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ON CENTRAL EXTENSIONS AND PROJECTIVE REPRESENTATIONS OF PLESKEN LIE ALGEBRAS

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Abstract In this article we describe bijective correspondence between second cohomology group and equivalent central extensions of Plesken Lie algebras. Further it is also established that the cocycles corresponding to the projectively equivalent projective representations of Plesken Lie algebras are cohomologous.

1 Introduction

For any finite group G, we have the associative algebra group algebra $\mathbb{F}[G]$ over a field $\mathbb{F}$. The group algebra $\mathbb{F}[G]$ can be made into a Lie algebra using a bracket product as a commutator and is the Lie algebra of the group algebra $\mathbb{F}[G]$. We are interested in the Lie algebra which is directly constructed from finite groups. In [3], Arjeh M. Cohen and D. E. Taylor found certain Lie algebra structure from a finite group and call it as Plesken Lie algebra. A Plesken Lie algebra $\mathcal{L}(G)$ of a finite group G over $\mathbb{F}$ is the linear span of elements $\hat{g} = g - g^{-1} \in \mathbb{F}[G]$ together with the Lie bracket $[\hat{g}, \hat{h}] = \hat{g}\hat{h} - \hat{h}\hat{g}$. In [3], they described the structure of Plesken Lie algebras and explicitly determine the groups for which the Plesken Lie algebra is simple and semisimple over the complex field. In the paper [4], John Cullinan and Mona Merling described the structural properties of Plesken Lie algebra over finite fields. More recently, Arjun and Romeo describe the linear representations of Plesken Lie algebras which are induced from the group representations in [1]. Also, they introduced the irreducible Plesken Lie algebra representations.

In group theory, Schur introduced projective representation to determine all the finite groups contained in $GL(V)$ where V is a finite dimensional complex vector space. The projective representation of a finite group G on a vector space V is a group homomorphism from that group into the projective linear group $PGL(V)$ (see cf. [5]). These representations occur commonly in the study of linear representation of groups and it have many applications in the areas of Physics and Mathematics. By definition, every linear representation of a group is projective, but the converse is not true. The concept of multipliers of finite groups were first studied by I. Schur in the early 1900's. Nowadays the multiplier of a group is defined to be the second cohomology group $H^2(G, \mathbb{C}^*)$. The notions of multipliers of Lie algebras were first introduced by Kaye Moneyhun in his dissertation using the concept of isoclinism. Later P. G. Batten constructed this structure and proved the results using special class of algebras and homomorphisms in his Ph.D Thesis [2]. Batten proved that the multilplier for a finite dimensional Lie algebra L is isomorphic to $H^2(L, \mathbb{C})$, the second cohomology group of L.

In this article, we will describe the projective representations of a Plesken Lie algebra $\mathcal{L}(G)$ using the multiplier $H^2(L, \mathbb{C})$. The theory of projective representations of a Lie algebra L involves understanding homomorphisms from the Lie algebra L to the quotient Lie algebra $\mathfrak{gl}(V)/\{kI_V : k \in \mathbb{C}\}$, V is a complex finite dimensional Lie algebra. Here we explicitly showed the correspondence between the Schur multiplier $H^2(\mathcal{L}(G), \mathbb{C})$, equivalent central extensions and projectively equivalent projective representations of the Plesken Lie algebra $\mathcal{L}(G)$.

Throughout this paper $\mathcal{L}(G)$ denote the Plesken Lie algebra of a finite group G over the field $\mathbb{C}$ of complex numbers.

2 Central extensions

In this section, we will describe the relations between a central extension of a Plesken Lie algebra $\mathcal{L}(G)$ by an abelian Plesken Lie algebra $\mathcal{L}(H)$ and $H^2(\mathcal{L}(G), \mathcal{L}(H))$ and it is shown that up to equivalence, the class of central extensions of $\mathcal{L}(G)$ by $\mathbb{C}$ and the second cohomology group of $\mathcal{L}(G)$ are essentially the same.

Definition 2.1. G and H are two finite groups such that $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are the corresponding Plesken Lie algebras of which $\mathcal{L}(H)$ is abelian. Then $(\mathcal{E}; f, g)$ is an extension of $\mathcal{L}(G)$ by $\mathcal{L}(H)$ if there exists a Lie algebra $\mathcal{E}$ such that the following is a short exact sequence :

$$0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0.$$

An extension $(\mathcal{E}; f, g)$ is *central* if $f(\mathcal{L}(H)) \subseteq Z(\mathcal{E})$ where $Z(\mathcal{E})$ is the center of $\mathcal{E}$.

Example 2.2. Consider the subgroups $G = \{\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{Z}_p\}$ and $H = \{\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbb{Z}_p\}$ of the Heisenberg group $\mathbb{H}(\mathbb{Z}_p)$. Then the Plesken Lie algebras of G and H are

$$\mathcal{L}(G) = span_{\mathbb{C}}\{\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : y, z \in \mathbb{Z}_p\} \text{ and}$$

$$\mathcal{L}(H) = span_{\mathbb{C}}\{\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{Z}_p\}$$

respectively. Consider the Plesken Lie algebra

$$\mathcal{E} = \mathcal{L}(\mathbb{H}(\mathbb{Z}_p)) = span_{\mathbb{C}}\{\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{Z}_p\}$$

of the Heisenberg group $\mathbb{H}(\mathbb{Z}_p)$. Choose the short exact sequence $0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0$ where $f : \mathcal{L}(H) \to \mathcal{E}$ is the inclusion such that $Im(f) = Ker(g)$, then g is given by

$$g(\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

Also $f(\mathcal{L}(H)) = Z(\mathcal{E})$, the center of $\mathcal{E}$. Thus $0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0$ is a central extension.

2.1 Equivalent central extensions

Two central extensions

$$0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0$$

and

$$0 \to \mathcal{L}(H) \xrightarrow{f'} \mathcal{E}' \xrightarrow{g'} \mathcal{L}(G) \to 0$$

of $\mathcal{L}(G)$ by $\mathcal{L}(H)$ are equivalent if there exists a Lie algebra homomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(H) & \xrightarrow{f} & \mathcal{E} & \xrightarrow{g} & \mathcal{L}(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathcal{L}(H) & \xrightarrow{f'} & \mathcal{E}' & \xrightarrow{g'} & \mathcal{L}(G) & \longrightarrow & 0 \end{array} \tag{2.1}$$

Remark 2.3. If such a homomorphism ϕ exists, then it must be an isomorphism (see [2]).

