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Solving the Multi-Objective Linear Plus Linear Fractional Programming Problem using Different Mathematical Programming Approaches

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Abstract In this study, we address the complexities of multi-objective optimization by formulating a mathematical model for the multi-objective linear plus linear fractional programming problem (MOL+LFPP). Multi-objective optimization is crucial in various real-world applications, including economics, engineering, and decision sciences, where multiple conflicting objectives must be optimized simultaneously. Integrating linear and fractional programming within a unified framework enhances the problem's applicability, particularly in trade-offs between efficiency and resource allocation scenarios. The objective functions presented in this problem are in linear and fractional forms, both critical in optimization, with linear programming being foundational for solving many real-world problems. In contrast, fractional programming is essential for optimizing ratios and efficiency measures. We proposed three different types of approaches, viz., fuzzy programming approach (FPA), weighted sum approach (WSA), and neutrosophic goal programming approach (NGPA), that generate the set of efficient solutions of the MOL+LFPP. Two numerical illustrations are solved to demonstrate the problem's efficiency and feasibility and get a compromise solution using LINGO 20.0 optimization software. The results reveal that each approach provides distinct compromise solutions, with the fuzzy goal programming approach (FGPA-I) and NGPA offering particularly efficient results in terms of the objective values. Managerial and practical implications are also discussed.

1 Introduction

Linear programming (LP) and fractional programming problems (FPP) are fundamental areas of optimization problems that have been extensively studied and applied various discipline [2, 3, 96], including operations research [4, 5], economics [6, 7, 80], engineering [8, 9, 1, 93], and supply chain management [10, 11]. These methodologies provide powerful tools for solving complex decision-making problems that involve optimizing certain objective functions to a set of constraints. The linear programming approach obtains a compromise solution for the mathematical programming problem whose requirements are represented by linear relationships [12, 81]. The standard form of a linear programming (LP) problem consists of an objective function linear expression involving decision variables that must be either maximized or minimized while adhering to a set of linear constraints [13, 14]. These constraints represent the limitations or requirements that the solution must satisfy. This feasible region, defined by these constraints, is a convex polyhedron, and the optimal solution lies at one of the vertices of this polyhedron

[15, 89].

Fractional programming (FP) is a branch of mathematical programming that deals with optimization problems. The objective function is a ratio of two functions, typically linear or non-linear [16, 17]. These problems arise naturally in many practical applications, where the objective is to maximize or minimize the ratio of two quantitates, such as cost-to-benefit ratios, efficiency ratios, and return-on-investment ratios. The general form of an FPP can be expressed as the optimization ratio of two functions, subject to the set of constraints. Specifically, the objective is to maximize or minimize the ratio $\frac{f(y)}{g(y)}$ where $f(y)$ and $g(y)$ are functions of the decision variables y [18, 19]. When both $f(y)$ and $g(y)$ are linear functions, the problem is referred to as the linear fractional programming problem [20]. Solving FPP is more complex than solving the LP problem due to the non-linearity introduced by the ratio. However, several methods have been developed to tackle these problems, such as Charnes-cooper transformation, which converts the fractional programming into an equivalent LP problem, and Dinkelbach's algorithm, which iteratively solves a series of parametric LP problems to find the optimal solution [21]. FP is particularly useful in various fields where efficiency and productivity measures are crucial. It provides a robust framework for optimizing ratios, enabling decision-makers to achieve more balanced and efficient outcomes in complex scenarios [22, 23, 90].

In mathematical optimization, linear fractional programming problem (LFPP) extends the scope of linear programming [24, 25, 26]. In LP, the objective functions are linear, and the objective functions of LFPP are ratios of two linear functions. Consequently, LFPP is a particular case of linear programming problems [27, 28, 29, 30], where the denominator is a constant function equal to one. The simplicity and versatility of LP make it a popular choice for a wide range of optimization problems. The multi-objective linear plus linear fractional programming problem (MOL+LFPP) encompasses multiple objectives that combine linear and linear fractional programming elements. The MOL+LFPP is crucial for decision-making and utilized in diverse scenarios with multiple objectives that are subject to a set of constraints. This type is also applicable to solving various real-world problems across different fields. The versatility and effectiveness of multi-objective linear fractional programming problems (MOLFPP) make it an indispensable tool in addressing complex, multi-objective decision-making problems. Given the practical importance of MOL+LFPP, researchers have developed various solution approaches, i.e., goal programming approach, fuzzy programming approach, evolutionary algorithm, genetic algorithm, and neutrosophic goal programming approach, to tackle these problems. Each of these methods has its advantages and limitations, and their effectiveness can vary depending on the specific characteristics of the problem. Therefore, this research undertakes a detailed comparison and evaluation of these methods to guide their application and identify the conditions under which each method performs best.

The remaining sections of the manuscript are arranged as follows: The literature review is discussed in Section 2. Section 3 presents a mathematical model of the multi-objective linear plus linear fractional programming problem, and Section 4 presents the methodology of the proposed FPA, WSA, and NGPA. Section 5 presents the two numerical examples of a proposed mathematical model. Finally, Section 6 discusses the conclusion and the managerial and practical implications.

2 Literature review

Fractional programming was introduced by Charnes & Cooper [31]. After that, Zoints [32] and Schaible [33] produced some valuable ideas in the field of FP. The Fractional Transportation Problem (FTP) represents a variant of the classical transportation problem where the objective function is fractional type [34]. It means that instead of minimizing or maximizing a linear combination of transportation costs, the objective is to optimize a fractional expression involving these costs. [35] introduced a stochastic programming model to address the issue of uncertainty in transportation problems.This model combines expected and variance considerations to achieve a balance. It introduces a mean-variance approach, which optimizes profit while considering shipping costs in uncertain scenarios with varying demands [82]. Liu [36] and Bas et al.[37] presented a linear fractional mathematical transportation problem. They constructed supply demand and iterative constraints of the proposed problem and solved this model using a fuzzy goal programming approach to find the optimal solution. Agrawal & Ganesh [38] and Chauhan et al.[39] introduced a new method for tackling linear fractional programming problems that involve fuzzy variables and parameters without constraints. They used a technique called -cut to solve these problems and provided both upper and lower bounds, giving a range of possible solutions. They demonstrated how their new approach works by applying it to a fuzzy linear fractional transportation problem and pointed out flaws in an existing method by showing an example. Joshi et al.[40] presented a novel method for optimizing transportation systems with multiple objectives under uncertain conditions. This method is beneficial for decision-making in manufacturing [83, 94, 95]. They introduced the MOLFPP as an essential tool for managing various operations in unpredictable environments.

Mekawy [41] presented a mathematical model of fuzzy MOLFPP and converted it into a precise problem. Edalatpanah [42] and Borza et al.[43] presented a novel approach for resolving fuzzy linear fractional programming problems. The approach utilizes horizontal membership functions and multi-dimensional relative-distance-measure fuzzy interval arithmetic. Sheikhi Ebadi [79] proposed a method for resolving linear fractional programming transportation problems utilizing fuzzy numbers. They focused on linear fractional programming, a specialized area within non-linear programming. Sheikhi Ebadi [4] presented a novel approach to solving linear interval fractional transportation problems (ILFTPs) with interval objective functions by transforming the ILFTP into a non-linear programming problem and then converting it into a linear programming problem with additional constraints and variables. Alburaikan et al.[44] presented an innovative approach to addressing multi-objective neutrosophic linear fractional programming problems. This approach involves representing parameters as neutrosophic numbers. They transformed these problems into equivalent crisp MOLPP using a variable transformation technique and a ranking function.

Given this importance, the potential research study of Farnam Darehmiraki [45] developed a solution approach for MOFPP in a hesitant fuzzy decision environment. The utilization of linguistic variables, interactive methods, and goal planning has enabled the creation of effective strategies, which have significantly enhanced decision-making methodologies in this context [84]. Akram et al. [46] developed a multi-objective fractional transportation problem (MOFTP) mathematical model under a fermatean fuzzy environment. They transformed this mathematical model into crisp mathematical form using the Trapezoidal fermatean fuzzy parameters and LR fully Pythagorean fuzzy programming approach [85]. They also provided some numerical examples to justify this approach. Veeramani et al. [47] presented the compromise solutions framework for the MOFTP using a neutrosophic goal programming approach. The framework addresses complex transportation scenarios by considering multiple objectives: cost, time, and environmental and social concerns. Pourofoghi and Darvishi Salokolaei [48] addressed the challenge of linear fractional programming problems with a linear fractional objective subject to a set of uncertain or grey constraints. They proposed a new method inspired by the variable change technique developed by Charnes and Cooper. They also merged Charnes and Cooper's variable change technique with the concept of convex intervals to solve this problem.

