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SOME ASPECTS OF E-SECONDARY SUBMODULES

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Abstract Let R be a commutative ring with unity and M be an R-module. The aim of this article is to introduce and investigate certain properties of a new class of submodules namely e-secondary submodules. A submodule N of M is said to be e-secondary if the endomorphism given by multiplication by a, a $\in$ R, that is, f : N $\rightarrow$ N such that f(N) = aN, then either aN $\leq_e$ M or a^tN = 0, for some t $\in \mathbb{N}$. This notion is then further extended to introduce fully e-secondary submodules of a module.

1 Introduction

Let R be a ring with unity and M be an R-module. A non-zero submodule N of M is called essential if N $\cap$ L $\neq \{0\}$ for every non-zero submodule L of M [4]. A non-zero submodule S of M is said to be secondary if for each a $\in$ R, the endomorphism of S given by multiplication by a is either surjective or nilpotent [12]. A non-zero submodule S of M is said to be second if for each a $\in$ R, the homomorphism of S given by multiplication by a is either surjective or zero [17]. A submodule N of M is called closed if N has no proper essential extention in M, i.e, if N $\leq_e$ K $\leq$ M then N = K [4]. A module M is called uniform if the intersection of any two non-zero submodules of M is non-zero [18]. The R-module M is called faithful if rM = 0 (r $\in$ R) implies r = 0 [6].
The first section of this article investigates various properties of e-secondary submodules and its relation with other classes of modules. The second section further extends the notion of e-secondary submodules to fully e-secondary submodules. The intersection and direct sum of fully e-secondary submodules have been discussed. A module M is called a multiplication module if every submodule N of M can be expressed as N = IM for some ideal I of R [7]. The third section of this paper deals with the behaviour of e-secondary submodules in multiplication modules.

Throughout this article, R will denote a commutative ring with unity and $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denotes the set of integers, rationals and reals respectively.

2 e-secondary submodules of a module

In this section we define e-secondary submodules of a module. Certain properties of this class of modules is studied and its connection with other types of modules is investigated.

Definition 2.1. Let M be an R-module. A submodule N of M is said to be e-secondary if the endomorphism given by multiplication by a, a $\in$ R, that is, f : N $\rightarrow$ N such that f(N) = aN, then aN $\leq_e$ M or a^tN = 0, for some t $\in \mathbb{N}$.

Examples:
1) R = $\mathbb{Z}$; M = $\mathbb{Z}$; N = n$\mathbb{Z}$ where n $\in \mathbb{Z}$. Consider f : n$\mathbb{Z} \rightarrow$ n$\mathbb{Z}$ such that f(n$\mathbb{Z}$) = an$\mathbb{Z}$,(a $\in \mathbb{Z}$). When a = 0, we get an$\mathbb{Z}$ = 0 and an$\mathbb{Z} \leq_e \mathbb{Z}$ when a $\neq$ 0.

2) Consider the $\mathbb{Z}$ module $\mathbb{Z}_4$. Submodules of $\mathbb{Z}_4$ are : $\{0\}$ = I, $\{0, 2\}$ = N, $\mathbb{Z}_4$. Consider f : N $\rightarrow$ N such that f(N) = aN, (a $\in \mathbb{Z}$). When a = 2m, m $\in \mathbb{Z}$, we have aN = 0 and when a = 2m

+ 1, m $\in \mathbb{Z}$, we get aN = $\{0, 2\} \leq_e \mathbb{Z}_4$.

3) Let M be a divisible R-module. Then M is an e-secondary submodule of itself. This follows from the definition that if M is a divisible module then rM = M for every non-zero r $\in$ R and the fact that M is always an essential submodule of itself.

4) The R-module (0) = M is trivially considered as e-secondary as for any a $\in$ R and any endomorphism f : M $\to$ M such that f(M) = aM, then we get a^tM = 0, t $\in \mathbb{N}$.

Remark 2.2. Submodule of e-secondary need not be e-secondary in general.