Now we recall the second cohomology group of a finite dimensional Lie algebra (cf.[2]). Let $\mathcal{L}(G)$ be a Plesken Lie algebra of G over $\mathbb{C}$. The set of 2-cocycles is given by

$$Z^2(\mathcal{L}(G), \mathbb{C}) = \{f : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C} : f \text{ is bilinear and } f([\hat{x}, \hat{y}], \hat{z}) \\ +f([\hat{y}, \hat{z}], \hat{x}) + f([\hat{z}, \hat{x}], \hat{y}) = 0 \; \forall \hat{x}, \hat{y}, \hat{z} \in \mathcal{L}(G)\}$$

and the set of 2-coboundaries are

$$B^2(\mathcal{L}(G), \mathbb{C}) = \{f : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C} : f \text{ is bilinear and there exists} \\ \sigma : \mathcal{L}(G) \to \mathbb{C} \text{ such that } f(\hat{x}, \hat{y}) = -\sigma([\hat{x}, \hat{y}])\}.$$

Then the second cohomology group of $\mathcal{L}(G)$ is given by

$$H^2(\mathcal{L}(G), \mathbb{C}) = \frac{Z^2(\mathcal{L}(G), \mathbb{C})}{B^2(\mathcal{L}(G), \mathbb{C})}.$$

and is called the *multiplier* of $\mathcal{L}(G)$ (see [2]).

Note that two 2-cocycles α_1 and α_2 are said to be *cohomologous* (i.e. they have the same cohomology class) if there exists a linear map $\sigma : \mathcal{L}(G) \to \mathbb{C}$ such that

$$\mu_2(\hat{x}, \hat{y}) - \mu_1(\hat{x}, \hat{y}) = -\sigma([\hat{x}, \hat{y}])$$

Lemma 2.4. *Let $\mu \in Z^2(\mathcal{L}(G), \mathbb{C})$. Then $\mu(\hat{x}, \hat{x}) = \mu(\hat{x}, 0) = \mu(0, \hat{x}) = 0$ for all $\hat{x} \in \mathcal{L}(G)$.*

Proof. Suppose $\mu \in Z^2(\mathcal{L}(G), \mathbb{C})$. Then for any $\hat{x}, \hat{y}, \hat{z} \in \mathcal{L}(G)$,

$$\mu([\hat{x}, \hat{y}], \hat{z}) + \mu([\hat{y}, \hat{z}], \hat{x}) + \mu([\hat{z}, \hat{x}], \hat{y}) = 0 \tag{2.2}$$

Take $\hat{x} = \hat{y} = \hat{z}$, then (2.2) implies that $3\mu(0, \hat{x}) = 0$. That is, $\mu(0, \hat{x}) = 0$. Then the alternating property of μ gives that $\mu(\hat{x}, 0) = 0$ and $\mu(\hat{x}, \hat{x}) = 0$. □

The following theorem gives us the relation between the multiplier $H^2(\mathcal{L}(G), \mathbb{C})$ and central extensions of $\mathcal{L}(G)$ by $\mathbb{C}$.

Theorem 2.5. *Let $\mathcal{L}(G)$ be a Plesken Lie algebra of a finite group G. Then there is a bijective correspondence between $H^2(\mathcal{L}(G), \mathbb{C})$ and equivalence class of central extensions of $\mathcal{L}(G)$ by $\mathbb{C}$.*

Proof. Let X denotes the equivalence classes of central extensions of $\mathcal{L}(G)$ by $\mathbb{C}$. Define $\Psi : X \to H^2(\mathcal{L}(G), \mathbb{C})$ by

$$\Psi([\mathcal{E}]) = [\mu]$$

For $(\mathcal{E}; f, g)$ be a central extension of $\mathcal{L}(G)$ by $\mathbb{C}$ and s be a section map of g (a linear map $s : \mathcal{E} \to \mathcal{L}(G)$ such that $g \circ s = I_{\mathcal{L}(G)}$). Define $\mu : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C}$ by

$$\mu(\hat{x}, \hat{y}) = [s(\hat{x}), s(\hat{y})] - s([\hat{x}, \hat{y}]).$$

Then $\mu(\hat{x}, \hat{y}) \in \mathbb{C}$ and for any $\hat{x}, \hat{y}, \hat{z} \in \mathcal{L}(G)$ and $c \in \mathbb{C}$

$$\begin{aligned} \mu(c\hat{x} + \hat{y}, \hat{z}) &= [s(c\hat{x} + \hat{y}), s(\hat{z})] - s([c\hat{x} + \hat{y}, \hat{z}]) \\ &= [s(c\hat{x}) + s(\hat{y}), s(\hat{z})] - s([c\hat{x}, \hat{z}] + [\hat{y}, \hat{z}]) \\ &= [cs(\hat{x}), s(\hat{z})] + [s(\hat{y}), s(\hat{z})] - cs([\hat{x}, \hat{z}]) - s([\hat{y}, \hat{z}]) \\ &= c\mu(\hat{x}, \hat{z}) + \mu(\hat{y}, \hat{z}) \end{aligned} \tag{2.3}$$

and $\mu(\hat{x}, c\hat{y}+\hat{z}) = c\mu(x,y)+\mu(\hat{x},\hat{z})$, hence μ is bilinear. Also

$$\begin{aligned}\mu([\hat{x},\hat{y}],\hat{z})+&\mu([\hat{y},\hat{z}],\hat{x})+\mu([\hat{z},\hat{x}],\hat{y})\\ &= [s([\hat{x},\hat{y}]),s(\hat{z})]-s([[\hat{x},\hat{y}],\hat{z}])+[s([\hat{y},\hat{z}],s(\hat{x})]\\ &\quad -s([[\hat{y},\hat{z}],\hat{x}])+[s([\hat{z},\hat{x}]),s(\hat{y})]-s([[\hat{z},\hat{x}],\hat{y}])\\ &= [[s(\hat{x}),s(\hat{y})]-\mu(\hat{x},\hat{y}),s(\hat{z})]+[[s(\hat{y}),s(\hat{z})]-\mu(\hat{y},\hat{z}),s(\hat{x})]\\ &\quad [[s(\hat{z}),s(\hat{x})]-\mu(\hat{z},\hat{x}),s(\hat{y})]-s([[\hat{x},\hat{y}],\hat{z}]+[[\hat{y},\hat{z}],\hat{x}]+[[\hat{z},\hat{x}],\hat{y}])\\ &= 0\end{aligned} \tag{2.4}$$

(using Jacobi identity in $\mathcal{E}$ and $\mathcal{L}(G)$ and also using the fact that the extension is central). Thus $\mu \in Z^2(\mathcal{L}(G),\mathbb{C})$.