Multi-objective fractional programming (MOFP) presents a valuable framework for addressing complex transportation planning challenges with conflicting objectives [49, 50, 91]. Cetin et al. [51] introduced a new type of MOFTP mathematical model, which had yet to be explored. They proposed a fuzzy approach to finding a compromise solution that is Pareto-optimal. Bhurjee Panda [52] addressed a broad mathematical problem involving multiple objectives and subject to constraints using uncertain parameters. The objective was to explore whether efficient solutions existed for this model and to devise a method to identify these efficient solutions. Lachhwani [53] introduced an innovative approach for addressing complex multi-level MOLFPP using a fuzzy goal programming approach. Instead, it utilizes individual linear membership functions for each objective function's numerator and denominator. The goal is to maximize the attainment of fuzzy goals by minimizing negative deviational variables. Sadia et al.[54] developed a mathematical multi-objective capacitated fractional transportation model with different membership function types: linear, exponential, hyperbolic, and mixed constraints. They applied a lexicographic goal programming approach to solve this mathematical model and obtained compromised solutions to the proposed problem. Pramyj [55] presented a technique to address fuzzy MOLFP. Subsequently, they transformed this MOLFP problem into a single objective LPP. By applying the regular simplex method, they also solved the simplified problem and derived an efficient solution for the original fuzzy MOLFP. Arya et al.[56] proposed a mathematical model of MOLFP using a fully fuzzy programming approach. They created computational techniques specifically designed for solving single-objective linear fractional optimization problems within fuzzy environments [86]. Additionally, they demonstrated how their algorithm could transform traditional numerical problems into fully fuzzy MOLFP.

Furthermore, Saini et al. [57] developed a multi-objective fractional capacitated linear transportation problem mathematical model using rough programming involving multiple objectives and fixed charges constraints. In essence, they explored various ways to balance different factors and find the optimal outcome for the transportation problem they were studying. El Sayed Abo-Sinna [58] and Midya et al.[59] developed the mathematical model to tackle complex transportation problems involving multiple objectives and fuzzy logic. Using different mathematical functions, they demonstrated how to connect this fuzzy model with a simpler, crisp (non-fuzzy) version. By doing this, they showed that even though the problem seems complicated with fuzzy logic involved, it is still possible to solve it effectively [87, 90]. Furthermore, they presented a solution approach that employs various mathematical functions to address the fuzziness inherent in the problem.

3 Mathematical model of multi-objective linear plus linear fractional programming problem

The formulation of the mathematical model MOL+LFPP is as follows;

$$\text{Minimize } F_i\left(\bar{y}\right) = \left(\bar{C}_i^T \bar{y} + d_i\right) + \frac{\bar{\alpha}_i^T \bar{y} + \gamma_i}{\bar{\beta}_i^T \bar{y} + \delta_i}, i = 1, 2, 3...., p \tag{3.1}$$

$$\text{S.t} \quad \bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} \geq \bar{0}\}$$

where, $\bar{C}_i^T, \bar{\alpha}_i^T, \bar{\beta}_i \in \bar{R}^N; i = 1, 2, ...p.$ and $\bar{A} \in \bar{R}^N \times M, \bar{b} \in \bar{R}^M$, and S represent the non empty, convex, and compact in $\bar{R}^N$and further, it is assumed that $\bar{y} \in S \; and \; \bar{\beta}_i^T + \delta_i > 0, i = 1, 2, 3...p$. The objective function $F_i(\bar{y})$ represents in Eq. 3.1 combines two terms; the first term is linear form, and the second is fractional form.

4 Solution methodology

The mathematical optimization of MOL+LFPP can be tacked using various solution methodologies to achieve a balanced trade-off among the objective functions.

(i) **Fuzzy programming approach**

(ii) **Weighted sum approach**

(iii) **Neutrosophic goal programming approach**

The mathematical optimization of the multi-objective linear plus linear fractional programming problem (MOL+LFPP) can be tackled using various solution methodologies to achieve a balanced trade-off among the objective functions. Firstly, the fuzzy programming approach incorporates uncertainty by using fuzzy sets and membership functions to represent the objectives and constraints, allowing for the consideration of satisfaction levels in decision-making. Secondly, the weighted sum approach involves assigning weights to each objective and summing them into a single objective function, thus converting the multi-objective problem into a single objective problem. Lastly, the neutrosophic goal programming approach focuses on minimizing the maximum deviation from the set goals for each objective, providing a way to handle competing objectives by finding a compromise solution. Each of these methodologies offers a unique way to address the complexities inherent in MOL+LFPP, enabling the identification of optimal solutions that balance the different objectives effectively.

4.1 Fuzzy programming approach (FPA)

In the proposed mathematical model to formulate the fuzzy programming approach of a MOL+LFPP, the objective functions $F_i(\bar{y}), i = 1, 2, ..., p$ are converted into fuzzy goals by assigning aspiration levels to each. According to Zimmermann [70], fuzzy programming is a robust approach to handling uncertainty in multi-objective optimization problems. Let $F_i^b = \max_{\bar{y}\in S} F_i(\bar{y})$ and $F_i^W = \min_{\bar{y}\in S} F_i(\bar{y}), i = 1, 2...p$ represent the best and worst values of the objective functions, respectively. The fuzzy goal can be expressed as $F_i(\bar{y}) \geq F_i^b, i = 1, 2, ..., p$. The membership function of the ith fuzzy objective goal can be constructed as follows:

$$\mu_i(\bar{y}) = \begin{cases} 1, & \textit{if } F_i(\bar{y}) \geq F_i^b \\ \dfrac{F_i(\bar{y}) \geq F_i^W}{F_i^b - F_i^W}, & \textit{if } F_i^W \leq F_i(\bar{y}) \leq F_i^b \\ 0, & \textit{if } F_i(\bar{y}) \leq F_i^b \end{cases} \tag{4.1}$$

Where F_i^W and F_i^b are the lower and upper bounds of the ith fuzzy objective goal, This mathematical programming problem reduces to maximizing the membership functions $\mu_i(\bar{y}), i = 1, 2..., p$ subject to the constraints $\bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0} \ \ i = 1, 2, ..., pright\}$. We assume that the objective function $F_i(\bar{y})$ and all the partial derivatives of the order less than equal to $N + 1$ are continuous on the feasible region S. So, the membership function $\mu_i(\bar{y}), i = 1, 2, .., p$, corresponding to the objective function $F_i(\bar{y}), i = 1, 2, .., p$ has the same property in the feasible region S.

Linear approximation for membership functions of MOL+LFPP

Let $\bar{y}_i^* = y_{i1}^*, y_{i2}^*, ..., y_{iN}^*$ be the individual best solutions of membership function $\mu_i(\bar{y})$ corresponding to the objective function $F_i(\bar{y})$ [60].The membership function $\mu_i(\bar{y})$ can be approximated linearly using the first-order Taylor series around $\bar{y}_i^*$:

$$\mu_i(\bar{y}) \approx \mu_i(\bar{y}_i)^* + (\bar{y}_1 - y_{i1}^*)\big(\frac{\partial}{\partial \bar{y}_1}\mu_i(\bar{y})\big)_{at \bar{y}=\bar{y}_i{}^*} + (\bar{y}_2 - y_{i2}^*)\big(\frac{\partial}{\partial \bar{y}_2}\mu_i(\bar{y})\big)_{at \bar{y}=\bar{y}_i{}^*} + ..., +$$

$$(_N - y_{iN}^*)\big(\tfrac{\partial}{\partial \bar{y}_N}\mu_i(\bar{y})\big)_{at \bar{y}=\bar{y}_i{}^*} = \bar{\mu}_i(\bar{y}), i = 1, 2, ..., p. \tag{4.2}$$