For example : Let us consider the $\mathbb{Z}$ module $\mathbb{R}$ .Then $\mathbb{R}$ is a submodule of itself. Consider f : $\mathbb{R} \to \mathbb{R}$ such that f($\mathbb{R}$) = a$\mathbb{R}$, (a $\in \mathbb{Z}$). Then for a non-zero a $\in \mathbb{Z}$, a$\mathbb{R} \leq_e \mathbb{R}$ and a$\mathbb{R}$ = 0 when a = 0. But the submodule $\mathbb{Q}$ of $\mathbb{R}$ is not e-secondary as the intersection of submodule a$\mathbb{Q}$ with the submodule $\mathbb{Q}^c \cup \{0\}$ of $\mathbb{R}$ is zero, that is, a$\mathbb{Q} \cap (\mathbb{Q}^c \cup \{0\})$ = 0 .

Proposition 2.3. *Every non-zero e-secondary submodule of a module is essential.*

Proof. Let M be an R-module and N be a non-zero e-secondary submodule of M. Since aN $\subseteq$ N and aN $\cap$ L $\neq \{0\}$, for all L $\leq$ M, this implies N $\cap$ L $\neq \{0\}$. Thus, N is essential in M.
Note: Here we have used the fact that aN $\leq_e$ M for atleast one a $\in$ R. Since aN $\leq_e$ M or a^tN = 0 holds for every a $\in$ R, so for a = 1(unity of R), we get a^tN = 1^tN = N $\neq$ 0. Therefore 1.N $\leq_e$ M.

□

Theorem 2.4. *Let M and X be R-modules and N be a non-zero e-secondary submodule of M. Let g: M $\to$ X be a homomorphism and f be the restriction of g on N. If f is a monomorphism, then so is g.*

Proof. As N is e-secondary, by previous result N is also essential. Given that f is the restriction of g on N, i.e, f: N $\to$ X. Let x $\in$ N $\cap$ Kerg. This implies x $\in$ N and x $\in$ Kerg. Therefore, x $\in$ N and g(x) = 0, which ultimately gives f(x) = 0 (since f is the restriction of g on N). As f is a monomorphism, so f(x) = 0 implies x = 0. Therefore, N $\cap$ Kerg = (0). Since N $\leq_e$ M and Kerg $\leq$ M, we get Kerg = $\{0\}$. Thus, g is a monomorphism. □

Remark 2.5. Essential submodule need not be e-secondary.

For example : Consider the $\mathbb{Z}$ module $\mathbb{Z}_6$ .The submodules are: $\{0\}$, $\{0, 2, 4\}$, $\{0, 3\}$, $\mathbb{Z}_6$ .Then $\mathbb{Z}_6 \leq_e \mathbb{Z}_6$. But consider the endomorphism f : $\mathbb{Z}_6 \to \mathbb{Z}_6$ such that f(x) = 2x , (x $\in \mathbb{Z}_6$). Then 2$\mathbb{Z}_6$ = $\{0, 2, 4\}$ and $\{0, 2, 4\} \cap \{0, 3\} = \{0\}$. Therefore, 2$\mathbb{Z}_6$ is not essential in $\mathbb{Z}_6$ and also $2^t\mathbb{Z}_6 \neq 0$ for any t $\in \mathbb{N}$. Thus, $\mathbb{Z}_6$ is an essential submodule of $\mathbb{Z}_6$ but is not an e-secondary submodule.

Proposition 2.6. *Let M be an R-module where R is a field. Let N be a cyclic submodule of M such that N is also essential in M. Then N is an e-secondary submodule of M.*

Proof. Since N is cyclic, so by definition there exists some m $\in$ M such that N is of the form N = $\{$rm : r $\in$ R$\}$. Consider the endomorphism f: N $\to$ N given by multiplication by a, a $\in$ R, i.e, f(N) = aN. Now aN = a(Rm)= (aR)m = Rm = N and since N is essential in M, so aN $\leq_e$ M. □

Remark 2.7. There are submodules which are both essential and e-secondary.

Example:
1) A simple module is both an essential as well as an e-secondary submodule (of itself).
2) Submodule N = $\{0, 2\}$ of $\mathbb{Z}_4$ is both essential and e-secondary.
3) Every uniform module is both essential and e-secondary.

In [21] Heuberger gave an equivalent definition of an essential submodule as : The submodule N of the right R-module M is essential in M if for all 0 $\neq$ x $\in$ M, r $\in$ R and n $\in \mathbb{Z}$ exists, such that 0 $\neq$ xr + nx $\in$ N.