To see that μ is independant of choice of section, consider a section $s' : \mathcal{L}(G) \to \mathcal{E}$ of g and ν the corresponding 2-cocycle, then

$$\begin{aligned} g(s'(\hat{x})-s(\hat{x}))=0 \quad &\Rightarrow s'(\hat{x})-s(\hat{x}) \in Ker(g)\\ &\Rightarrow s'(\hat{x})-s(\hat{x}) \in Im(f) \cong \mathbb{C}\\ &\Rightarrow s'(\hat{x})-s(\hat{x}) = c_{\hat{x}} \text{ for some } c_{\hat{x}} \in \mathbb{C}\end{aligned} \tag{2.5}$$

Define $\sigma : \mathcal{L}(G) \to \mathcal{E}$ by

$$\sigma(\hat{x}) = c_{\hat{x}}$$

then σ is linear and

$$\begin{aligned}\nu(\hat{x},\hat{y}) &= [s'(\hat{x}),s'(\hat{y})]-s'([\hat{x},\hat{y}])\\ &= [c_{\hat{x}}+s(\hat{x}),c_{\hat{y}}+s(\hat{y})]-(c_{[\hat{x},\hat{y}]}+s([\hat{x},\hat{y}]))\\ &= [c_{\hat{x}},c_{\hat{y}}]+[c_{\hat{x}},s(\hat{y})]+[s(\hat{x}),c_{\hat{y}}]+[s(\hat{x}),s(\hat{y})]-s([\hat{x},\hat{y}])-c_{[\hat{x},\hat{y}]}\\ &= \mu(\hat{x},\hat{y})-\sigma([\hat{x},\hat{y}])\end{aligned} \tag{2.6}$$

thus μ and ν are cohomologous. i.e., $[\mu]=[\nu]$. ie., μ does not depend on the choice of the section.

Consider two equivalent central extensions $(\mathcal{E}; f, g)$ and $(\mathcal{E}'; f', g')$ of $\mathcal{L}(G)$ by $\mathbb{C}$. Then there is a Lie algebra homomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \xrightarrow{f} & \mathcal{E} & \xrightarrow{g} & \mathcal{L}(G) & \longrightarrow & 0\\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathbb{C} & \xrightarrow{f'} & \mathcal{E}' & \xrightarrow{g'} & \mathcal{L}(G) & \longrightarrow & 0\end{array} \tag{2.7}$$

i.e. $\phi \circ f = f'$ and $g = g' \circ \phi$. Let $s : \mathcal{L}(G) \to \mathcal{E}$ be a section of g. Then there is a 2-cocycle μ such that

$$\mu(\hat{x},\hat{y}) = [s(\hat{x}),s(\hat{y})]-s([\hat{x},\hat{y}])$$

Define $s' = \phi \circ s$, then $s' : \mathcal{L}(G) \to \mathcal{E}'$ is a linear map such that

$$g' \circ s' = g' \circ \phi \circ s = g \circ s = I_{\mathcal{L}(G)}.$$

i.e. s' is a section of g' and there is a 2-cocycle $\nu \in Z^2(\mathcal{L}(G),\mathbb{C})$ such that

$$\begin{aligned}\nu(\hat{x},\hat{y}) &= [s'(\hat{x}),s'(\hat{y})]-s'([\hat{x},\hat{y}])\\ &= [(\phi\circ s)(\hat{x}),(\phi\circ s)(\hat{y})]-(\phi\circ s)([\hat{x},\hat{y}])\\ &= \phi([s(\hat{x}),s(\hat{y})]-s([\hat{x},\hat{y}]))\\ &= \phi(\mu(\hat{x},\hat{y}))\\ &= \mu(x,y) \ \text{ as } \mu(x,y) \in \mathbb{C}\end{aligned} \tag{2.8}$$

hence two equivalent central extensions maps same element in $H^2(\mathcal{L}(G), \mathbb{C})$ via ϕ and thus Ψ is well defined.

Let $(\mathcal{E}; f, g)$ and $(\mathcal{E}'; f', g')$ are two central extensions of $\mathcal{L}(G)$ by $\mathbb{C}$ such that $\Psi([\mathcal{E}]) = [\mu]$ and $\Psi([\mathcal{E}']) = [\nu]$. Suppose that $\Psi([\mathcal{E}]) = \Psi([\mathcal{E}'])$. i.e. $[\mu] = [\nu]$. Let s and s' be the sections of g and g' respectively. Then we have

$$\mu(\hat{x}, \hat{y}) = [s(\hat{x}), s(\hat{y})] - s([\hat{x}, \hat{y}]) \text{ and } \nu(\hat{x}, \hat{y}) = [s'(\hat{x}), s'(\hat{y})] - s'([\hat{x}, \hat{y}])$$

Since μ and ν are cohomologous, there is a linear map $\sigma : \mathcal{L}(G) \to \mathbb{C}$ such that

$$\mu(\hat{x}, \hat{y}) - \nu(\hat{x}, \hat{y}) = -\sigma([\hat{x}, \hat{y}])$$

Note that every element $\hat{x}$ in $\mathcal{E}$ can be uniquely written as $\hat{x} = c_{\hat{y}} + s(\hat{y})$ for some $c_{\hat{y}} \in \mathbb{C}$ (since g is a projection of $\mathcal{E}$, $\mathcal{E} = Ker(g) \oplus Im(g)$). To prove $\mathcal{E}$ and $\mathcal{E}'$ are equivalent, we have to find a Lie algebra homomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ such that the diagram (2.7) commutes.

For define $\phi : \mathcal{E} \to \mathcal{E}'$ by

$$\phi(c_{\hat{y}} + s(\hat{y})) = c_{\hat{y}} + s'(\hat{y}) + \sigma(\hat{y})$$

Then for any $c_{\hat{y}} + s(\hat{y}), c_{\hat{y'}} + s(\hat{y'}) \in \mathcal{E}$ and $k \in \mathbb{C}$,

$$\begin{aligned} \phi(k(c_{\hat{y}} + s(\hat{y})) + (c_{\hat{y'}} + s(\hat{y'}))) &= \phi(kc_{\hat{y}} + c_{\hat{y'}} + s(k\hat{y} + \hat{y'})) \\ &= kc_{\hat{y}} + c_{\hat{y'}} + s'(k\hat{y} + \hat{y'}) + \sigma(k\hat{y} + \hat{y'}) \\ &= kc_{\hat{y}} + c_{\hat{y'}} + ks'(\hat{y}) + s'(\hat{y'}) + k\sigma(\hat{y}) + \sigma(\hat{y'}) \\ &= k\phi(c_{\hat{y}} + s(\hat{y})) + \phi(c_{\hat{y'}} + s(\hat{y'})) \end{aligned} \tag{2.9}$$