Fuzzy goal programming formulation of MOL+LFPP

The problem reduces to maximizing the linearized membership function;

$$\text{Maximize } \hat{\mu}_i(\bar{y}), i = 1, 2...., p \tag{4.3}$$

$$\text{S.t } \bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\}$$

Given that the maximum value of a membership function is one, in Eq. 4.3, the flexible membership goal with an aspired level of one can be formulated as follows;

$$\hat{\mu}_i(\bar{y}) + d_i^- - d_i^+ = 1, i = 1, 2...., p \tag{4.4}$$

where, d_i^- and d_i^+ are negative and positive deviation variables, respectively. According to [61, 62], the Eq. 4.4 can be rewritten as;

$$\hat{\mu}_i(\bar{y}) + d_i^- = 1, i = 1, 2...., p \tag{4.5}$$

The fuzzy goal programming approach -I can be formulated as follows:

$$\text{Minimize } \lambda$$

$$\begin{aligned} \text{S.t} \quad & \hat{\mu}_i(\bar{y}) + d_i^- = 1 \\ & \bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\} \\ & \lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1, 2, ..., p. \end{aligned} \tag{4.6}$$

Alternatively, the fuzzy goal programming approach -I can be explicitly as:

$$\text{Minimize } \lambda$$

$$\text{S.t } \mu_i(\bar{y}) \approx \mu_i(\bar{y_i}^*) + (\bar{y_1} - y_{i1}^*)\left(\frac{\partial}{\partial y_1}\mu_i(\bar{y})\right)_{at\bar{y}=\bar{y_i}^*} + (\bar{y_2} - y_{i2}^*)\left(\frac{\partial}{\partial y_2}\mu_i(\bar{y})\right)_{at\bar{y}=\bar{y_i}^*} +, ..., +$$

$$(\bar{y_N} - y_{iN}^*)\left(\frac{\partial}{\partial y_N}\mu_i(\bar{y})\right)_{at\bar{y}=\bar{y_i}^*} + d_i^- = 1 \tag{4.7}$$

$$\bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\}$$

$$\lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1, 2, ..., p.$$

The fuzzy goal programming approach -II for solving the MOL+LFPP can be represented as;

$$\begin{aligned} \text{Minimize } \xi = \sum_{i=1}^{p} w_i d_i^- \\ \text{S.t} \quad \hat{\mu}_i(\bar{y}) + d_i^- = 1 \\ \bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\} \\ 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1, 2, ..., p. \end{aligned} \tag{4.8}$$

Alternatively, the fuzzy goal programming approach -II can be explicitly formulated as follows;

$$\text{Minimize } \xi = \sum_{i=1}^{p} w_i d_i^-$$

$$\text{S.t}$$

$$\mu_i(\bar{y}) \approx \mu_i(\bar{y_i}^*) + (\bar{y_1} - y_{i1}^*)\left(\frac{\partial}{\partial y_1}\mu_i(\bar{y})\right)_{at\bar{y}=\bar{y_i}^*} + (\bar{y_2} - y_{i2}^*)\left(\frac{\partial}{\partial y_2}\mu_i(\bar{y})\right)_{at\bar{y}=\bar{y_i}^*} +, ..., +$$

$$(\bar{y_N} - y_{iN}^*)\left(\frac{\partial}{\partial y_N}\mu_i(\bar{y})\right)_{at\bar{y}=\bar{y_i}^*} + d_i^- = 1 \tag{4.9}$$

$$\bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\}$$

$$\lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1, 2, ..., p.$$

Here, the decision-makers can take the normalized weights such that $\sum_{i=1}^{p} w_i = 1$, with $w_i = \frac{1}{p}$ or any preference weight in the decision-making situations. This approach provides flexibility in reflecting the decision-maker's preferences and priorities.

Selection of optimal compromise solutions

The concept of the ideal point and the use of distance functions for group decision-making problems were initially explored by Yu [63]. This approach has been extensively applied in various multi-objective decision-making problems to achieve satisfactory solutions [64, 2]. Given that different fuzzy goal programming approaches yield distinct optimal solutions, the Euclidean distance function is employed to select the FGPA-I and FGPA-II that provide the best optimal solutions. The Euclidean distance function is represented as follows:

$$E^2 = \sum_{i=1}^{p} [(1 - \hat{\mu_i}(\bar{y}))^2]^{\frac{1}{2}} \tag{4.10}$$

where, $\hat{\mu_i}(\bar{y})$ represents the membership function of the ith fuzzy objective functions, the solution for which E^2 is minimal is considered the best optimal solution.

Fuzzy goal programming algorithms for MOL+LFPP

The proposed FGPA algorithm for solving the MOL+LFPP is outlined as follows:

Step 1: Determine the best and worst solutions for each objective function $F_i(\bar{y})$ and subject to constraints.

Step 2: Construct the membership function $\mu_i(\bar{y}), i = 1, 2..., p$ for each objective goal, as previously defined in Eq. 4.1

Step 3: Find the individual best solution for each membership function $\mu_i(\bar{y}), i = 1, 2..., p$ and subject to the set of constraints.

Step 4: Transform the $\mu_i(\bar{y}), i = 1, 2..., p$, into the equivalent linear membership function $\mu_i(\bar{y}), i = 1, 2..., p$ using the first order Taylor series approximation at the best solution points $\bar{y}_i^* = y_{i1}^*, y_{i2}^*, ..., y_{iN}^*$.

Step 5: Construct the FGPA-I and FGPA-II as Eq represents. 4.7 and 4.9.

Step 6: Solve the mathematical model of FGPA-I and FGPA-II.

Step 7: Compute E^2 for solutions obtained from the mathematical model of the FGPA-I and FGPA-II.

Step 8: Select the solution in which E^2 is minimal as the best optimal compromise solution.

4.2 Weighted sum approach (WSA)

The weighted sum approach [65, 66, 67] is a widely used technique for solving multi-objective optimization problems, including multi-objective linear plus linear fractional programming problems (MOL+LFPP) [47]. This method converts a multi-objective problem into a single-objective optimization problem by assigning weights to each objective and summing them to form a composite objective function. The solution is obtained by optimizing these single composite functions.

The formulation of WSA for MOL+LFPP is represented as follows:

$$\text{Minimize } F = \sum_{i=1}^{p} w_i F_i(\bar{y}) = \sum_{i=1}^{p} w_i \left(\bar{C}_i^T \bar{y} + d_i + \frac{\bar{\alpha}_i^T \bar{y} + \gamma_i}{\bar{\beta}_i^T \bar{y} + \delta_i} \right), i = 1, 2, 3...., p \tag{4.11}$$

$$\text{S.t } \bar{y} \in S = \{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\}$$

In the WSA, weights w_i are assigned to each objective such that $\sum_{i=1}^{p} w_i = 1,$ and w$_i \geq 0, i = 1, 2..., p.$

Algorithm of the proposed weighted sum approach:

To achieve the best compromise solution for multi-objective optimization problems [68], the problem is transformed into a single-objective optimization problem using the following approach [69].

Let the successive resolution of the mathematical model of the MOL+LFPP be represented as:

$$\text{Minimize } F_i(\bar{y}) = [F_1(\bar{y}), F_2(\bar{y}), ..., F_p(\bar{y}), i = 1, 2, ..., p] \tag{4.12}$$

$$\text{S.t } \bar{y} \in S$$

Where $\bar{y}$ represents the decision variables, and S is the feasible set. The goal is to find the balance, not necessarily the optimal one, for each objective function. The multi-objective optimization problem can be transformed into a single objective optimization problem by selecting the objective function as;

$$\text{Minimize } \bar{\lambda} = \sum \lambda(1 - w_i) \tag{4.13}$$

Where, w_i is the weight assigned to the pth objective function, and λ shows the general deviation variable. Here, $F_i^* + \lambda(1 - w_i)$ represents the upper class of every objective function. The ideal objective F_i^* for each objective F_i can be obtained independently of the other objectives.

The multi-objective optimization problem is represented by the following single-objective optimization problem;

$$\text{Minimize } \bar{\lambda} = \sum_{i=1}^{p} \lambda(1 - w_i) \tag{4.14}$$

$$\text{S.t } F_i \leq F_i^* + (1 - w_i)$$

$$\bar{y} \in S \text{ and } \bar{y} \geq \bar{0}$$

In this model, rather than using the deviation variable directly, we established a deviation function $\lambda(1 - w_i)$. Furthermore, by multiplying the λ, we assume the weight w_i of the pth objective is assigned a priority, reducing the value of the deviation function and yielding a solution closer to the ideal objectives. When preferences are defined, this approach provides a compromise solution suitable for all types of multi-objective optimization problems, including MOL+LFPP.

