\sigma_\Omega(ab) \setminus \{0\} = \sigma_\Omega(ba) \setminus \{0\}? \tag{2.1}$$

We shall show, in Section 3, the failure of (2.1) for condition spectrum. In the Section 4, we shall show that even the Ransford spectrum corresponding to the simplest Ransford set $\Omega (= \mathcal{A} \setminus \{0\})$ fails to commute in the Banach algebra $\mathcal{A} = \mathcal{B}(\ell^p)$, $1 \leq p < \infty$.

The following simple lemma regarding invariance of Ransford sets under non-zero scalar multiplication is needed for our results. We give the proof for convenience.

Lemma 2.4. *The equation* (2.1) *is equivalent to the following:*

$$\text{for all } a, b \in \mathcal{A}, \quad 1 \in \sigma_\Omega(ab) \text{ if and only if } 1 \in \sigma_\Omega(ba).$$

Proof. In order to verify (2.1), we need to check that for some $\lambda \in \mathbb{C} \setminus \{0\}$ and for some arbitrary $x, y \in \mathcal{A}$, the following holds

$$\lambda \in \sigma_\Omega(xy) \text{ if and only if } \lambda \in \sigma_\Omega(yx).$$

That is, we need to check $(\lambda - xy) \notin \Omega$ if and only if $(\lambda - yx) \notin \Omega$. But since Ω is a Ransford set, it is invariant under nonzero scalar multiplication. So, dividing by λ, we get that it is equivalent to check

$$(1 - \frac{x}{\lambda} y) \notin \Omega \text{ if and only if } (1 - y\frac{x}{\lambda}) \notin \Omega.$$

Since $x, y \in \mathcal{A}$ are arbitrary, replacing $\frac{x}{\lambda}$ and y by general elements a, b respectively, it is equivalent to check

$$(1 - ab) \notin \Omega \text{ if and only if } (1 - ba) \notin \Omega.$$

That is, we need to check for all $a, b \in \mathcal{A}$, $1 \in \sigma_\Omega(ab)$ if and only if $1 \in \sigma_\Omega(ba)$. □

In view of Lemma 2.4, in order to check (2.1), it is enough to check for all $a, b \in \mathcal{A}$, $(1 - ab) \in \Omega$ if and only if $(1 - ba) \in \Omega$. Also it goes without saying that for getting counterexample to (2.1), we need to consider non-commutative Banach algebras.

3 Non-commutativity of condition spectrum

The following well known easy-to-prove lemma regarding non-invertible but left (or right) invertible elements [7] is necessary for our discussion. We prove it for convenience.

Lemma 3.1. *For $a, b \in \mathcal{A}$, if b is non-invertible but left invertible with left inverse a, that is, $ab = 1$ but $ba \neq 1$ then $ba \neq \lambda 1$ for any $\lambda \in \mathbb{C}$.*

Proof. Given $ab = 1$. Let if possible, for some $\lambda \in \mathbb{C}$, $ba = \lambda 1$. Then multiplying a on the left side of both, we get $a(ba) = \lambda a$ which gives $(ab)a = \lambda a$, i.e., $a = \lambda a$. But according to the given condition, $a \neq 0$. So $\lambda = 1$, which is a contradiction to the assumption that $ba \neq 1$. □

Remark 3.2. From Lemma 3.1, replacing a and $b \in \mathcal{A}$ by $\frac{a}{\mu}$ and b, we can get a general statement that, if $ab = \mu 1$ but $ba \neq \mu 1$ for some $\mu \in \mathbb{C} \setminus \{0\}$, then $ba \neq \lambda 1$ for any $\lambda \in \mathbb{C}$.

We will now discuss about the non-commutativity of condition spectrum [6]. We start with its definition.

Definition 3.3. [6] ($\epsilon-$condition spectrum) Let $0 < \epsilon < 1$. The $\epsilon-$condition spectrum of an element $a \in \mathcal{A}$ is defined by

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : (\lambda - a) \notin \mathcal{G}(\mathcal{A}) \text{ or } \|\lambda - a\| \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\}.$$

Note that for any $\epsilon \in (0, 1)$, $\sigma(a) \subseteq \sigma_\epsilon(a)$. In the definition of $\epsilon-$condition spectrum, $\epsilon \in (0, 1)$ is considered intentionally, otherwise for $\epsilon \geq 1$, we get $\sigma_\epsilon(a) = \mathbb{C}$. In [6], $\epsilon-$condition spectrum is shown as a Ransford spectrum with the corresponding Ransford set defined by

$$\Omega_\epsilon := \{a \in \mathcal{G}(\mathcal{A}) : \|a\| \|a^{-1}\| < \frac{1}{\epsilon}\}.$$