Thus ϕ is linear. Also

$$\begin{aligned} [\phi(c_{\hat{y}} + s(\hat{y})), \phi(c_{\hat{y}'} + s(\hat{y'}))] &= [c_{\hat{y}} + s'(\hat{y}) + \sigma(\hat{y}), c_{\hat{y'}} + s(\hat{y'}) + \sigma(\hat{y'})] \\ &= [c_{\hat{y}}, c_{\hat{y'}}] + [c_{\hat{y}}, s'(\hat{y'})] + [c_{\hat{y}}, \sigma(\hat{y'})] \\ &\quad + [s'(\hat{y}), c_{\hat{y'}}] + [s'(\hat{y}), s'(\hat{y'})] + [s'(\hat{y}), \sigma(\hat{y'})] \\ &\quad + [\sigma(\hat{y}), c_{\hat{y'}}] + [\sigma(\hat{y}), s'(\hat{y'})] + [\sigma(\hat{y}), \sigma(\hat{y'})] \\ &= [s'(\hat{y}), s'(\hat{y'})] \end{aligned} \tag{2.10}$$

(since $\sigma : \mathcal{L}(G) \to \mathbb{C}$ and $\mathbb{C} \cong f(\mathbb{C}) \subseteq Z(\mathcal{E})$, $[\sigma(y), s(y')] = 0$ for all $s(y') \in \mathcal{E}$) and

$$\begin{aligned} \phi([c_{\hat{y}} + s(\hat{y}), c_{\hat{y'}} + s(\hat{y'})]) &= \phi([c_{\hat{y}}, c_{\hat{y'}}] + [c_{\hat{y}}, s(\hat{y'})] + [s(\hat{y}), c_{\hat{y'}}] + [s(\hat{y}), s(\hat{y'})] \\ &= \phi([s(\hat{y}), s(\hat{y'})]) \\ &= \phi(\mu(\hat{y}, \hat{y'}) + s([\hat{y}, \hat{y'}])) \\ &= \mu(\hat{y}, \hat{y'}) + s'([\hat{y}, \hat{y'}]) + \sigma([\hat{y}, \hat{y'}]) \\ &= \nu(\hat{y}, \hat{y'}) + s'([\hat{y}, \hat{y'}]) \\ &= [s'(\hat{y}), s'(\hat{y'})] \end{aligned} \tag{2.11}$$

$(9), (10)$ and (11) implies that ϕ is a Lie algebra homomorphism. Also

$$\begin{aligned} ((g' \circ \phi) - g)(c_{\hat{y}} + s(\hat{y})) &= (g' \circ \phi)(c_{\hat{y}} + s(\hat{y})) - g_(c_{\hat{y}} + s(\hat{y})) \\ &= g'(c_{\hat{y}} + s'(\hat{y}) + \sigma(\hat{y})) - g(c_{\hat{y}} + s(\hat{y})) \\ &= 0 \end{aligned} \tag{2.12}$$

(since, $g'(c_{\hat{y}}) = g'(\sigma(\hat{y})) = g(c_{\hat{y}}) = 0$ in $\mathcal{L}(G)$ because $c_{\hat{y}}, \sigma(\hat{y}) \in \mathbb{C}$). Thus $g' \circ \phi = g$. Also $\phi \circ f = f'$. That we got a Lie algebra homomorphism ϕ such that the diagram (2.7) commutes and thus $\mathcal{E}$ and $\mathcal{E}'$ are equivalent. Therefore, Ψ is one-one.

Let μ be a 2-cocycle. Define a set $\mathcal{E}_\mu = \mathcal{L}(G) \oplus \mathbb{C}$ with Lie bracket is given by

$$[(\hat{x},c),(\hat{y},d)] = ([\hat{x},\hat{y}],\mu(\hat{x},\hat{y}))$$

Then for any $(\hat{x},c),(\hat{y},d),(\hat{z},k) \in \mathcal{E}_\mu$ and $p \in \mathbb{C}$,

$$\begin{aligned}[p(\hat{x},c)+(\hat{y},d),(\hat{z},k)] &= [(p\hat{x}+\hat{y},pc+d),(\hat{z},k)] \\ &= ([p\hat{x}+\hat{y},\hat{z}],\ \mu(p\hat{x}+\hat{y},\hat{z})] \\ &= (p[\hat{x},\hat{z}]+[\hat{y},\hat{z}],\ p\mu(\hat{x},\hat{z})+\mu(\hat{y},\hat{z}) \\ &= p[(\hat{x},c),(\hat{z},k)]+[(\hat{y},d),(\hat{z},k)]\end{aligned} \tag{2.13}$$

Similarly, $[(\hat{x},c),p(\hat{y},d)+(\hat{z},k)] = [(\hat{x},c),(\hat{y},d)]+p[(\hat{x},c),(\hat{z},k)]$ and thus bracket operation is bilinear.
For any $(\hat{x},c) \in \mathcal{E}_\mu$,

$$[(\hat{x},c),(\hat{x},c)] = ([\hat{x},\hat{x}],\mu(\hat{x},\hat{x})) = (0,0)$$

Also,

$$\begin{aligned}&[[(\hat{x},c),(\hat{y},d)],(\hat{z},k)]+[[(\hat{y},d),(\hat{z},k)],(\hat{x},c)]+[[(\hat{z},k),(\hat{x},c)],(\hat{y},d)] \\ &\quad= [([\hat{x},\hat{y}],\mu(\hat{x},\hat{y})),\ (\hat{z},k)]+[([\hat{y},\hat{z}],\mu(\hat{y},\hat{z})),\ (\hat{x},c)] \\ &\qquad+[([\hat{z},\hat{x}],\mu(\hat{z},\hat{x})),\ (\hat{y},d)] \\ &\quad= ([[\hat{x},\hat{y}],\hat{z}],\ [\mu(\hat{x},\hat{y}),k])+([[\hat{y},\hat{z}],\hat{x}],\ [\mu(\hat{y},\hat{z}),c]) \\ &\qquad+([[\hat{z},\hat{x}],\hat{y}],\ [\mu(\hat{z},\hat{x}),d]) \\ &\quad= (0,0)\end{aligned} \tag{2.14}$$