The proposed approach for MOL+LFPP is represented as follows:

$$\text{Minimize } \bar{\lambda} = \sum_{i=1}^{p} \lambda(1 - w_i)$$

$$\text{S.t } \left(\bar{C}_i^T \bar{y} + d_i + \frac{\bar{\alpha}_i^T \bar{y} + \gamma_i}{\bar{\beta}_i^T \bar{y} + \delta_i} \right) \leq F_i^* + \lambda(1 - w_i) \tag{4.15}$$

$$0 \leq w_i \leq 1, \; \bar{y} \in S = \left\{ \bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0} \right\}, i = 1, 2, ..., p.$$

4.3 Neutrosophic goal programming approach

This section introduces a novel strategy for solving the multi-objective linear plus linear fractional programming problem (MOL+LFPP) using the neutrosophic goal programming approach. This method builds on Zimmermann's [70] neutrosophic extension. The proposed neutrosophic compromise technique provides a fresh way to handle uncertainty in mathematical optimization problems [71].It aims to optimize three aspects of a neutrosophic decision: the degree of truth (satisfaction), the degree of falsity (dissatisfaction), and the degree of indeterminacy (partial satisfaction).Bellman & Zadeh [72] worked on three critical methodologies for fuzzy sets: the fuzzy decision, the fuzzy goal, and the fuzzy constraints. This methodology has been widely applied in decision-making scenarios involving fuzziness. Here is a brief explanation: Fuzzy decision (Fd): A decision that incorporates the fuzziness of the problem's parameters. Fuzzy goal (Fg): The desired outcome expressed in fuzzy terms. Fuzzy constraints (Fc): The limitations or restrictions of the problem are described using fuzzy sets. This new methodology leverages these foundational concepts to enhance decision-making where indeterminacy is a significant factor. The fuzzy decision is defined as follows:

$$Fd = (Fg \cap Fc) \tag{4.16}$$

Accordingly, the neutrosophic decision set$(Fd)^N$ which represents a combination of neutrosophic objectives and is subject to the constraints, is defined as follows;

$$(Fd)^N = \left(\bigcap_{i=1}^{p} (Fg)_i \right) \left(\bigcap_{t=1}^{T} (Fc)_t \right) = (y, \varphi_{Fd}(y), \theta_{Fd}(y), \phi_{Fd}(y)) \tag{4.17}$$

where,

$$\varphi_{Fd}(y) = \min \begin{cases} \varphi^1_{Fg}, \varphi^2_{Fg}, ..., \varphi^t_{Fg} & \forall y \in Y \\ \varphi^1_{Fc}, \varphi^2_{Fc}, ..., \varphi^t_{Fc} \end{cases}$$

$$\theta_{Fd}(y) = \min \begin{cases} \theta^1_{Fg}, \theta^2_{Fg}, ..., \theta^t_{Fg} & \forall y \in Y \\ \theta^1_{Fc}, \theta^2_{Fc}, ..., \theta^t_{Fc} \end{cases}$$

$$\phi_{Fd}(y) = \min \begin{cases} \phi^1_{Fg}, \phi^2_{Fg}, ..., \phi^t_{Fg} & \forall y \in Y \\ \phi^1_{Fc}, \phi^2_{Fc}, ..., \phi^t_{Fc} \end{cases} \tag{4.18}$$

where $\phi_{Fd}(y)$ represents the truth membership function, $\theta_{Fd}(y)$ denotes the indeterminacy membership function, and $\phi_{Fd}(y)$ signifies the falsity membership function of neutrosophic decision set $(Fd)^N$.

To formulate the membership functions for the MOL+LFPP, we start by determining the bounds for each objective function. Each objective's lower and upper bounds are F^l_i and F^U_i, respectively. These bounds are calculated by optimizing each objective as a single objective function and subject to the relevant constraints. Solving each p objective independently means we obtain p solutions, $y_1, y_2, ..., y_p$. These solutions are then substituted into each objective function to determine the bounds for each objective as follows;

$$F^l_i = \min \{F_i(\bar{y})\}^p_{i=1}$$
$$F^U_i = \max \{F_i(\bar{y})\}^p_{i=1} \tag{4.19}$$

Next, the bounds within the neutrosophic environment are determined as follows:

$$F^l_i(\varphi) = F^l_i, F^U_i(\varphi) = F^U_i, \text{for the truth membership} F^l_i(\theta) = F^l_i(\theta), F^U_i(\theta) =$$
$$F^U_i(\varphi) + S_i(F^U_i(\varphi)) - F^l_i(\varphi), \text{ for the indeterminacy membership}, F^l_i(\phi) = F^l_i(\varphi) +$$
$$t_i(F^U_i(\varphi) - F^l_i(\varphi)), F^U_i(\phi) = F^U_i(\varphi), \text{for the falsity membership} \tag{4.20}$$

Where, t_i and s_i are predetermined real members within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_i(F_i(\bar{y})) = \begin{cases} 1, & if\ F_i(\bar{y}) < F^l_i(\varphi) \\ 1 - \dfrac{F_i(\bar{y}) - F^l_i(\varphi)}{F^U_i(\varphi) - F^l_i(\varphi)}, & if\ F^l_i(\varphi) \le F_i(\bar{y}) \le F^U_i(\varphi) \\ 0, & if\ F_i(\bar{y}) > F^U_i(\varphi) \end{cases} \tag{4.21}$$

$$\theta_i(F_i(\bar{y})) = \begin{cases} 1, & if\ F_i(\bar{y}) < F^l_i(\theta) \\ 1 - \dfrac{F_i(\bar{y}) - F^l_i(\theta)}{F^U_i(\theta) - F^l_i(\theta)}, & if\ F^l_i(\theta) \le F_i(\bar{y}) \le F^U_i(\theta) \\ 0, & if\ F_i(\bar{y}) > F^U_i(\theta) \end{cases} \tag{4.22}$$

$$\phi_i(F_i(\bar{y})) = \begin{cases} 1, & if\ F_i(\bar{y}) < F^l_i(\phi) \\ 1 - \dfrac{F_i(\bar{y}) - F^l_i(\phi)}{F^U_i(\phi) - F^l_i(\phi)}, & if\ F^l_i(\phi) \le F_i(\bar{y}) \le F^U_i(\phi) \\ 0, & if\ F_i(\bar{y}) > F^U_i(\phi) \end{cases} \tag{4.23}$$

where,$F^U_i(*) \neq F^l_i(*)$ for all objectives. $F^U_i(*) = F^l_i(*)$ for any membership function, then the value of this membership is set to 1. Utilizing Eq. (4.21 - 4.23) and following the principle outlined by Bellman & Zadeh [72], the neutrosophic optimization mathematical model for MOL+LFPP can be expressed as follows:

$$Max\ Min \sum_{i=1,2,..,p} \varphi_i(F_i(\bar{y}))$$

$$
\begin{aligned}
&Max\ Min \sum_{i=1,2,..,p} \theta_i(F_i(\bar{y})) \\
&Max\ Min \sum_{i=1,2,..,p} \phi_i(F_i(\bar{y})) \\
&\text{S.t} \\
&\bar{y} \in S = \left\{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\right\}, i = 1, 2, ..., p.
\end{aligned} \tag{4.24}
$$

Through the utilization of auxiliary parameters, the mathematical problem (4.24) can be reformulated as the following:

$$
\begin{aligned}
&Max\ \zeta, Max\ \eta, Max\ \delta \\
&\varphi_{F_i}(y) \geq \zeta, \theta_{F_i}(y) \geq \eta, \phi_{F_i}(y) \geq \delta \\
&\text{S.t} \\
&\bar{y} \in S = \left\{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\right\}, \zeta \geq \eta, \zeta \geq \delta, \zeta + \eta + \delta \leq 3, \zeta, \eta, \delta \in [0, 1] \\
&i = 1, 2, ..., p.
\end{aligned} \tag{4.25}
$$

