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Proper $BN(k)$ Rings are BZS

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Abstract An associative ring R, not necessarily commutative and not necessarily with identity, is called a *BN ring* if every element of R is either idempotent or nilpotent. If the index of nilpotency of the elements is bounded by k, then we call a BN ring R a ***BN(k)*** *ring*. A BN ring is *proper* if it is neither Boolean nor nil. We show that any proper $BN(k)$ ring is a BZS, that is a $BN(2)$, ring.

1 Introduction

This paper generalizes the study of *Boolean-zero square* or ***BZS*** rings, those rings in which every element is either idempotent or nilpotent of index 2, initiated by the authors in [4] and [5]. BZS rings generalize both the well-known class of Boolean rings, in which all elements are idempotent, and zero square rings, in which every nonzero element is nilpotent of index two (see [9] and [8] for more information about Boolean rings and zero square rings, respectively). BZS rings are a special case of BZS near-rings, which are studied in [3]; BZS near-rings also capture the Malone trivial near-rings introduced in [7] and studied, *inter alia*, in [1]. A *proper* BZS ring is a BZS ring that is neither a Boolean ring nor a zero square ring.

In this paper we investigate *BN rings*, which are rings in which each element is either idempotent or nilpotent. If the index of nilpotency of the elements in a BN ring is bounded by k, then we call the ring a ***BN(k)*** *ring*. In this notation a BZS ring R is a $BN(2)$ ring. A *proper* ***BN*** *ring* is a BN ring which is neither Boolean nor nil. Note that a $BN(k)$ ring is also a $BN(k+1)$ ring by definition.

Throughout the paper, R denotes a BN ring, N denotes the set of nilpotent elements of R, for any integer $t \geq 2$ N_t denotes the set of elements of N having index of nilpotency at most t, and E denotes the set of nonzero idempotent elements of R. We focus mainly on the case in which R is a proper $BN(k)$ ring.

2 Preliminary Results

In this section we extend two of the results of [4] from BZS rings to BN rings. The following result is proven for $BN(2)$ rings in [4, Proposition 2.1].

Proposition 2.1. *Let $(R, +, \cdot)$ be a proper BN ring such that $(R, +)$ is a cyclic group of order $n \geq 2$. Then either R is isomorphic to the ring of integers modulo 2 or it is a ring with identically zero multiplication.*

Proof. Suppose that g generates R additively. Since R is a BN ring, either $g^k = 0$ for some $k \in \mathbb{Z}$ or $g \cdot g = g$. If $g^k = 0$, then for all integers $0 \leq \alpha, \beta < n-1$ we have $(\alpha g^p) \cdot (\beta g^q) = \alpha\beta g^{p+q}$ since R is distributive; i.e., R is a nil ring. If $g \cdot g = g$, then $((n-1)g) \cdot ((n-1)g) = (n-1)^2 g \cdot g = 1g \cdot g = g \neq 0$, so $((n-1)g) \cdot ((n-1)g) = (n-1)g = -g$ implies $g + g = 0$, so that $n = 2$ and R is the ring of integers modulo 2. □

The following result is proven for $BN(2)$ rings in [4, Lemma 3.3] and is used extensively in the sequel.

Lemma 2.2. *If R is a BN ring and if $e \in E$, then $2e = 0$.*

Proof. Let $e \in E$. Then, for any integer $n \geq 1$, $(-e)^n = \begin{cases} e \text{ if } n \text{ is even} \\ -e \text{ if } n \text{ is odd} \end{cases}$ so that $-e \notin N$. Hence $-e = (-e)^2 = e$. □

3 Proper $BN(3)$ Rings are BZS

In this section we show that any proper $BN(3)$ ring is BZS. In what follows, we use, for an element $a \in R$, the notation $\rho(a)$ to indicate multiplication of both sides of an equation by a on the right and $\lambda(a)$ to indicate multiplication of both sides of an equation by a on the left.