(using tha Jacobi identity of $\mathcal{L}(G)$ and also the fact that $\mathbb{C}$ is an abelian Lie algebra). Thus $\mathcal{E}_\mu$ becomes a Lie algebra.
Define

$$0 \to \mathbb{C} \xrightarrow{f} \mathcal{E}_\mu \xrightarrow{g} \mathcal{L}(G) \to 0$$

as follows:

$$f(c) = (0,c) \text{ and } g(\hat{x},c) = \hat{x}$$

Then

$$f(kc+d) = (0,kc+d) = kf(c)+f(d)$$

i.e. f is linear. Also

$$[f(c),f(d)] = [(0,c),(0,d)] = ([0,0],\alpha(0,0)) = 0$$

and

$$f([c,d]) = f(0) = 0$$

together implies $f([c,d]) = [f(c),f(d)]$. Thus f is a Lie algebra homomorphism. Also

$$\begin{aligned}g(k(\hat{x},c)+(\hat{y},d)) &= g(k\hat{x}+\hat{y},kc+d) \\ &= k\hat{x}+\hat{y} \\ &= kg(\hat{x},c)+g(\hat{y},d)\end{aligned}$$

and

$$[g(\hat{x},c),g(\hat{y},d)] = [\hat{x},\hat{y}] \text{ and } g([(\hat{x},c),(\hat{y},d)]) = [\hat{x},\hat{y}]$$

implies that $[g(\hat{x},c),g(\hat{y},d)] = g([(\hat{x},c),(\hat{y},d)])$. Thus g is a Lie algebra homomorphism. Also,

$$\begin{aligned} Ker(g) &= \{(\hat{x},c) \in \mathcal{E}_\mu : g(\hat{x},c) = 0\} \\ &= \{(\hat{x},c) \in \mathcal{E}_\mu : \hat{x} = 0\} \\ &= \{(0,c) : c \in \mathbb{C}\} \\ &= Im(f) \end{aligned} \tag{2.15}$$

and for $(0,c) \in f(\mathbb{C})$ and $(\hat{x},d) \in \mathcal{E}_\mu$,

$$[(0,c),(\hat{x},d)] = ([0,\hat{x}],\mu(0,\hat{x})) = (0,0) \tag{2.16}$$

ie., $f(\mathbb{C}) \subseteq Z(\mathcal{E}_\mu)$.Therefore, $(\mathcal{E}_\mu; f, g)$ is a central extension of $\mathcal{L}(G)$ by $\mathbb{C}$.
Consider a section $s : \mathcal{L}(G) \to \mathcal{E}_\mu$ defined by $s(\hat{x}) = (\hat{x},0)$. Then $(g \circ s)(\hat{x}) = g(\hat{x},0) = \hat{x} = I_{\mathcal{L}(G)}(\hat{x})$ and

$$\begin{aligned} [s(\hat{x}),s(\hat{y})] &= [(\hat{x},0),(\hat{y},0)] \\ &= ([\hat{x},\hat{y}],[0,0] + \mu(\hat{x},\hat{y})) \\ &= ([\hat{x},\hat{y}],0) + (0,\mu(\hat{x},\hat{y})) \\ &= s([\hat{x},\hat{y}]) + f(\mu(\hat{x},\hat{y})) \\ &= s([\hat{x},\hat{y}]) + \mu(\hat{x},\hat{y}) \end{aligned} \tag{2.17}$$

(since, f is an inclusion). i.e.

$$\mu(\hat{x},\hat{y}) = [s(\hat{x}),s(\hat{y})] - s([\hat{x},\hat{y}])$$

Thus, $(\mathcal{E}_\mu; f, g)$ is a central extension with $\Psi([\mathcal{E}_\mu]) = [\mu]$ and hence Ψ is onto. □

The following is an illustration of the above theorem.

Example 2.6. Let G and H be the subgroups of $\mathbb{H}(\mathbb{Z}_p)$ as in example 2.2. Then the Plesken Lie algebras of G and H are

$$\mathcal{L}(G) = span_{\mathbb{C}}\{\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : y,z \in \mathbb{Z}_p\} \text{ and}$$

$$\mathcal{L}(H) = span_{\mathbb{C}}\{\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : y \in \mathbb{Z}_p\}$$

respectively. Here $\mathcal{L}(H)$ is a one dimensional abelian Plesken Lie algebra and thus it is isomorphic to the Lie algebra $\mathbb{C}$ with trivial Lie bracket. Also $\mathcal{L}(G)$ is a two dimensional abelian Lie algebra. We have, a Lie algebra L of dimension n is abelian if and only if $\dim(\mathcal{M}(L)) = \frac{n(n-1)}{2}$ where $\mathcal{M}(L)$ is the multiplier of L (see [6]). Also P. G. Batten proved that, if L is a finite dimensional Lie algebra then $\mathcal{M}(L) \cong H^2(L,\mathbb{C})$ (see [2]). Thus here for the two dimensional abelian Plesken Lie algebra $\mathcal{L}(G)$,

$$\dim(H^2(\mathcal{L}(G),\mathbb{C})) = \dim(\mathcal{M}(\mathcal{L}(G))) = 1$$

i.e. $H^2(\mathcal{L}(G),\mathbb{C})$ is a linear space spanned by one element. Then by Theorem 2.5, there is only one non-trivial central extension of $\mathcal{L}(G)$ by $\mathbb{C}$ up to equivalence. Since the central extension $(\mathcal{E}; f, g)$ of $\mathcal{L}(G)$ by $\mathbb{C}$ in example 2.2 is a non-split extension, the cohomology class $[\alpha]$ in $H^2(\mathcal{L}(G),\mathbb{C})$ corresponding to $[\mathcal{E}]$ is non zero. Hence any non-trivial central central extension of $\mathcal{L}(G)$ by $\mathbb{C}$ is equivalent to the extension $(\mathcal{E}; f, g)$ in example 2.2.

3 Projective representations of Plesken Lie algebras

A *projective representation* of a Plesken Lie algebra $\mathcal{L}(G)$ is a Lie algebra homomorphism $\phi : \mathcal{L}(G) \to \mathfrak{pgl}(V)$ (for some finite dimensional vector space V), where $\mathfrak{pgl}(V)$ is the quotient Lie algebra $\mathfrak{gl}(V)/\{kI_V : k \in \mathbb{C}\}$.

Every linear representation of $\mathcal{L}(G)$ is a projective representation. For, consider a linear representation $\rho : \mathcal{L}(G) \to \mathfrak{gl}(V)$ of $\mathcal{L}(G)$ and the natural homomorphism $\pi : \mathfrak{gl}(V) \to \mathfrak{pgl}(V)$, then the composition $\pi \circ \rho : \mathcal{L}(G) \to \mathfrak{pgl}(V)$ is a projective representation.