Theorem 2.8. *Let K, N be submodules of an R-module M such that N $\leq$ K $\leq$ M. If K is an e-secondary submodule of M, then the submodule $\frac{K}{N}$ is an e-secondary submodule of the quotient module $\frac{M}{N}$.*

Proof. Consider the endomorphism *f* from $\frac{K}{N}$ to $\frac{K}{N}$ given by multiplication by a, a $\in$ R, ie, *f* (x+N) = a(x+N), x $\in$ K.
Case 1: aK $\leq_e$ M
Then using the above mentioned definition, we get for all non-zero x in M, r $\in$ R and n $\in \mathbb{Z}$ exists, such that 0 $\neq$ xr + nx $\in$ aK. This implies (xr + nx) + N $\in$ aK + N. Thus, a($\frac{K}{N}$) $\leq_e \frac{M}{N}$
Case 2: If a^tx = 0, then a^t(x+N) = a^tx + a^tN = 0 + N = N. Therefore, a^t(x+N) = 0 in $\frac{M}{N}$. □

Remark 2.9. The concept of secondary and e-secondary need not imply each other.

Example: (i) $\mathbb{Z}$ as a $\mathbb{Z}$ module is e-secondary but not secondary as n$\mathbb{Z} \leq_e \mathbb{Z}$ for all n $\in \mathbb{Z}$ but n$\mathbb{Z} \neq \mathbb{Z}$ for any n $\neq$ 1, -1 and n$\mathbb{Z}$ = 0 only when n = 0.
(ii) Consider the $\mathbb{Z}$ module $\mathbb{Z}_6$ .The submodule N = {0, 3} of $\mathbb{Z}_6$ is secondary as for every a $\in \mathbb{Z}$, aN = N or a^tN = 0 but N is not e-secondary.

Proposition 2.10. *Let N be a non-zero secondary submodule of M. Then N is an e-secondary submodule of M if and only if N is essential in M.*

Proof. Let M be an R module and N be a secondary submodule of M. Then for any endomorphism f : N $\to$ N such that f(N) = aN (a $\in$ R), we have either aN = N (i.e surjective) or a^tN = 0. Suppose N is an essential submodule of M. If a^tN = 0 then result holds trivially. If aN = N, then N being essential, we get aN $\leq_e$ M. Thus, N is an e-secondary submodule of M.
Conversely, let N be an e-secondary submodule of M. Result follows from prop 2.3. □

Corollary 2.11. *Let N be a non-zero second submodule of M. Then N is an e-secondary submodule of M if and only if N is essential in M.*

Proof. Since every second submodule is also a secondary submodule, so result follows from the previous proposition. □

Proposition 2.12. *Let M be an R-module and N be an e-secondary submodule of M. If every submodule of N is a closed submodule of N, then N is a secondary submodule of M.*

Proof. Consider the endomorphism from N to N given by multiplication by a, for any a $\in$ R, i.e, f : N $\to$ N such that f(N) = aN. Since N is e-secondary, if a^tN = 0, then result holds trivially. If not, then aN $\leq_e$ M. This implies aN $\leq_e$ N. But aN being a closed submodule of N, aN $\leq_e$ N $\leq$ M implies that aN = N. Thus, N is a secondary submodule of M. □

Theorem 2.13. *Let M_1, M_2 be R-modules and g: $M_1 \to M_2$ be an isomorphism. If N is an e-secondary submodule of M_1, then g(N) is an e-secondary submodule of M_2.*

Proof. Since N is an e-secondary submodule of M_1, so for any endomorphism f : N $\to$ N such that f(N) = aN, a $\in$ R, then aN $\leq_e M_1$ or a^tN = 0. Suppose a^tN = 0. As g is one-one homomorphism, g(a^tN) = g(0) = 0, i.e a^tg(N) = 0. Now, suppose aN $\leq_e M_1$. This implies aN $\cap$ L $\neq$ {0} for any non-zero submodule L of M_1. As g is an isomorphism, so g(aN $\cap$ L) $\neq$ g(0) = {0}. This implies g(aN) $\cap$ g(L) $\neq$ {0}. Since g is a homomorphism, this gives ag(N) $\cap$ g(L) $\neq$ {0}. As g(L) is an arbitrary submodule of M_2, thus we get g(N) is an e-secondary submodule of M_2. □

Recall that any monomorphism f : A $\to$ B is said to be an essential monomorphism if Imf $\leq_e$ B [4].