Lemma 3.1. *Let R be a proper $BN(3)$ ring, let $0 \neq e \in E$, and let $x \in N_2$. Then the element $e + x$ is idempotent.*

Proof. If $x = 0$ then the result is trivial. Assume $x \neq 0$.

The proof is by contradiction. Suppose that $e+x$ is nilpotent. Then we must have $(e+x)^3 = 0$.

Now by Lemma 2.2 and since $x \in N_2$,

$$e = ex + xe + exe + xex \tag{3.1}$$

Applying $\lambda(e)$ and $\rho(x)$ to (3.1) we get

$$\begin{aligned} ex &= ex^2 + exex + exex + exex^2 \\ &= exex + exex \\ &= 0 \end{aligned} \tag{3.2}$$

by Lemma 2.2. Substituting $ex = 0$ into (3.1) we get $e = xe$. But then $e = xe = x(xe) = x^2 e = 0$, contradiction! □

Lemma 3.2. *Let R be a proper $BN(3)$ ring, and let $e \in E, x \in N_2$. Then*

a) $exe = 0$*;*

b) $xex = 0$*;*

c) $ex + xe = x$.

Proof. If $x = 0$ then the results are trivial. Otherwise, $e + x$ is idempotent by Lemma 3.1, and we get

$$e + ex + xe = e + x,$$

yielding

$$ex + xe = x. \tag{3.3}$$

Applying $\rho(e)$ to (3.3) and simplifying we get $exe = 0$. Applying $\lambda(x)$ to (3.3) and simplifying we get $xex = 0$. □

In the next results we apply the preceding lemma to show that a proper $BN(3)$ ring is actually BZS.

Proposition 3.3. *Let R be a proper $BN(3)$ ring, $e \in E, x \in N_3$, and suppose $x \notin N_2$. Then $e + x \notin E$.*

Proof. The proof is by contradiction. Suppose $e + x \in E$. Then $(e + x)^2 = e + ex + xe + x^2$ so that $e + x = e + ex + xe + x^2$, yielding

$$x = ex + xe + x^2 \tag{3.4}$$

Applying $\lambda(x)$ to (3.4) we get

$$x^2 = xex + x^2 e. \tag{3.5}$$

Applying $\rho(x)$ to (3.4) we get

$$x^2 = ex^2 + xex. \tag{3.6}$$

From (3.5) and (3.6) we get $x^2e = ex^2$. But then by Lemma 3.2 c) (since $0 \neq x^2 \in N_2$), we have $x^2 = ex^2 + x^2e = 0$, contradiction! □

Proposition 3.4. *Let R be a proper $BN(3)$ ring, $e \in E, x \in N_3$, and suppose $x \notin N_2$. Then $e + x \notin N$.*

Proof. The proof is by contradiction.

Suppose that $(e + x)^3 = 0$. Expanding this equation we get

$$e = ex + xe + exe + xex + ex^2 + x^2e. \tag{3.7}$$

Applying both $\rho(e)$ and $\lambda(e)$ to (3.7) we get

$$\begin{aligned} e &= exe + exe + exe + exexe + ex^2e + ex^2e \\ &= exe + exexe, \end{aligned} \tag{3.8}$$

by Lemma 2.2. Applying $\rho(x)$ to (3.8) we get

$$ex = (ex)^2 + (ex)^3. \tag{3.9}$$

If $ex \in N$ then $(ex)^3 = 0$ which implies $ex = (ex)^2$. The only idempotent element in N is 0. Hence $ex = 0$. From (3.8) we get $e = 0$, contradiction! If $ex \in E$ then by (3.9) we have that $ex = ex + ex = 0$, contradiction! □

Theorem 3.5. *If R is a proper BN(3) ring, then R is a BZS ring.*

Proof. This follows from Propositions 3.3 and 3.4. □

4 Proper $BN(k)$ Rings are BZS

In this section we complete the proof of our main result: that a proper $BN(k)$ ring is BZS for any integer $k \geq 2$.