The problem presented in (4.25) can be depicted as follows:

$$
\begin{aligned}
&Max\ \zeta - \delta + \eta \\
&F_i(\bar{y}) + \Big(F_i^U(\varphi) - F_i^l(\varphi)\Big)\zeta \leq F_i^U(\varphi) \\
&F_i(\bar{y}) + \Big(F_i^U(\theta) - F_i^l(\theta)\Big)\eta \leq F_i^U(\theta) \\
&F_i(\bar{y}) + \Big(F_i^U(\phi) - F_i^l(\phi)\Big)\delta \leq F_i^U(\phi) \\
&\text{S.t} \\
&\bar{y} \in S = \left\{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\right\}, \zeta \geq \eta, \zeta \geq \delta, \zeta + \eta + \delta \leq 3, \zeta, \eta, \delta \in [0, 1] \\
&i = 1, 2, ..., p.
\end{aligned} \tag{4.26}
$$

The problem presented in (4.26) can be written as follows:

$$
\begin{aligned}
&Max\ \zeta - \delta + \eta \\
&F_i(\bar{y}) + \Big(F_i^U(\varphi) - F_i^l(\varphi)\Big)\zeta - F_i^U(\varphi) \leq 0 \\
&F_i(\bar{y}) + \Big(F_i^U(\theta) - F_i^l(\theta)\Big)\eta - F_i^U(\theta) \leq 0 \\
&F_i(\bar{y}) + \Big(F_i^U(\phi) - F_i^l(\phi)\Big)\delta - F_i^U(\phi) \leq 0 \\
&\text{S.t} \\
&\bar{y} \in S = \left\{\bar{y} \in \bar{R} | \bar{A}\bar{y} \leq = \geq \bar{b}, \bar{y} > \bar{0}\right\}, \zeta \geq \eta, \zeta \geq \delta, \zeta + \eta + \delta \leq 3, \zeta, \eta, \delta \in [0, 1] \\
&i = 1, 2, ..., p
\end{aligned} \tag{4.27}
$$

5 Numerical illustrations

5.1 Numerical example 1

Consider the numerical example with three objective functions as follows;

$$\text{Maximize } F_1(\bar{y}) = (-y_2 - 1) + \frac{(-5y_1 + 4y_2)}{(2y_1 + y_2 + 5)}$$

$$\text{Maximize } F_2(\bar{y}) = (y_2 + 1) + \frac{(9y_1 + 2y_2)}{(7y_1 + 3y_2 + 1)}$$

$$\text{Maximize } F_3(\bar{y}) = (y_1 + 1) + \frac{(3y_1 + 8y_2)}{(4y_1 + 5y_2 + 3)}$$

S.t

$$y_1 - y_2 \geq 2, 4y_1 + 5y_2 \leq 25, y_1 + 9y_2 \leq 9, y_1 \geq 5 \text{ and } y_1, y_2 \geq 0.$$

Now, we apply the LINGO 20.0 optimization software in this numerical example and obtain the solution of each objective function individually using the fuzzy goal programming approach. Each objective function's best and worst solution was identified and summarized in the following pay-off matrix.

$$\text{Pay-off matrix=} \begin{array}{r c} & \begin{array}{ccc} \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{array} \\ \begin{array}{r} \bar{y}_1(5,1) \\ \bar{y}_2(5,1) \\ \bar{y}_3(5.804, 0.3548) \end{array} & \begin{pmatrix} -7.3125 & 3.2051 & 6.8214 \\ -7.3125 & 3.2051 & 6.8214 \\ -8.4333 & 2.5950 & 7.5299 \end{pmatrix} \end{array}$$

Solutions by the FGPA

First, we determine the individual best and worst solutions. The solutions are represented here, The individual best and worst solutions are obtained as follows: F_1^b = −7.312 at (5,1), F_2^b = 3.205 at (5,1), and F_3^b= 7.53 at (5.806, 0.3550), F_1^w = −8.433 at (5.806, 0.355), F_2^w = 2.595 at (5.806, 0.355) and F_3^w= 6.736 at (5, 0.444). Then, the fuzzy goal appears in the following form: $F_1(\bar{y}) \geq -7.312, F_1(\bar{y}) \geq 3.205, F_3(\bar{y}) \geq 7.53$. The membership function of the problem is represented as follows;

$$\mu_1(\bar{y}) = \frac{F_1(\bar{y}) + 8.43}{1.121} = \frac{(-y_2 - 1) + \dfrac{(-5y_1 + 4y_2)}{(2y_1 + y_2 + 5)} + 8.433}{1.121}$$

$$\mu_1(\bar{y}) = \frac{F_2(\bar{y}) - 2.595}{0.610} = \frac{(y_2 + 1) + \dfrac{(9y_1 + 2y_2)}{(7y_1 + 3y_2 + 1)} - 2.595}{0.610}$$

$$\mu_1(\bar{y}) = \frac{F_3(\bar{y}) - 6.736}{0.794} = \frac{(y_1 + 1) + \dfrac{(3y_1 + 8y_2)}{(4y_1 + 5y_2 + 3)} - 6.736}{0.794}$$

The membership functions $\mu_1(\bar{y})$, $\mu_2(\bar{y})$, $\mu_3(\bar{y})$ are maximal at the points (5, 1), (5, 1), (5.806, 0.355). The membership functions are transformed into equivalent linear membership functions at the best solution points by the first-order Taylor series as follows;

$$\hat{\mu}_1(\bar{y}) = \mu_1(5,1) + (y_1 - 5)\left(\frac{\partial}{\partial \bar{y_1}}\mu_1(\bar{y})\right)_{\text{at } \bar{y}=(5,1)} + (y_2 - 1)\left(\frac{\partial}{\partial \bar{y_2}}\mu_2(\bar{y})\right)_{\text{at } \bar{y}=(5,1)}$$

$$\hat{\mu}_1(\bar{y}) = 1 - 1.024(y_1 - 5) + 0.192(y_2 - 1),\ \text{similarly other membership function}$$

$$\hat{\mu}_2(\bar{y}) = 1 + 1.024(y_1 - 5) + 1.571(y_2 - 1),\ \hat{\mu}_3(\bar{y}) = 1 - 1.115(y_1 - 5.806) + 0.197(y_2 - 0.355)$$

The fuzzy goal programming approach -I can be represented as

$$\begin{gathered}
\text{Minimize } \lambda \\
\text{S.t} \\
1 - 1.024(y_1 - 5) + 0.192(y_2 - 1) + d_1^- = 1 \\
1 + 1.024(y_1 - 5) + 1.571(y_2 - 1) + d_2^- = 1 \\
1 - 1.115(y_1 - 5.806) + 0.197(y_2 - 0.355) + d_3^- = 1 \\
y_1 - y_2 \geq 2, 4y_1 + 5y_2 \leq 25, y_1 + 9y_2 \leq 9, y_1 \geq 5 \text{ and } y_1, y_2 \geq 0 \\
\lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1, 2, 3.
\end{gathered} \tag{5.1}$$

This approach provides the best compromise solutions using LINGO 20.0 optimization software $F_1 = -7.807$, $F_2 = 2.939$, $F_3 = 7.131$ at the points $y_1 = 5.352$ and $y_2 = 0.718$

The fuzzy goal programming approach -II can be represented as:

$$\begin{gathered}
\text{Minimize } \xi = \frac{1}{3}(d_1^- + d_2^- + d_3^-) \\
\text{S.t} \\
1 - 1.024(y_1 - 5) + 0.192(y_2 - 1) + d_1^- = 1 \\
1 + 1.024(y_1 - 5) + 1.571(y_2 - 1) + d_2^- = 1 \\
1 - 1.115(y_1 - 5.806) + 0.197(y_2 - 0.355) + d_3^- = 1 \\
y_1 - y_2 \geq 2, 4y_1 + 5y_2 \leq 25, y_1 + 9y_2 \leq 9, y_1 \geq 5 \text{ and } y_1, y_2 \geq 0 \\
\lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1, 2, 3.
\end{gathered} \tag{5.2}$$

Now, we solve the above mathematical model using LINGO 20.0 optimization software and provide the best compromise solutions $F_1 = -7.312$, $F_2 = 3.025$, $F_3 = 6.821$ at the points $y_1 = 5$, $y_2 = 1$.