Lemma 2.14. *Let M be an R module and f be an endomorphism from M to M given by multiplication by a, for any a $\in$ R. If f is an essential monomorphism, then M is an e-secondary submodule of itself.*

Proof. Follows from the definition. □

Recall that a module M is called a Quasi-dedekind module if for each endomorphism f of M, f $\neq$ 0, then kerf = 0 [19].

Proposition 2.15. *Let M be an R-module, where R is a boolean ring. If M is a quasi dedekind module, then M is also e-secondary.*

Proof. Consider an endomorphism f : M $\rightarrow$ M given by multiplication by a, a $\in$ R. Claim: f^2 = f . Let x $\in$ M. Consider f^2(x) = $f(f$(x)) = f(ax) = a.(ax) = a^2x = ax (since R is a boolean ring so a^2 = a). Also, f(x) = ax. This implies f^2(x) = f(x) $\Rightarrow$ f^2 = f. Let x $\in$ M; then f(x) $\in$ M. Let y = x - f(x) $\Rightarrow$ f(y) = f(x - f(x)) = f(x) - f^2(x) = f(x) - f(x) (since f^2 = f). Thus f(y) = 0 $\Rightarrow$ y $\in$ Ker$f$. Since Kerf = 0, we get y = 0 $\Rightarrow$ x - f(x) = 0 $\Rightarrow$ f(x) = x $\Rightarrow$ f is an identity mapping and therefore f(m) = m for every m $\in$ M which implies f(M) = aM = M $\leq_e$ M. □

Corollary 2.16. *If M is a finitely generated quasi-dedekind module, then M is e-secondary.*

Proof. Follows from the fact that if M is a finitely generated quasi-dedekind module then M is uniform and every uniform submodule is e-secondary. □

Theorem 2.17. *Let M be an R-module and N be a submodule of M. The following are equivalent:*
(1) N is an e-secondary submodule of M.
(2) For every ideal I of R, either I^tN = 0 for some t $\in \mathbb{N}$ or IN $\leq_e$ M where I^t = {i^t : i $\in$ I}.

Proof. (1) $\Rightarrow$ (2) : If I^tN = 0, then result holds trivially. If not, then there exists some a $\in$ I such that a^tN $\neq$ 0. This implies aN $\neq$ 0 $\Rightarrow$ aN $\leq_e$ M $\Rightarrow$ aN $\leq_e$ IN. Thus, we get IN $\leq_e$ M (using Lemma 2.3 [20]).
(2) $\Rightarrow$ (1) : Clearly. □

3 Fully e-secondary submodules

In this section we study a particular case of e-secondary submodule and name it as fully e-secondary submodule of a module. Various properties exhibited by this class of modules is explored along with its interaction with other types of modules.

Definition 3.1. Let M be an R-module. A submodule N of M is said to be fully e-secondary if the endomorphism given by multiplication by a, a $\in$ R, that is, f : N $\rightarrow$ N such that f(N) = aN, then aN $\leq_e$ M and if a^tN = 0 for any t $\in \mathbb{N}$, then a = 0.
Example: $\mathbb{Z}$ as a $\mathbb{Z}$ module is fully e-secondary.

Proposition 3.2. *Let M be an R-module. If N is a fully e-secondary submodule of M, then every submodule of M containing N is also fully e-secondary.*

Proof. Let N be a fully e-secondary submodule of M and K be a submodule of M containing N. So N $\leq$ K implies aN $\leq$ aK (a $\in$ R). As aN $\leq_e$ M, so aN $\cap$ L $\neq$ {0}, for all L $\leq$ M. Since aN $\leq$ aK, therefore aK $\cap$ L $\neq$ {0}. Thus aK $\leq_e$ M.
□

Remark 3.3. Every fully e-secondary submodule is e-secondary. However, the converse need not be true in general.
Example: Consider the submodule N = {0, 2} of the $\mathbb{Z}$ module $\mathbb{Z}_4$. Then N is e-secondary but not fully e-secondary as 2N = 0 but 2 $\neq$ 0.

Lemma 3.4. *Every fully e-secondary submodule is faithful.*

Proof. Since N is fully-secondary, so ann(N) = {0} which implies N is faithful. □

The converse need not be true in general.
Example: The submodule $\mathbb{Q}$ of the $\mathbb{Z}$ -module $\mathbb{R}$ is faithful but it is not e-secondary and thus is not fully e-secondary.