Lemma 4.1. *Let $k \geq 2$ be an integer. If R is a proper $BN(k)$ ring, then R does not contain identity.*

Proof. The proof is by contradiction. Suppose R contains identity 1. Since for some $t \geq 2$ N_t contains a nonzero element, we have that N_2 contains a nonzero element. So let $0 \neq x \in N_2$. If $(1+x)^2 = 1+x$, then $1+2x = 1+x$, which implies $x = 0$, contradiction! Thus, we must have $(1+x)^k = 0$. Since $(1+x)^k = 1+kx$, it follows that $-kx = 1$, so that $0 = (-kx)^2 = 1^2 = 1$, contradiction! □

Corollary 4.2. *Let $k \geq 2$ be an integer. If R is a proper $BN(k)$ ring and $e \in E$, the ring eRe is Boolean.*

Proof. By the previous result we know that eRe is not a proper $BN(k)$ ring. Since $eRe \subseteq R$, eRe consists of idempotents and nilpotents. Since eRe has identity, eRe consists only of idempotents, and hence is Boolean. □

Lemma 4.3. *Let a, b be two elements in a ring R, and $m \geq 1$ an integer. Then of the 2^m summands in the formal expansion of $(a+b)^m$, f_{m+2} do not have any consecutive b's in them, where f_n is the n^{th} Fibonacci number using the convention $f_1 = f_2 = 1$; of those f_{m+2} summands that do not have any consecutive b's in them, f_{m+1} have a as their leftmost element and f_m have b as their leftmost element.*

Proof. The proof is by induction. The basis case is $m = 1$, in which case $(a+b)^m = a+b$, and $f_3 = 2$ terms do not have any consecutive b's in them, $f_2 = 1$ of which have leftmost element a and $f_1 = 1$ of which have leftmost element b. For the inductive step, suppose that the result is true for $m = j \geq 1$. Then the expansion of $(a+b)^j$ can be written as $\alpha_1 + \alpha_2 + \cdots + \alpha_{f_{j+1}} + \beta_1 + \beta_2 + \cdots + \beta_{f_j} + \gamma_1 + \gamma_2 + \cdots + \gamma_{(2^j - f_{j+2})}$, where each α_i has leftmost element a and no consecutive b's, each β_i has leftmost element b and no consecutive b's, and each γ_i contains consecutive b's. Then $(a+b)^{j+1} = (a+b)(a+b)^j = a\alpha_1 + a\alpha_2 + \cdots + a\alpha_{f_{j+1}} + a\beta_1 + a\beta_2 + \cdots + a\beta_{f_j} + b\alpha_1 + b\alpha_2 + \cdots + b\alpha_{f_{j+1}} + [a\gamma_1 + a\gamma_2 + \cdots + a\gamma_{(2^j - f_{j+2})} + b\beta_1 + b\beta_2 + \cdots + b\beta_{f_j} + b\gamma_1 + b\gamma_2 + \cdots + b\gamma_{(2^j - f_{j+2})}]$. The terms in brackets will all have consecutive b's while those not in brackets, $f_{j+1} + f_j = f_{j+2}$ of which have leftmost element a and f_{j+1} of which have leftmost element b, will not. Thus the proof is complete. □

Proposition 4.4. *Let R be a proper $BN(k)$ ring, $k \geq 3$, $e \in E$, $x \in N_k$, $x \notin N_{k-1}$. Then $e + x^{k-1} \notin N$.*