$\epsilon-$condition spectrum is a compact subset of $\mathbb{C}$ [6]. If $a \in \mathcal{A}$ is a scalar multiple of identity, i.e., $a = \mu 1$ for some $\mu \in \mathbb{C}$, then $\sigma_\epsilon(a) = \{\mu\}$ [6]. From now onwards, we consider $\epsilon \in (0, 1)$ and write simply condition spectrum instead of $\epsilon-$condition spectrum, trusting that it will not create any confusion. The following theorem is crucial for our arguments to prove some results. We state it for the sake of convenience.

Theorem 3.4. *[6, Theorem 3.1] Let $\mathcal{A}$ be complex unital Banach algebra and $a \in \mathcal{A}$ be such that $a \neq \lambda 1$ for any $\lambda \in \mathbb{C}$. Then $\sigma_\epsilon(a)$ has no isolated points.*

By the statement of Theorem 3.4, we have that if $a \in \mathcal{A}$ be such that $a \neq \lambda 1$ for any $\lambda \in \mathbb{C}$, then $\sigma_\epsilon(a)$ has infinitely many points as it is a non-empty compact subset of $\mathbb{C}$. By this fact, we will show that condition spectrum fails to commute by the following examples.

Example 3.5. Let us consider the Banach algebra $\mathcal{A} = \mathcal{B}(\ell^p)$, $1 \leq p < \infty$ and the left shift operator $\mathcal{L}$ and right shift operator $\mathcal{R}$ in $\mathcal{A}$. So $\mathcal{RL} = 1 - \mathcal{P}$ and $\mathcal{LR} = 1$. So by [6], $\sigma_\epsilon(\mathcal{LR}) = \{1\}$. Again $\mathcal{RL} = 1 - \mathcal{P} \neq 1$, where $\mathcal{P}$ is the projection onto the first component given by $\mathcal{P}(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, \ldots)$. So by Lemma 3.1, $\mathcal{RL} = 1 - \mathcal{P} \neq \lambda 1$ for any $\lambda \in \mathbb{C}$. So by Theorem 3.4, $\sigma_\epsilon(\mathcal{RL})$ has no isolated points, i.e., it has infinitely many points. Hence

$$\sigma_\epsilon(\mathcal{RL}) \setminus \{0\} \neq \sigma_\epsilon(\mathcal{LR}) \setminus \{0\}.$$

Example 3.6. In $\mathcal{A} = \mathcal{B}(\ell^p)$, $1 \leq p < \infty$, let $T_2 \in \mathcal{A}$ be defined by

$$T_2(x_1, x_2, x_3, \ldots) = (x_1, x_1, x_2, x_3, \ldots).$$

It can be checked that $\mathcal{L}T_2 = 1$, but $T_2\mathcal{L} \neq 1$, where $\mathcal{L}$ is the left shift operator. So by [6], $\sigma_\epsilon(\mathcal{L}T_2) = \{1\}$. Again by Lemma 3.1, $T_2\mathcal{L} \neq \lambda 1$ for any $\lambda \in \mathbb{C}$. So by Theorem 3.1 of [6], $\sigma_\epsilon(T_2\mathcal{L})$ has infinitely many points. Hence

$$\sigma_\epsilon(\mathcal{L}T_2) \setminus \{0\} \neq \sigma_\epsilon(T_2\mathcal{L}) \setminus \{0\}.$$

The following theorem regarding left invertible but non-invertible elements is a generalization of Examples 3.5 and 3.6.

Theorem 3.7. *If there exist* $a, b \in \mathcal{A}$ *such that* $ab = 1$ *and* $ba \neq 1$*, then* $\sigma_\epsilon(ab) \setminus \{0\} \neq \sigma_\epsilon(ba) \setminus \{0\}$.

Proof. Since $ab = 1$ and $ba \neq 1$, by [6], we have that $\sigma_\epsilon(ab) = \{1\}$ and by virtue of Theorem 3.4, $\sigma_\epsilon(ba)$ contains infinitely many points. So we have that $\sigma_\epsilon(ab) \setminus \{0\} \neq \sigma_\epsilon(ba) \setminus \{0\}$. □

Remark 3.8. From the statement of Lemma 3.1 and Theorem 3.7, in a general sense, replacing a and $b \in \mathcal{A}$ by $\frac{a}{\mu}$ and b, we get that, if there exist $a, b \in \mathcal{A}$ such that $ab = \mu 1$ and $ba \neq \mu 1$ for some $\mu \in \mathbb{C} \setminus \{0\}$, then by [6], $\sigma_\epsilon(ab) = \{\mu\}$ and by virtue of Theorem 3.4, $\sigma_\epsilon(ba)$ contains infinitely many points. So we have that $\sigma_\epsilon(ab) \setminus \{0\} \neq \sigma_\epsilon(ba) \setminus \{0\}$. So condition spectrum does not commute in those Banach algebras which contain one sided invertible but singular elements.