Solution by the WSA

In this section, we apply the weighted sum approach to the proposed numerical example and get the optimal compromise solutions. Let us assign the different weights of the problem,$W_1 = 0.1$, $W_2= 0.9$,$W_3 = 0.2$. The general formulation of the problem using WSP is represented as follows:

$$\text{Max F} = 0.1\left((-y_2 - 1) + \frac{(-5y_1 + 4y_2)}{(2y_1 + y_2 + 5)}\right) + 0.9\left((y_2 + 1) + \frac{(9y_1 + 2y_2)}{(7y_1 + 3y_2 + 1)}\right) +$$

$$0.2\left((y_1 + 1) + \frac{(3y_1 + 8y_2)}{(4y_1 + 5y_2 + 3)}\right)$$

$$\text{S.t } y_1 - y_2 \geq 2, 4y_1 + 5y_2 \leq 25, y_1 + 9y_2 \leq 9, y_1 \geq 5 \text{ and } y_1, y_2 \geq 0$$

The problem is solved using the LINGO 20.0 optimization software, and the best compromise solutions are as follows: $F_1 = -6.971$, $F_2 = 3.017$, $F_3 = 6.9731$ at the points $y_1 = 5$ and $y_2 = 1$.

Solution by NGPA

We have solved the numerical example 1 individually as a single objective optimization problem using the proposed NGPA. We construct the pay-off matrix by evaluating three objective functions with the three individual solutions.

$$\text{Pay-off matrix}=\begin{array}{c} \\ \bar{y}_1(5,1) \\ \bar{y}_2(5,1) \\ \bar{y}_3(5.804,0.3548) \end{array}\begin{array}{c} \begin{array}{ccc} \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{array} \\ \begin{pmatrix} -7.3125 & 3.2051 & 6.8214 \\ -7.3125 & 3.2051 & 6.8214 \\ -8.4333 & 2.5950 & 7.5299 \end{pmatrix} \end{array}$$

Determine the lower and upper bounds for each objective function. These bounds are assigned using the following formula, $F_i^l = \min\{F_i(\bar{y})\}_{i=1}^3, F_i^U = \max\{F_i(\bar{y})\}_{i=1}^3$.

The bounds of each objective represented as follows; $-7.3125 \leq F_1 \leq 6.5431$, $3.2051 \leq F_2 \leq 6.8732$, $7.5299 \leq F_3 \leq 7.7395$. Now, we define the membership function of each objective function using the NGPA.

For the first objective function F_1: $F_1^l(\varphi) = -7.3125$, $F_1^U(\varphi) = 6.5431$, for the truth membership, $F_1^l(\theta) = -7.3125$, $F_1^U(\theta) = F_1^U(\varphi) + s_1\left(F_1^U(\varphi) - F_1^l(\varphi)\right) = -7.3125 + s_1$ for the indeterminacy membership, $F_1^l(\varphi) = F_1^l(\varphi) + t_1\left(F_1^U(\varphi) - F_1^l(\varphi)\right) = -7.3125 + t_1$, $F_1^U(\varphi) = 6.5431$, for the falsity membership, where t_1 and s_1 are predetermined real numbers within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_1(F_1(\bar{y})) = \begin{cases} 1, & if\ \ F_1(\bar{y}) < -7.3125 \\ 1 - \dfrac{(-y_2-1) + \dfrac{(-5y_1+4y_2)}{(2y_1+y_2+5)} + 7.3125}{13.8556}, & if\ \ -7.3125 \leq F_1(\bar{y}) \leq 6.5431 \\ 0, & if\ F_1(\bar{y}) > 6.5431 \end{cases}$$

$$\theta_1(F_1(\bar{y})) = \begin{cases} 1, & if\ \ F_1(\bar{y}) < -7.3125 \\ 1 - \dfrac{(-y_2-1) + \dfrac{(-5y_1+4y_2)}{(2y_1+y_2+5)} + 7.3125}{s_1}, & if\ \ -7.3125 \leq F_1(\bar{y}) \leq -7.3125 + s_1 \\ 0, & if\ F_1(\bar{y}) > -7.3125 + s_1 \end{cases}$$

$$\phi_1(F_1(\bar{y})) = \begin{cases} 1, & if\ \ F_1(\bar{y}) > 6.5431 \\ 1 - \dfrac{6.5431 - (-y_2-1) + \dfrac{(-5y_1+4y_2)}{(2y_1+y_2+5)} + 7.3125}{13.8556 - t_1}, & if\ \ -7.3125 + t_1 \leq F_1(\bar{y}) \leq 6.543 \\ 0, & if\ F_1(\bar{y}) > -7.3125 + t_1 \end{cases}$$

For the second objective function F_2: $F_2^l(\varphi) = 3.2051$, $F_2^U(\varphi) = 6.8732$, for the truth membership, $F_2^l(\theta) = 3.2051$, $F_2^U(\theta) = F_2^U(\varphi) + s_2\left(F_2^U(\varphi) - F_2^l(\varphi)\right) = 3.2051 + s_2$ for the indeterminacy membership, $F_1^l(\varphi) = F_2^l(\varphi) + t_2\left(F_1^U(\varphi) - F_2^l(\varphi)\right) = 3.2051 + t_2$, $F_2^U(\varphi) = 6.8732$, for the falsity membership, where t_2 and s_2 are predetermined real numbers within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_2(F_2(\bar{y})) = \begin{cases} 1, & if \;\; F_2(\bar{y}) < 3.2051 \\ 1 - \dfrac{(y_2+1) + \dfrac{(9y_1+2y_2)}{(7y_1+3y_2+1)} - 3.2051}{3.6681}, & if \; 3.2051 \le F_2(\bar{y}) \le 6.8732 \\ 0, & if \; F_2(\bar{y}) > 6.8732 \end{cases}$$

$$\theta_2(F_2(\bar{y})) = \begin{cases} 1, & if \;\; F_2(\bar{y}) < 3.2051 \\ 1 - \dfrac{(y_2+1) + \dfrac{(9y_1+2y_2)}{(7y_1+3y_2+1)} - 3.2051}{s_2}, & if \; 3.2051 \le F_2(\bar{y}) \le 3.2051 + s_2 \\ 0, & if \; F_2(\bar{y}) > 3.2051 + s_2 \end{cases}$$

$$\phi_2(F_2(\bar{y})) = \begin{cases} 1, & if \;\; F_2(\bar{y}) < 3.2051 \\ 1 - \dfrac{6.8732 - (y_2+1) + \dfrac{(9y_1+2y_2)}{(7y_1+3y_2+1)}}{3.6681 - t_2}, & if \; 3.2051 + t_2 \le F_2(\bar{y}) \le 6.8732 \\ 0, & if \; F_2(\bar{y}) < 3.2051 + t_2 \end{cases}$$

For the second objective function F_3: $F_3^l(\varphi) = 7.5299$, $F_3^U(\varphi)$ =7.7395, for the truth membership, $F_3^l(\theta) = 7.5299$, $F_3^U(\theta) = F_3^U(\varphi) + s_3\left(F_3^U(\varphi) - F_3^l(\varphi)\right)$ =7.5299+ s_3 for the indeterminacy membership, $F_3^l(\varphi) = F_3^l(\varphi) + t_3\left(F_3^U(\varphi) - F_3^l(\varphi)\right)$ = 7.5299 +t_3, $F_3^U(\varphi)$ = 7.7395, for the falsity membership, where t_3 and s_3 are predetermined real numbers within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_3(F_3(\bar{y})) = \begin{cases} 1, & if \;\; F_3(\bar{y}) < 7.5299 \\ 1 - \dfrac{(y_1+1) + \dfrac{(3y_1+8y_2)}{(4y_1+5y_2+3)} - 7.5299}{0.2096}, & if \; 7.5299 \le F_3(\bar{y}) \le 7.739 \\ 0, & if \; F_3(\bar{y}) > 7.739 \end{cases}$$

$$\theta_3(F_3(\bar{y})) = \begin{cases} 1, & if \;\; F_3(\bar{y}) < 7.5299 \\ 1 - \dfrac{(y_1+1) + \dfrac{(3y_1+8y_2)}{(4y_1+5y_2+3)} - 7.5299}{s_3}, & if \; 7.5299 \le F_3(\bar{y}) \le 7.5299 + s_3 \\ 0, & if \; F_3(\bar{y}) > 7.5299 + s_3 \end{cases}$$

$$\phi_3(F_3(\bar{y})) = \begin{cases} 1, & if\ \ F_3(\bar{y}) > 7.7395 \\ 1 - \dfrac{7.7395 - (y_1+1) + \dfrac{(3y_1+8y_2)}{(4y_1+5y_2+3)}}{0.2096 - t_3}, & if\ 7.5299 + t_3 \leq F_3(\bar{y}) \leq 7.739 \\ 0, & if\ F_3(\bar{y}) < 7.5299 + t_3 \end{cases}$$