The following proposition gives a characterization for the projective representation of Lie algebras.

Proposition 3.1. *Let G be a finite group and ϕ a projective representation of $\mathcal{L}(G)$ on V. Then there is a linear map $\Phi : \mathcal{L}(G) \to \mathfrak{gl}(V)$ and a bilinear map $\mu : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C}$ such that*

$$[\Phi(\hat{x}), \Phi(\hat{y})] = \mu(\hat{x}, \hat{y})I_V + \Phi([\hat{x}, \hat{y}]) \text{ for all } \hat{x}, \hat{y} \in \mathcal{L}(G). \tag{3.1}$$

Conversely, if there is a linear map Φ and a bilinear map μ satisfying (3.1), then $\pi \circ \Phi : \mathcal{L}(G) \to \mathfrak{pgl}(V)$ where $\pi : \mathfrak{gl}(V) \to \mathfrak{pgl}(V)$ is the canonical homomorphism, is a projective representation of $\mathcal{L}(G)$.

Proof. Suppose $\phi : \mathcal{L}(G) \to \mathfrak{pgl}(V)$ is a projective representation and π is the natural homomorphism from $\mathfrak{gl}(V)$ to $\mathfrak{pgl}(V)$. Let X be the coset representatives of $\mathfrak{gl}(V)$ in $\mathfrak{pgl}(V)$. Define $\Phi : \mathcal{L}(G) \to \mathfrak{gl}(V)$ by choosing for each $x \in \mathcal{L}(G)$, an element A_x in $\mathfrak{gl}(V)$ such that $\pi(A_{\hat{x}}) = \phi(\hat{x})$. Then ,

$$\Phi(\hat{x}) = A_{\hat{x}}$$

is a linear map. Also,

$$\begin{aligned} [\Phi(\hat{x}), \Phi(\hat{y})] - \Phi([\hat{x}, \hat{y}]) &= [A_{\hat{x}}, A_{\hat{y}}] - A_{[\hat{x},\hat{y}]} \\ &= \mu(\hat{x}, \hat{y})I_V \end{aligned} \tag{3.2}$$

(since, $\pi([A_{\hat{x}}, A_{\hat{y}}] - A_{[\hat{x},\hat{y}]}) = [\phi(\hat{x}), \phi(\hat{y})] - \phi([\hat{x}, \hat{y}]) = 0$ in $\mathfrak{pgl}(V)$) where $\mu : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C}$ is a bilinear map. Conversely given a linear map $\Phi : \mathcal{L}(G) \to \mathfrak{gl}(V)$ satisfying (3.1).

Consider

$$\mathcal{L}(G) \xrightarrow{\Phi} \mathfrak{gl}(V) \xrightarrow{\pi} \mathfrak{pgl}(V)$$

where π is the natural homomorphism. Then

$$\begin{aligned} [(\pi \circ \Phi)(\hat{x}), (\pi \circ \Phi)(\hat{y})] &= [\pi(\Phi(\hat{x})), \pi(\Phi(\hat{y}))] \\ &= \pi([\Phi(\hat{x}), \Phi(\hat{y})]) \\ &= \pi(\mu(\hat{x}, \hat{y})I_V + \Phi([\hat{x}, \hat{y}])) \\ &= \pi(\Phi([\hat{x}, \hat{y}])) = (\pi \circ \Phi)([\hat{x}, \hat{y}]) \end{aligned} \tag{3.3}$$

That is, $\pi \circ \Phi$ is a projective representation of $\mathcal{L}(G)$. □

Now we can define the projective representation of $\mathcal{L}(G)$ as follows:

Definition 3.2. A linear map $\Phi : \mathcal{L}(G) \to \mathfrak{gl}(V)$ is a projective representation of $\mathcal{L}(G)$ if there exists a bilinear map $\mu : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C}$ such that

$$[\Phi(\hat{x}), \Phi(\hat{y})] = \mu(\hat{x}, \hat{y})I_V + \Phi([\hat{x}, \hat{y}]).$$

Recall that an extension $0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0$ *splits* if there exist a Lie algebra homomorphism $s : \mathcal{L}(G) \to \mathcal{E}$ such that $g \circ s = 1_{\mathcal{L}(G)}$. It is also known that an extension splits if and only if $\mathcal{E}$ is isomorphic to the semidirect product Lie algebra $\mathcal{L}(H) \rtimes \mathcal{L}(G)$ (see [7]).

Example 3.3. Consider the central extension $0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0$ in example 2.2. Then the semidirect product $\mathcal{L}(H) \rtimes \mathcal{L}(G)$ of $\mathcal{L}(H)$ and $\mathcal{L}(G)$ is the Lie algebra with underlying vector space $\mathcal{L}(H) \times \mathcal{L}(G)$ and the Lie bracket $[(\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2)] = (\hat{y}_1\hat{x}_2 - \hat{y}_2\hat{x}_1, [\hat{y}_1, \hat{y}_2])$ (see [7]). Note that for every $\hat{x} \in \mathcal{L}(H), \hat{y} \in \mathcal{L}(G), \hat{y}\hat{x} = 0$ and also $\mathcal{L}(G)$ is abelian. Thus $\mathcal{L}(H) \rtimes \mathcal{L}(G)$ becomes an abelian Lie algebra. Since $\mathcal{E}$ is a non-abelian Lie algebra, $\mathcal{E} \ncong \mathcal{L}(H) \rtimes \mathcal{L}(G)$ and thus $0 \to \mathcal{L}(H) \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{L}(G) \to 0$ is not a split extension. Then there exists a linear map $s : \mathcal{L}(G) \to \mathcal{E}$ such that $g \circ s = 1_{\mathcal{L}(G)}$ and s is not a Lie algebra homomorphism. Now define $\mu : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathcal{L}(H)$ by

$$\mu(\hat{x}, \hat{y}) = [s(\hat{x}), s(\hat{y})] - s([\hat{x}, \hat{y}])$$

and $\Phi : \mathcal{L}(G) \to \mathfrak{gl}(V)$ by

$$\Phi(\hat{x}) = s(\hat{x})I_V$$

Then μ is a bilinear map and Φ is a linear map such that

$$[\Phi(\hat{x}), \Phi(\hat{y})] = \mu(\hat{x}, \hat{y})I_V + \Phi([\hat{x}, \hat{y}]).$$

and hence Φ is a projective representation of $\mathcal{L}(G)$.