The following example shows failure of commutativity of condition spectrum in $\mathcal{B}(\ell^p)$ and as well as in its corresponding Calkin algebra $\mathcal{C} = \mathcal{C}al(\ell^p) = \mathcal{B}(\ell^p)/\mathcal{B}_0(\ell^p)$(see [2]).

Example 3.9. For $1 \leq p < \infty$, let $\mathcal{A} = \mathcal{B}(\ell^p)$ and $\mathcal{C} = \mathcal{C}al(\ell^p)$ be the corresponding Calkin algebra. Let $\pi : \mathcal{A} \to \mathcal{C}$ be the natural map and $x = (x_1, x_2, x_3, \ldots) \in \ell^p$ be arbitrary. Let us define $S, T \in \mathcal{A}$ by

$$S(x) = S(x_1, x_2, \ldots) = (x_1, x_3, x_5, \ldots)$$

and

$$T(x) = T(x_1, x_2, \ldots) = (x_1, 0, x_2, 0, x_3, 0, \ldots).$$

So

$$ST(x) = S(x_1, 0, x_2, 0, x_3, 0, \ldots) = (x_1, x_2, \ldots) = x,$$

$$TS(x) = T(x_1, x_3, x_5, \ldots) = (x_1, 0, x_3, 0, x_5, 0, \ldots) = x - (0, x_2, 0, x_4, 0, \ldots).$$

So $ST = 1$ and $TS = 1 - W$, where $W(x) = (0, x_2, 0, x_4, 0, \ldots)$. So $TS \neq 1$ and by virtue of Theorem 3.7, we have

$$\sigma_\epsilon(ST) \setminus \{0\} \neq \sigma_\epsilon(TS) \setminus \{0\}.$$

Again W is not a compact operator because it is an infinite dimensional projection. So in the Calkin algebra, $\pi(S)\pi(T) = \pi(ST) = \pi(1)$ but since W is not compact operator, we get $\pi(T)\pi(S) = \pi(TS) = \pi(1 - W) \neq \pi(1)$. So again by Theorem 3.7, we conclude that

$$\sigma_\epsilon(\pi(S)\pi(T)) \setminus \{0\} \neq \sigma_\epsilon(\pi(T)\pi(S)) \setminus \{0\}.$$

The following simple lemma regarding one sided, non-zero zero-divisors (see [7]) is recorded for future references. We prove it for convenience.

Lemma 3.10. *If there exist* $a, b \in \mathcal{A}$ *such that* $ab = 0$ *and* $ba \neq 0$*, then* $ba \neq \lambda 1$ *for any* $\lambda \in \mathbb{C}$.

Proof. Let $ab = 0$ and $ba \neq 0$. Let if possible, $ba = \lambda 1$ for some $\lambda (\neq 0) \in \mathbb{C}$. Then $(ba)b = \lambda b$ which gives $b(ab) = \lambda b$, i.e., $\lambda b = 0$. Since $\lambda \neq 0$, we get $b = 0$ which gives $ba = 0$, which is a contradiction. Hence $ba \neq \lambda 1$ for any $\lambda \in \mathbb{C}$. □

Remark 3.11. If there exist $a, b \in \mathcal{A}$ such that $ab = 0$ and $ba \neq 0$, then $\sigma_\epsilon(ab) \setminus \{0\} \neq \sigma_\epsilon(ba) \setminus \{0\}$. Because $\sigma_\epsilon(ab) = \{0\}$ by [6] and by Lemma 3.10, we have $ba \neq \lambda 1$ for any $\lambda \in \mathbb{C}$. So by Theorem 3.4, we have that $\sigma_\epsilon(ba)$ has infinitely any points and hence $\sigma_\epsilon(ab) \setminus \{0\} \neq \sigma_\epsilon(ba) \setminus \{0\}$.

The following is an another example of non-commutativity of condition spectrum in the operator algebra $\mathcal{B}(\ell^p)$ and also in the corresponding Calkin algebra.