The construction of an equivalent neutrosophic mathematical model for the proposed problem is represented as follows;

$$Max\ \zeta - \delta + \eta$$

S.t

$$y_1 - y_2 \geq 2, 4y_1 + 5y_2 \leq 25, y_1 + 9y_2 \leq 9, y_1 \geq 5 \text{ and } y_1, y_2 \geq 0$$

$$(-y_2 - 1) + \frac{(-5y_1 + 4y_2)}{(2y_1 + y_2 + 5)} + 13.8556\ \zeta \leq 6.5431$$

$$(y_2 + 1) + \frac{(9y_1 + 2y_2)}{(7y_1 + 3y_2 + 1)} + 3.6681\ \zeta \leq 6.8732$$

$$(y_1 + 1) + \frac{(3y_1 + 8y_2)}{(4y_1 + 5y_2 + 3)} + 0.2096\ \zeta \leq 7.7395$$

$$(-y_2 - 1) + \frac{(-5y_1 + 4y_2)}{(2y_1 + y_2 + 5)} + s_1\eta - s_1 \leq -7.3125$$

$$(y_2 + 1) + \frac{(9y_1 + 2y_2)}{(7y_1 + 3y_2 + 1)} + s_2\eta - s_2 \leq 3.2051$$

$$(y_1 + 1) + \frac{(3y_1 + 8y_2)}{(4y_1 + 5y_2 + 3)} + s_3\eta - s_3 \leq 7.5299$$

$$(-y_2 - 1) + \frac{(-5y_1 + 4y_2)}{(2y_1 + y_2 + 5)} - (13.8556 - t_1)\delta - t_1 \leq -7.3125$$

$$(y_2 + 1) + \frac{(9y_1 + 2y_2)}{(7y_1 + 3y_2 + 1)} - (3.6681 - t_2)\delta - t_1 \leq 3.2051$$

$$(y_1 + 1) + \frac{(3y_1 + 8y_2)}{(4y_1 + 5y_2 + 3)} - (0.2096 - t_3)\delta - t_3 \leq 7.5299$$

$$\zeta \geq \eta, \zeta \geq \delta, \zeta + \eta + \delta \leq 3, \zeta, \eta, \delta \in [0, 1], i = 1, 2, 3.$$

Now, solve the above numerical illustration using LINGO 20.0 optimization software and obtain the best compromise solutions. The compromise solutions are represented as $F_1 = -6.3123$, $F_2 = 3.025$, $F_3 = 7.912$.

Table 1. Compromise solutions of MOL+LFPP (Example 1).

Approach/ Methods	y_1	y_2	F_1	F_2	F_3
FGPA-I	5.352	0.718	−7.807	2.939	7.131
FGPA-II	5	1	−7.131	3.025	6.821
WSA	5	1	−6.971	3.017	6.9731
NGPA	5	1	−6.3123	3.025	7.912

Based on the analysis, NGPA provides the better compromise solution among the given approaches as it achieves the highest values in all three objective functions. Table 1 presents the compromise solutions, for example, 1 of the MOL+LFPP, dealing with the solution points (y_1, y_2) and their corresponding objective values (F_1, F_2, F_3)were obtained through the different approaches: FGPA-I, FGPA -II, WSA, and NGPA. This table showcases the variations in the results achieved by each method, highlighting the trade-offs and efficiencies of the different solution approaches in addressing the problem objectives.

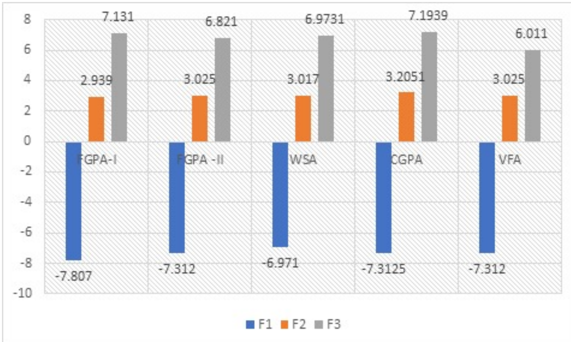

Figure 1. Shows the compromise solutions using different approaches.

Figure 1 illustrates the compromise solutions for each objective function using different approaches, for example, 1 of the MOL+LFPP. The figure highlights the performance of the FGPA-I, FGPA-II, WSA, CGPA, and VFA are used to optimize the given objectives, providing a comparative visual analysis of their effectiveness.

5.2 Numerical example 2

We consider another numerical example as follows;

$$\text{Maximize } F_1(\bar{y}) = (-y_1 - 1) + \frac{(-y_1 + 2y_2 - 5)}{(7y_1 + 3y_2 + 1)}$$

$$\text{Maximize } F_2(\bar{y}) = (-2y_2 - 1) + \frac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)}$$

$$\text{Maximize } F_3(\bar{y}) = (-3y_1 - 1) + \frac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)}$$

S.t

$y_1 \leq 6, y_2 \leq 6, 2y_1 + y_2 \leq 9, -2y_1 + y_2 \leq 5, y_1 - y_2 \leq 5$ and $y_1, y_2 \geq 0$.

The pay-off matrix obtains the individual best and worst solutions as follows:

$$\text{Pay-off matrix} = \begin{array}{c} \\ \bar{y}_1(0,5) \\ \bar{y}_1(4.5,0) \\ \bar{y}_3(0,5) \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{array} \\ \begin{pmatrix} -0.6875 & -31.0 & -1.45 \\ -5.7923 & -0.2727 & -15.9 \\ -0.6875 & -31.0 & -1.45 \end{pmatrix} \end{array}$$

The individual best and worst solutions are obtained as follows;
F_1^b=-0.688 at (0, 5), F_2^b=-0.272 at (4.5, 0) and F_3^b=-1.45 at (0, 5)
F_1^w=-5.792 at (4.5, 0), F_2^w= 31 at (0, 5) and F_3^w=-15.9 at (4.5, 0).

Solutions by FPA

We use the best and worst solutions. The compromise solutions are obtained and summarized as follows;

Then, the fuzzy goal appears in the following form: $F_1(\bar{y}) \geq -0.688$, $F_2(\bar{y}) \geq -0.272$, $F_3(\bar{y}) \geq -1.45$.

The membership functions are formulated as follows:

$$\mu_1(\bar{y}) = \frac{F_1(\bar{y}) + 5.792}{5.104} = \frac{(-y_1 - 1) + \dfrac{(-y_1 + 2y_2 - 5)}{(7y_1 + 3y_2 + 1)} + 5.792}{5.104}$$

$$\mu_2(\bar{y}) = \frac{F_2(\bar{y})+31}{31.272} = \frac{(-2y_2-1)+\dfrac{(2y_1-3y_2-5)}{(y_1+1)}+31}{31.272}$$

$$\mu_3(\bar{y}) = \frac{F_3(\bar{y})+15.9}{14.45} = \frac{(-3y_1-1)+\dfrac{(5y_1+2y_2-19)}{(-5y_1+20)}+15.9}{14.45}$$

The membership functions $\mu_1(\bar{y}), \mu_2(\bar{y}), \mu_3(\bar{y})$ are maximal at the points (0, 5), (4.5, 0), (0, 5). The membership functions are converted into equivalent linear membership functions at the best solution points by the first order Taylor series as follows;

$$\hat{\mu}_1(\bar{y}) = \mu_1(0,5) + (y_1-0)\left(\frac{\partial}{\partial \bar{y}_1}\mu_1(\bar{y})\right)_{\text{at } \bar{y}=(0,5)} + (y_2-5)\left(\frac{\partial}{\partial \bar{y}_2}\mu_2(\bar{y})\right)_{\text{at } \bar{y}=(0,5)}$$

$$\hat{\mu}_1(\bar{y}) = 1 - 0.235(y_1-0) + 0.013(y_2-5), \text{ similarly other membership function}$$

$$\hat{\mu}_2(\bar{y}) = 1 + 0.667(y_1-4.5) - 0.163(y_2-0), \hat{\mu}_3(\bar{y}) = 1 - 0.198(y_1-0) + 0.007((y_2-5)$$