Theorem 3.5. *Let M be an R-module where R is a reduced ring. Then every e-secondary submodule N of M is fully e-secondary if and only if N is faithful.*

Proof. Let N be fully e-secondary. Then N is faithful by Prop 3.4.
Conversely, let N be a faithful submodule of M. Since N is e-secondary, so either aN $\leq_e$ M or a^tN = 0 for any a $\in$ R. If aN $\leq_e$ M for all non-zero a $\in$ R, then result is true. Suppose for some a $\in$ R, a^tN = 0. As N is faithful so a^tN = 0 $\Rightarrow$ a^t = 0 $\Rightarrow$ a = 0 (as R is a reduced ring). Thus, N is fully e-secondary. □

Remark 3.6. Intersection of two non-zero submodules of a module may be fully e-secondary even if the submodules themselves are not fully e-secondary.
Example: Consider the $\mathbb{Z}$ module $\mathbb{Z}_6$. Intersection of its submodule N_1 = {0, 2, 4} and N_2 = {0, 3} is $N_1 \cap N_2$ = {0} which is fully e-secondary (trivially) but neither N_1 nor N_2 is fully e-secondary.

Theorem 3.7. *Direct sum of fully e-secondary submodules is again fully e-secondary.*

Proof. Let M be an R-module and N_1 and N_2 be two fully e-secondary submodules of M. Then a$N_1 \leq_e$ M and a$N_2 \leq_e$ M. Since direct sum of essential is again essential, we get a$N_1 \bigoplus$ aN_2 $\leq_e$ M. This implies a($N_1 \bigoplus N_2$) $\leq_e$ M. □

In [5], Ahmed introduced the notion of essential second modules and defined it as: Let M be an R-module. Then M is said to be essential second when for any a $\in$ R, either Ma = 0 or Ma $\leq_e$ M.

Proposition 3.8. *Every fully e-secondary submodule of M is essential second.*

Proof. Consider a fully e-secondary submodule N of an R-module M. Then for any non-zero a $\in$ R, aN $\leq_e$ M which implies aN $\leq_e$ N. □

The converse however need not be true in general.
Example: The submodule N = {0, 3} of the $\mathbb{Z}$ module $\mathbb{Z}_6$ is essential second as aN $\leq_e$ N for every a $\in \mathbb{Z}$. Now consider the submodule K = {0, 2, 4} of $\mathbb{Z}_6$. Since N $\cap$ K = {0}, so 1.N $\nleq_e$ M. Thus, N is not e-secondary and therefore, not fully e-secondary.

4 e-secondary submodules in multiplication modules

This section is dedicated to the study of e-secondary submodules and its behaviour in multiplication modules. The following two propositions discusses its relation with the essential ideals of the ring.

Proposition 4.1. *Let M be a faithful multiplication R-module. Let N be a submodule of M such that N = IM. If N is fully e-secondary then I is an essential ideal of R.*

Proof. Let if possible I is not an essential ideal of R. Then there exists some ideal S of R such that I $\cap$ S = 0. Since M is a faithful multiplication module, so (0) = (I $\cap$ S)M = IM $\cap$ SM = N $\cap$ SM [22, Th. 1.7]. As N is fully e-secondary, so N is essential and thus N $\cap$ SM = 0 implies SM = 0 which implies S = 0 (as M is faithful). □

Proposition 4.2. *Let M be a faithful multiplication R-module. If every ideal of R is essential in R then M is fully e-secondary.*

Proof. Suppose M is not fully e-secondary. Then there exists a submodule S of M such that aM $\cap$ S = 0 for some non-zero a $\in$ R, which implies $<$ a $>$ M $\cap$ S = 0. Denoting the ideal $<$ a $>$ = I, we get IM $\cap$ S = 0. Since M is a multiplication module, there exists some ideal K such that S = KM. Therefore, we get IM $\cap$ KM = 0 $\Rightarrow$ (I $\cap$ K)M = 0 $\Rightarrow$ I $\cap$ K = 0 as M is faithful. But I being an essential ideal, I $\cap$ K = 0 implies K = 0. Thus, S = 0 and so aM $\leq_e$ M. □

Theorem 4.3. *Let M be a non-zero multiplication R-module having a unique maximal submodule N. If N is secondary, then N is also e-secondary.*