Example 3.12. For $1 \le p < \infty$, let us define $T_1, T_2 \in \mathcal{B}(\ell^p)$ by

$$T_1(x) = (x_1, x_3, x_5, x_7, \ldots),$$

$$T_2(x) = (0, x_1, 0, x_2, 0, x_3, 0, x_4, \ldots)$$

which gives

$$T_1T_2(x) = T_1(0, x_1, 0, x_2, 0, x_3, 0, \ldots) = (0, 0, 0, \ldots)$$

and

$$T_2T_1(x) = T_2(x_1, x_3, x_5, \ldots) = (0, x_1, 0, x_3, 0, x_5, 0, \ldots) = W_1(x)(say).$$

So we get $T_1T_2 = 0$, which is a compact operator and $T_2T_1 = W_1$, which is not compact operator because W_1 is an infinite dimensional projection. Also in the Calkin algebra, we get $\pi(T_1)\pi(T_2) = \pi(T_1T_2) = \pi(0)$ and $\pi(T_2)\pi(T_1) = \pi(T_2T_1) = \pi(W_1) \neq \pi(0)$, since W_1 is not a compact operator. So $\sigma_\epsilon(T_1T_2) = \sigma_\epsilon(\pi(T_1)\pi(T_2)) = \{0\}$, which is a finite set. By the same argument of Theorem 3.7 and Remark 3.11,

$$\sigma_\epsilon(T_1T_2) \setminus \{0\} \neq \sigma_\epsilon(T_2T_1) \setminus \{0\}$$

and

$$\sigma_\epsilon(\pi(T_1)\pi(T_2)) \setminus \{0\} \neq \sigma_\epsilon(\pi(T_2)\pi(T_1)) \setminus \{0\}.$$

Next, we shall show that for $n(> 1) \in \mathbb{N}$, in $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$, the Banach algebra of $n \times n$ complex matrices, the condition spectrum fails to commute. In the following example, we shall show it for 2×2 matrices by considering one sided zero-divisors in $\mathcal{A}$.

Example 3.13. Let in $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$, let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. By Remark 3.11, we get that,

$$\sigma_\epsilon(AB) \setminus \{0\} \neq \sigma_\epsilon(BA) \setminus \{0\}.$$

Remark 3.14. The fact of non-commutativity of condition spectrum in Example 3.13 can be easily generalised for matrices of order $n > 2$.

4 Non-commutativity of Ransford spectrum corresponding to Ransford set $\Omega = \mathcal{A} \setminus \{0\}$

This section is mainly devoted to some counterexamples to (2.1) for Ransford spectrum, corresponding to the Ransford set $\Omega := \mathcal{A} \setminus \{0\}$ along with some related theoritical facts. So for $a \in \mathcal{A}$, the corresponding Ransford spectrum is

$$\sigma_\Omega(a) = \{\lambda \in \mathbb{C} : (\lambda - a) \notin \Omega\} = \{\lambda \in \mathbb{C} : (\lambda - a) = 0\}.$$

We state the following remark to begin with.

Remark 4.1. For the Ransford set $\Omega = \mathcal{A} \setminus \{0\}$, to check (2.1), in view of Lemma 2.4, it is enough to check that for all $a, b \in \mathcal{A}$, $1 - ab = 0$ if and only if $1 - ba = 0$.

The following lemma provides an example of (2.1) corresponding to the Ransford set $\Omega = \mathcal{A} \setminus \{0\}$ in the non-commutative Banach algebra $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$, for $n(> 1) \in \mathbb{N}$.

Lemma 4.2. *For $n(> 1) \in \mathbb{N}$, let us consider the Ransford set $\Omega = \mathcal{A} \setminus \{0\}$ in the Banach algebra $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$ of $n \times n$ complex matrices. Then (2.1) holds in $\mathcal{A}$.*

Proof. By Remark 4.1, it is enough to show that for all $B, C \in \mathcal{A}$,

$$BC = I_n \text{ if and only if } CB = I_n,$$

where I_n denotes $n \times n$ identity matrix.