Next we proceed to explore the connection between projective representations and cohomology groups. Consider the projective representation $\Phi : \mathcal{L}(G) \to \mathfrak{gl}(V)$, then for any $\Phi(\hat{x}), \Phi(\hat{y}), \Phi(\hat{z}) \in \mathfrak{gl}(V)$,

$$[[\Phi(\hat{x}), \Phi(\hat{y})], \Phi(\hat{z})] + [[\Phi(\hat{y}), \Phi(\hat{z})], \Phi(\hat{x})] + [[\Phi(\hat{z}), \Phi(\hat{x})], \Phi(\hat{y})] = 0 \tag{3.4}$$

(by Jacobi identity in $\mathfrak{gl}(V)$). Also

$$\begin{aligned}[][\Phi(\hat{x}), \Phi(\hat{y})], \Phi(\hat{z})] &= [\mu(\hat{x}, \hat{y})I_V + \Phi([\hat{x}, \hat{y}]), \Phi(\hat{z})] \\ &= \mu(\hat{x}, \hat{y})[I_V, \Phi(\hat{z})] + [\Phi([\hat{x}, \hat{y}]), \Phi(\hat{z})] \\ &= [\Phi([\hat{x}, \hat{y}]), \Phi(\hat{z})] \\ &= \mu([\hat{x}, \hat{y}], \hat{z})I_V + \Phi([[\hat{x}, \hat{y}], \hat{z}])\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&[[\Phi(\hat{x}), \Phi(\hat{y})], \Phi(\hat{z})] + [[\Phi(\hat{y}), \Phi(\hat{z})], \Phi(\hat{x})] + [[\Phi(\hat{z}), \Phi(\hat{x})], \Phi(\hat{y})] \\ &= \mu([\hat{x}, \hat{y}], \hat{z})I_V + \Phi([[\hat{x}, \hat{y}], \hat{z}]) + \mu([\hat{y}, \hat{z}], \hat{x})I_V \\ &\quad + \Phi([[\hat{y}, \hat{z}], \hat{x}]) + \mu([\hat{z}, \hat{x}], \hat{y})I_V + \Phi([[\hat{z}, \hat{x}], \hat{y}]) \\ &= (\mu([\hat{x}, \hat{y}], \hat{z}) + \mu([\hat{y}, \hat{z}], \hat{x}) + \mu([\hat{z}, \hat{x}], \hat{y}))I_V \\ &\quad + \Phi([[\hat{x}, \hat{y}], \hat{z}] + [[\hat{y}, \hat{z}], \hat{x}] + [[\hat{z}, \hat{x}], \hat{y}]) \\ &= (\mu([\hat{x}, \hat{y}], \hat{z}) + \mu([\hat{y}, \hat{z}], \hat{x}) + \mu([\hat{z}, \hat{x}], \hat{y}))I_V\end{aligned} \tag{3.6}$$

Then from (3.6),

$$\mu([\hat{x}, \hat{y}], \hat{z}) + \mu([\hat{y}, \hat{z}], \hat{x}) + \mu([\hat{z}, \hat{x}], \hat{y}) = 0 \tag{3.7}$$

μ is a bilinear map and satisfies the 2-cocycle condition. That is, $\mu \in Z^2(\mathcal{L}(G), \mathbb{C})$. Thus the projective representation Φ is also referred to as an *μ-representation* on the vector space V.

3.1 Projectively equivalent α-representations of Plesken Lie algebras

Here we proceed to define projectively equivalent projective representations of $\mathcal{L}(G)$ and to explicitly show the relation between equivalence class of projective representations of $\mathcal{L}(G)$ and the multiplier $H^2(\mathcal{L}(G), \mathbb{C})$.

Definition 3.4. Let Φ_1 be an μ_1-representation and Φ_2 be an μ_2-representation of $\mathcal{L}(G)$ on the complex vector spaces V and W respectively. Then Φ_1 and Φ_2 are projectively equivalent if there exists an isomorphism $f : V \to W$ and a linear map $\delta : \mathcal{L}(G) \to \mathbb{C}$ such that

$$\Phi_2(\hat{x}) = f \circ \Phi_1(\hat{x}) \circ f^{-1} + \delta(\hat{x}) I_W \text{ for all } \hat{x} \in \mathcal{L}(G)$$

If $\delta(\hat{x}) = 0$ for all $\hat{x} \in \mathcal{L}(G)$, then Φ_1 and Φ_2 are linearly equivalent.

Example 3.5. Let Φ_1 and Φ_2 be two projective representations of $\mathcal{L}(D_8)$ (where $D_8 =< a, b : a^4 = b^2 = e, bab = a^{-1} >$) where :
$\Phi_1 : \mathcal{L}(D_8) \to \mathfrak{gl}(\mathbb{C}^2)$ is defined by

$$\Phi_1(c\hat{a}) = c \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

and $\Phi_2 : \mathcal{L}(D_8) \to \mathfrak{gl}(\mathbb{C}^2)$ is defined by

$$\Phi_2(c\hat{a}) = c \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be

$$f(v_1, v_2) = \left(\frac{v_1 - iv_2}{2}, \frac{v_1 + iv_2}{2} \right)$$

and $\delta : \mathcal{L}(D_8) \to \mathbb{C}$ be $\delta(x) = 0$ for all $x \in \mathcal{L}(D_8)$. It is easy to verify that

$$\Phi_2(x)(v_1, v_2) = (f \circ \Phi_1(x) \circ f^{-1})(v_1, v_2) + \delta(x) I_{\mathbb{C}^2}(v_1, v_2) \text{ for all } x \in \mathcal{L}(D_8)$$

thus Φ_1 and Φ_2 are linearly equivalent.

Next we proceed to establish the correspondence between two projectively equivalent projective representations of $\mathcal{L}(G)$ and $H^2(\mathcal{L}(G), \mathbb{C})$.