Proof. The proof is by contradiction. Assume that $e + x^{k-1} \in N$. Then both

$$(e + x^{k-1})^k = 0 \tag{4.1}$$

and

$$(e + x^{k-1})^{k+1} = 0. \tag{4.2}$$

By the previous lemma, (at least) one of the expansions of the left-hand sides of (4.1) or (4.2) contains an odd number of summands that do not contain consecutive products of x^{k-1}. Applying $\lambda(e)$ and $\rho(e)$ to that equation, and noting that terms containing consecutive products of x^{k-1} equal zero when $k \geq 3$, we find by using Corollary 4.2 that the remaining terms are e and products of the form

$$ex^{k-1}e,$$

so that

$$e = Aex^{k-1}e,$$

where A is an even number. Since A is even, we obtain from Lemma 2.2 that

$$e = 0,$$

contradiction! □

Proposition 4.5. *Let R be a proper $BN(k)$ ring, $k \geq 4$, $e \in E$, $x \in N_k$, $x \notin N_{k-1}$. Then $e + x^{k-2} \notin N$.*

Proof. The proof follows *mutatis mutandis* from the previous result since $(x^{k-2})^2 = 0$ for $k \geq 4$. □

Proposition 4.6. *Let R be a proper $BN(k)$ ring, $k \geq 4$, $e \in E$, $x \in N_k$, $x \notin N_{k-1}$. Then $e + x^{k-1} \notin E$.*

Proof. By the previous two results, we must have $e + x^{k-1} \in E$ and $e + x^{k-2} \in E$. So $(e + x^{k-1})^2 = e + x^{k-1}$, implying

$$x^{k-1} = ex^{k-1} + x^{k-1}e \tag{4.3}$$

and $(e + x^{k-2})^2 = e + x^{k-2}$, implying

$$x^{k-2} = ex^{k-2} + x^{k-2}e. \tag{4.4}$$

(Note that here we use $k \geq 4$ to conclude that $(x^{k-2})^2 = 0$.) Applying $\rho(x)$ and $\lambda(x)$ separately to (4.3) yields, respectively,

$$x^{k-1}ex = 0 \tag{4.5}$$

and

$$xex^{k-1} = 0. \tag{4.6}$$

Applying $\lambda(e)$ to (4.3) yields

$$ex^{k-1}e = 0 \tag{4.7}$$

Now substituting (4.4) into (4.3) we find

$$x^{k-1} = ex(ex^{k-2} + x^{k-2}e) + x^{k-1}e = exex^{k-2} + ex^{k-1}e + x^{k-1}e, \tag{4.8}$$

so that by (4.7) we have

$$x^{k-1} = exex^{k-2} + x^{k-1}e, \tag{4.9}$$

whence comparison of this last equation to (4.3) implies that

$$exex^{k-2} = ex^{k-1}. \tag{4.10}$$

Applying $\lambda(ex)$ to this last equation now gives

$$exexex^{k-2} = exex^{k-1} = 0 \tag{4.11}$$

by (4.6), so that Corollary 4.2 and (4.10) give us

$$exex^{k-2} = 0 = ex^{k-1}. \tag{4.12}$$

Now combining (4.9) with (4.4) gives us

$$x^{k-1} = exex^{k-2} + (ex^{k-2} + x^{k-2}e)xe = exex^{k-2} + ex^{k-1}e + x^{k-2}exe, \tag{4.13}$$

so that (4.7) implies

$$x^{k-1} = exex^{k-2} + x^{k-2}exe. \tag{4.14}$$

Comparing (4.14) with (4.3) using (4.10) yields

$$x^{k-2}exe = x^{k-1}e. \tag{4.15}$$

Applying $\rho(xe)$ to this last equation implies

$$x^{k-2}exexe = x^{k-1}exe = 0 \tag{4.16}$$

by (4.5), so that Corollary 4.2 and (4.15) give us

$$x^{k-2}exe = 0 = x^{k-1}e. \tag{4.17}$$

Finally, (4.3), (4.12), and (4.17) together imply $x^{k-1} = 0$, contradiction! □

We can now summarize our results in the following theorem.

Theorem 4.7. *If R is a proper $BN(k)$ ring for $k \geq 2$, then R is BZS,*

Proof. This follows directly from Theorem 3.5, Proposition 4.4, and Proposition 4.6. □

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