Proof. Let N be a secondary submodule of M and a be any arbitrary element of ring R.
Case 1: If a^tN = 0 for some t $\in$ N, then result holds clearly.
Case 2: If a^tN $\neq$ 0, then aN = N.
Suppose N is not e-secondary. Then there exists a submodule K of M such that aN $\cap$ K = 0 $\Rightarrow$ N $\cap$ K = 0. Since M is a multiplication module, so by [4] K is contained in some maximal submodule of M. But N is the only maximal submodule of M and so K $\subseteq$ N. Therefore N $\cap$ K = 0 $\Rightarrow$ K = 0 $\Rightarrow$ N is essential in M. Using Prop 2.10, N is an e-secondary submodule of M. □

References

[1] Wisbauer,R. Foundations of Module and Ring Theory, Gordon and Breach Science Publishers.

[2] Ansari-Toroghy, H., Farshadifar, F. (2007). The dual notion of multiplication modules. Taiwanese J. Math.11(4):1189–1201. DOI: 10.11650/twjm/1500404812.

[3] Atiyah,M., MacDonald,I.G. Introduction to Commutative Algebra.

[4] Goodearl , K. R., " Ring theory " Marcel Dekker , New York, 1972.

[5] Ahmed, G., Essential Second Modules; Iraqi Journal of Science, 2019, Vol. 60, No. 3, pp: 633-637 DOI: 10.24996/ijs.2019.60.3.21.

[6] Shahad,H., Al-Mothafar,N., On P-Essential Submodules; Iraqi Journal of Science, 2021, Vol. 62, No. 12, pp: 4916-4922 DOI: 10.24996/ijs.2021.62.12.29.

[7] Barnard, A. (1981). Multiplication modules. J. Algebra 71(1):174–178. DOI: 10.1016/0021-8693(81)90112-5.

[8] Farshadifar,F. S-secondary submodules of a module, Communications in Algebra, 49:4, 1394-1404, DOI: 10.1080/00927872.2020.1836189.

[9] Farshadifar, F. (2000). S-second submodules of a module. Algebra Discret. Math. (in press).

[10] Ahmed,M., Ibrahiem,T., H-essential Submodules and Homessential Modules; Iraqi Journal of Science · June 2019 DOI: 10.24996/ijs.2019.60.6.22.

[11] Ahmed,M., Weak essential submodules, Um-Salama Science Journal Vol.6(1)2009.

[12] Macdonald, I. G. (1973). Secondary representation of modules over a commutative ring. Symp. Math. 11:23–43.

[13] Yousef A. Qasim, Sahira M. Yaseen; Annihilator Essential Submodules; Iraqi Academics Syndicate International Conference for Pure and Applied Sciences Journal of Physics: Conference Series 1818 (2021) 012213 IOP Publishing doi:10.1088/1742-6596/1818/1/012213.

[14] Ansari-Toroghy, H., Farshadifar,F., Pourmortazavi, S.S., and Khaliphe,F., On Secondary Modules; International Journal of Algebra, Vol. 6, 2012, no. 16, 769 - 774.

[15] Matsumura H, Commutative ring theory, Cambridge University Press (2002).

[16] Wang, F., Kim, H. (2016). Foundations of Commutative Rings and Their Modules, Singapore: Springer.

[17] Yassemi, S. (2001). The dual notion of prime submodules. Arch. Math. (Brno). 37:273–278.

[18] Abdullah, N.K. " Semi-essential submodules and semi-uniform modules" M. Sc. Thesis. University of Tikrit, 2005.

[19] Diallo,A., Diop,P., Barry,M., On S-quasi-Dedekind Modules, Journal of Mathematics Research; Vol. 9, No. 5; October 2017 ISSN 1916-9795 E-ISSN 1916-9809.

[20] Dungh, N. V., Huynh, D. V., Smith P. F. and Wisbauer, R. 2008. Extending Modules. Pitman Research Notes in Math Serie, longman, Harlow.

[21] Heuberger, D.A., New proofs of some properties of essential submodules, CARPATHIAN J. MATH 29(2013) No.1, 19-26, Online edition ISSN 1843-4401.

[22] El-Bast, Z. A. and Smith, P. F., (1988), Multiplication modules, Comm. In Algebra, 16:755-779.

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