Theorem 3.6. *Let Φ_1 be an μ_1-representation and Φ_2 be an μ_2-representation of $\mathcal{L}(G)$. If Φ_1 is projectively equivalent to Φ_2, then μ_1 and μ_2 are cohomologous.*

Proof. Suppose Φ_1 and Φ_2 are projectively equivalent projective representations of $\mathcal{L}(G)$. Then there exists an isomorphism $f : V \to W$ and a linear map $\delta : \mathcal{L}(G) \to \mathbb{C}$ such that

$$\Phi_2(\hat{x})(v) = f \circ \Phi_2(\hat{x}) \circ f^{-1} + \delta(\hat{x}) I_W \text{ for all } \hat{x} \in \mathcal{L}(G)$$

Then,

$$\begin{aligned}
\mu_2(\hat{x}, \hat{y}) I_W &= [\Phi_2(\hat{x}), \Phi_2(\hat{y})] - \Phi_2([\hat{x}, \hat{y}]) \\
&= [f \circ \Phi_2(\hat{x}) \circ f^{-1} + \delta(\hat{x}) I_W, f \circ \Phi_2(\hat{y}) \circ f^{-1} + \delta(\hat{x}) I_W] \\
&\qquad - (f \circ \Phi_2([\hat{x}, \hat{y}]) \circ f^{-1} + \delta([\hat{x}, \hat{y}]) I_W) \\
&= [f \circ \Phi_2(\hat{x}) \circ f^{-1}, f \circ \Phi_2(\hat{y}) \circ f^{-1}] - f \circ \Phi_2([\hat{x}, \hat{y}]) \circ f^{-1} - \delta([\hat{x}, \hat{y}]) I_W \\
&= f \circ ([\Phi_1(\hat{x}), \Phi_1(\hat{y})] - \Phi_1([\hat{x}, \hat{y}])) \circ f^{-1} - \delta([\hat{x}, \hat{y}]) I_W \\
&= f \circ \mu_1(\hat{x}, \hat{y}) I_V \circ f^{-1} - \delta([\hat{x}, \hat{y}]) I_W \\
&= \mu_1(\hat{x}, \hat{y}) I_W - \delta([\hat{x}, \hat{y}]) I_W
\end{aligned}$$

Thus μ_1 and μ_2 are cohomologous. □

Theorem 3.7. *Let Φ_1 be an μ_1-representation of $\mathcal{L}(G)$ and μ_2 is a 2-cocycle cohomologous to μ_1, then there exists an μ_2-representation Φ_2 of $\mathcal{L}(G)$ which is projectively equivalent to Φ_1.*

Proof. Suppose Φ_1 is an μ_1-representation of $\mathcal{L}(G)$ and μ_2 is a 2-cocycle which is cohomologous to μ_1. Then there exists a linear map $\sigma : \mathcal{L}(G) \to \mathbb{C}$ such that

$$\mu_1(\hat{x}, \hat{y}) - \mu_2(\hat{x}, \hat{y}) = -\sigma([\hat{x}, \hat{y}]). \tag{3.8}$$

Define $\Phi_2 : \mathcal{L}(G) \to \mathfrak{gl}(V)$ by

$$\Phi_2(\hat{x}) = \Phi_1(\hat{x}) - \sigma(\hat{x}) I_V.$$

Clearly Φ_2 is a linear map and

$$\begin{aligned}[\Phi_2(\hat{x}), \Phi_2(\hat{y})] &= [\Phi_1(\hat{x}) - \sigma(\hat{x})I_v, \Phi_1(\hat{y}) - \sigma(\hat{y})I_v] \\ &= [\Phi_1(\hat{x}), \Phi_1(\hat{y})] \\ &= \mu_1(\hat{x}, \hat{y})I_V + \Phi_1([\hat{x}, \hat{y}]) \\ &= \mu_1(\hat{x}, \hat{y})I_V + \Phi_2([\hat{x}, \hat{y}]) + \sigma([\hat{x}, \hat{y}])I_V \\ &= \mu_2([\hat{x}, \hat{y}])I_V + \Phi_2([\hat{x}, \hat{y}])\end{aligned} \tag{3.9}$$

i.e. Φ_2 is an μ_2-representation projectively equivalent to Φ_1 (take $f = I_V$). □

The following gives an example for projectively equivalent projective representations.

Example 3.8. Consider the μ-representation Φ of $\mathcal{L}(G)$ as in example 3.3, where

$$\Phi(\hat{x}) = s(\hat{x})I_V \text{ and } \mu(\hat{x}, \hat{y}) = [s(\hat{x}), s(\hat{y})] - s([\hat{x}, \hat{y}]).$$

Define $\Psi : \mathcal{L}(G) \to \mathfrak{gl}(V)$ by

$$\Psi(\hat{x}) = s(\hat{x})I_V - \sigma(\hat{x})I_V$$

and $\nu : \mathcal{L}(G) \times \mathcal{L}(G) \to \mathbb{C}$ by

$$\nu(\hat{x}, \hat{y}) = \mu(\hat{x}, \hat{y}) - \sigma([\hat{x}, \hat{y}])$$

where $\sigma : \mathcal{L}(G) \to \mathbb{C}$ is defined by $\sigma(\hat{x}) = \sigma(\sum_{i=1}^{n} a_i\hat{g}_i) = \sum_{i=1}^{n} a_i$. Then Ψ is a ν-representation. Let $f = I_V$ and $\delta : \mathcal{L}(G) \to \mathbb{C}$ be given by $\delta(\hat{x}) = -\sigma(\hat{x})$, then

$$\Phi(\hat{x}) = f \circ \Psi(\hat{x}) \circ f^{-1} + \delta(\hat{x})I_V \text{ for all } \hat{x} \in \mathcal{L}(G).$$

Hence Φ and Ψ are projectively equivalent.

From theorem 3.7, we can conclude that any projective representation up to projective equivalence defines an element of the second cohomology group $H^2(\mathcal{L}(G), \mathbb{C})$.

4 Conclusions

- The classical First and Second Whitehead Lemmas state that the first, respectively second cohomology group of a finite-dimensional semisimple Lie algebra with coefficients in any finite-dimensional module vanishes (see [8]). Thus from section 3, we can conclude that for semisimple Plesken Lie algebras, there exists no projectively equivalent projective representations. In other words, there exists only linearly equivalent projective representations for a semisimple Plesken Lie algebra. Also we can conclude that the central extensions of a semisimple Plesken Lie algebra $\mathcal{L}(G)$ by $\mathbb{C}$ are equivalent and it is the Lie algebra $\mathcal{L}(G)\oplus\mathbb{C}$ with the Lie bracket $[(\hat{x}, c), (\hat{y}, d)] = ([\hat{x}, \hat{y}], [c, d])$ for $(\hat{x}, c), (\hat{y}, d) \in \mathcal{L}(G) \oplus \mathbb{C}$.
- We have there is a bijective correspondence between set of equivalence class of central extensions of $\mathcal{L}(G)$ and $\mathbb{C}$ and $H^2(\mathcal{L}(G), \mathbb{C})$. Also there is a bijective correspondence between set of projective equivalence class of projective representations of $\mathcal{L}(G)$ and $H^2(\mathcal{L}(G), \mathbb{C})$. Thus we can classify the projectively equivalent projective representations of $\mathcal{L}(G)$ by central extensions of $\mathcal{L}(G)$ by $\mathbb{C}$.

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