Let $B, C \in \mathcal{A}$ be arbitrary and $BC = I_n$. Then taking determinant on both sides, we get $det(BC) = 1$ which gives $det(B)det(C) = 1$. So $det(B) \neq 0$ and $det(C) \neq 0$, i.e., both B, C are invertible matrices. Now

$$(CB)^{-1} = B^{-1}C^{-1} = B^{-1} \cdot I_n \cdot C^{-1} = B^{-1} \cdot (BC) \cdot C^{-1} = (B^{-1}B) \cdot (CC^{-1}) = I_n \cdot I_n = I_n.$$

So we get $CB = ((CB)^{-1})^{-1} = (I_n)^{-1} = I_n$. We have shown one way implication, the other way can be derived similarly by swapping the roles of B and C. □

The failure of (2.1) for the Ransford set $\Omega(= \mathcal{A} \setminus \{0\})$ is exemplified by the following example:

Example 4.3. Let us consider the Banach algebra $\mathcal{A} = \mathcal{B}(\ell^p)$, $1 \leq p < \infty$ and the Ransford set $\Omega = \mathcal{A} \setminus \{0\}$. Let us consider the left shift operator $\mathcal{L}$ and right shift operator $\mathcal{R}$ in $\mathcal{A}$. So $\mathcal{RL} = 1 - \mathcal{P}$ and $\mathcal{LR} = 1$, which gives $1 - \mathcal{RL} = \mathcal{P} \in \Omega$ and $1 - \mathcal{LR} = 0 \notin \Omega$, where $\mathcal{P}$ denotes the projection operator onto the first component. So

$$1 \in \sigma_\Omega(\mathcal{LR}) \text{ but } 1 \notin \sigma_\Omega(\mathcal{RL})$$

and hence

$$\sigma_\Omega(\mathcal{LR}) \setminus \{0\} \neq \sigma_\Omega(\mathcal{RL}) \setminus \{0\}.$$

Remark 4.4. From the discussion of Example 4.3, we can conclude that if a Ransford set Ω_1 of $\mathcal{B}(\ell^p)$ contains $\mathcal{P}$, then the corresponding Ransford spectrum will not commute because $1 - \mathcal{RL} = \mathcal{P} \in \Omega_1$ and $1 - \mathcal{LR} = 0 \notin \Omega_1$ and hence

$$\sigma_{\Omega_1}(\mathcal{LR}) \setminus \{0\} \neq \sigma_{\Omega_1}(\mathcal{RL}) \setminus \{0\}.$$

The following remark is a generalization of Example 4.3.

Remark 4.5. For the Ransford set $\Omega = \mathcal{A} \setminus \{0\}$, if there exist $a, b \in \mathcal{A}$ such that $ab = 1$ but $ba \neq 1$, then $(1 - ab) = 0 \notin \Omega$ but $(1 - ba) \neq 0$, which gives $(1 - ba) \in \Omega$. So we have $\sigma_\Omega(ab) \setminus \{0\} \neq \sigma_\Omega(ba) \setminus \{0\}$. More generally, we can say that, if there exist $a, b \in \mathcal{A}$ such that $ab = \lambda 1$ and $ba \neq \lambda 1$, for some $\lambda \in \mathbb{C} \setminus \{0\}$, then $\sigma_\Omega(ab) \setminus \{0\} \neq \sigma_\Omega(ba) \setminus \{0\}$.

5 Conclusion

We conclude by recording our findings in a formal way as follows.

If there exist $a, b \in \mathcal{A}$, such that $ab = \mu 1$ for some $\mu \in \mathbb{C}$ and $ba \neq \mu 1$. Then we have the following:

- $ba \neq \lambda 1$, for any $\lambda \in \mathbb{C}$ (by Remark 3.2),
- $\sigma_\epsilon(ab) \setminus \{0\} \neq \sigma_\epsilon(ba) \setminus \{0\}$ (by Remark 3.8),
- $\sigma_\Omega(ab) \setminus \{0\} \neq \sigma_\Omega(ba) \setminus \{0\}$, if $\mu \neq 0$ where $\Omega = \mathcal{A} \setminus \{0\}$ (by Remark 4.5),
- if there is a Ransford set $\Omega(\subseteq \mathcal{A})$, such that $(\mu 1 - ba) \in \Omega$ and $\mu \neq 0$, then the corresponding Ransford spectrum does not commute, i.e.,

$$\sigma_\Omega(ab) \setminus \{0\} \neq \sigma_\Omega(ba) \setminus \{0\}.$$

It would be interesting to investigate sufficient conditions on Ransford sets to satisfy (2.1) in various non-commutative Banach algebras. So in this context, as an immediate offshoot, we would like to end with the following question.

Question 5.1. For a non-commutative Banach algebra $\mathcal{A}$, does there exist any Ransford set $\Omega(\subseteq \mathcal{A})$, other than $\mathcal{G}(\mathcal{A})$, for which the corresponding Ransford spectrum always commute, i.e.,

$$\forall a, b \in \mathcal{A}, \;\; \sigma_\Omega(ab) \setminus \{0\} = \sigma_\Omega(ba) \setminus \{0\}?$$

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