The fuzzy goal programming approach -I can be represented as

Minimize λ

S.t

$$1 - 0.235(y_1-0) + 0.013(y_2-5) + d_1^- = 1$$

$$1 + 0.667(y_1-4.5) + 0.163(y_2-0) + d_2^- = 1$$

$$1 - 0.198(y_1-0) + +0.007(y_2-5) + d_3^- = 1 \tag{5.3}$$

$$y_1 \leq 6, y_2 \leq 6, 2y_1+y_2 \leq 9, -2y_1+y_2 \leq 5, y_1-y_2 \leq 5 \text{ and } y_1, y_2 \geq 0.$$

$$\lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1,2,3.$$

Eq. 5.3 provides the best compromise solutions $F_1 = -3.306$, $F_2 = -7.51$,$F_3 = -1.92$ at the pints $y_1 = 0$ and $y_2 = 0.302$. The fuzzy goal programming approach -II can be represented as:

$$\text{Minimize } \xi = \frac{1}{3}(d_1^- + d_2^- + d_3^-)$$

S.t

$$1 - 0.235(y_1-0) + 0.013(y_2-5) + d_1^- = 1$$

$$1 + 0.667(y_1-4.5) + 0.163(y_2-0) + d_2^- = 1$$

$$1 - 0.198(y_1-0) + +0.007(y_2-5) + d_3^- = 1 \tag{5.4}$$

$$y_1 \leq 6, y_2 \leq 6, 2y_1+y_2 \leq 9, -2y_1+y_2 \leq 5, y_1-y_2 \leq 5 \text{ and } y_1, y_2 \geq 0.$$

$$\lambda \geq d_i^-, 0 \leq d_i^- \leq 1, d_i^- \geq 0, i = 1,2,3.$$

Eq. 5.4 provides the best compromise solutions using LINGO 20.0 optimization software $F_1 = -6.0$, $F_2 = -6.0$, $F_3 = -1.95$ at $y_1 = 0$ and y_2= 0.

Solution by the WSA

In this section, we apply the weighted sum approach to the proposed numerical example and get the optimal compromise solutions. Let us assign the different weights of the problem, w_1 = 0.1, w_2 = 0.9,w_3= 0.2. The general formulation of the problem using WSP is represented as follows:

$$\text{Max F} = 0.1(-y_1-1)+\frac{(-y_1+2y_2-5)}{(7y_1+3y_2+1)}+0.9(-2y_2-1)+\frac{(2y_1-3y_2-5)}{(y_1+1)}+0.2(-3y_1-1)+\frac{(5y_1+2y_2-19)}{(-5y_1+20)}$$

S.t $y_1 \leq 6, y_2 \leq 6, 2y_1+y_2 \leq 9, -2y_1+y_2 \leq 5, y_1-y_2 \leq 5$ and $y_1, y_2 \geq 0$.

The problem is solved using the LINGO 20.0 optimization software, and the best compromise solutions are as follows: $F_1 = -7.031$, $F_2 = -2.1321$, $F_3 = -1.972$ at the points $y_1 = 5$, $y_2 = 1$.

Solution by NGPA

We have solved the numerical example 2 individually as a single objective optimization problem using the proposed NGPA. Construct the pay-off matrix by evaluating objective functions and constraints with the three solutions obtained.

$$\text{Pay-off matrix} = \begin{array}{c} \\ \bar{y_1}(0,5) \\ \bar{y_1}(4.5,0) \\ \bar{y_3}(0,5) \end{array} \begin{array}{c} \begin{array}{ccc} \text{F}_1 & \text{F}_2 & \text{F}_3 \end{array} \\ \begin{pmatrix} -0.6875 & -31.0 & -1.45 \\ -5.7923 & -0.2727 & -15.9 \\ -0.6875 & -31.0 & -1.45 \end{pmatrix} \end{array}$$

Determine the lower and upper bounds for each objective function. These bounds are assigned using the following formula,$F_i^l = \min\{F_i(\bar{y})\}_{i=1}^3, F_i^U = \max\{F_i(\bar{y})\}_{i=1}^3$.

The bounds of each objective represented as follows; $-0.6875 \leq F_1 \leq -1.237$, $-0.2727 \leq F_2 \leq -15.67$, $-1.45 \leq F_3 \leq -1.375$. Now, we define the membership function of each objective function using the NGPA.

For the first objective function F_1: $F_1^l(\varphi) = -0.6875$, $F_1^U(\varphi) = -1.237$, for the truth membership, $F_1^l(\theta) = -0.6875$, $F_1^U(\theta) = F_1^U(\varphi) + s_1\left(F_1^U(\varphi) - F_1^l(\varphi)\right) = -0.6875 + s_1$ for the indeterminacy membership, $F_1^l(\varphi) = F_1^l(\varphi) + t_1\left(F_1^U(\varphi) - F_1^l(\varphi)\right) = -0.6875 + t_1$, $F_1^U(\varphi) = -1.237$, for the falsity membership, where t_1 and s_1 are predetermined real numbers within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_1(F_1(\bar{y})) = \begin{cases} 1, & if \;\; F_1(\bar{y}) < -0.6875 \\ 1 - \dfrac{(-y_1-1)+\dfrac{(-y_1+2y_2-5)}{(7y_1+3y_2+1)}+0.6875}{-0.5495}, & if \;\; -0.6875 \leq F_1(\bar{y}) \leq -1.237 \\ 0, & if \; F_1(\bar{y}) > -1.237 \end{cases}$$

$$\theta_1(F_1(\bar{y})) = \begin{cases} 1, & if \;\; F_1(\bar{y}) < -0.6875 \\ 1 - \dfrac{(-y_1-1)+\dfrac{(-y_1+2y_2-5)}{(7y_1+3y_2+1)}+0.6875}{s_1}, & if \;\; -0.6875 \leq F_1(\bar{y}) \leq -0.6875+s_1 \\ 0, & if \; F_1(\bar{y}) > -0.6875+s_1 \end{cases}$$

$$\phi_1(F_1(\bar{y})) = \begin{cases} 1, & if \;\; F_1(\bar{y}) > -1.237 \\ 1 - \dfrac{-1.23-(-y_1-1)+\dfrac{(-y_1+2y_2-5)}{(7y_1+3y_2+1)}+0.6875}{-0.5495-t_1}, & if \;\; -0.6875+t_1 \leq F_1(\bar{y}) \leq -1.237 \\ 0, & if \; F_1(\bar{y}) > -0.6875+t_1 \end{cases}$$

For the second objective function F_1: $F_1^l(\varphi) = -0.2727$, $F_1^U(\varphi) = -15.67$, for the truth membership, $F_1^l(\theta) = -0.2727$, $F_1^U(\theta) = F_1^U(\varphi) + s_1\left(F_1^U(\varphi) - F_1^l(\varphi)\right) = -0.2727 + s_1$ for the indeterminacy membership, $F_1^l(\varphi) = F_1^l(\varphi) + t_1\left(F_1^U(\varphi) - F_1^l(\varphi)\right) = -0.2727 + t_1$, $F_1^U(\varphi) = -15.67$, for the falsity membership, where t_1 and s_1 are predetermined real numbers within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_2(F_2(\bar{y})) = \begin{cases} 1, & if\ \ F_2(\bar{y}) < -0.2727 \\ 1 - \dfrac{(-2y_2 - 1) + \dfrac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)} + 0.2727}{-15.3973}, & if\ \ -0.2727 \leq F_2(\bar{y}) \leq -15.67 \\ 0, & if\ F_2(\bar{y}) > -15.67 \end{cases}$$

$$\theta_2(F_2(\bar{y})) = \begin{cases} 1, & if\ \ F_2(\bar{y}) < -0.2727 \\ 1 - \dfrac{(-2y_2 - 1) + \dfrac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)} + 0.2727}{s_2}, & if\ \ -0.2727 \leq F_2(\bar{y}) \leq -0.2727 + s_2 \\ 0, & if\ F_2(\bar{y}) > -0.2727 + s_2 \end{cases}$$

$$\phi_2(F_2(\bar{y})) = \begin{cases} 1, & if\ \ F_2(\bar{y}) > -15.67 \\ 1 - \dfrac{-15.67 - (-2y_2 - 1) + \dfrac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)}}{-15.3973 - t_2}, & if\ \ -0.2727 \leq F_2(\bar{y}) \leq -15.67 \\ 0, & if\ F_2(\bar{y}) < -0.2727 + t_2 \end{cases}$$

For the third objective function F_3: $F_3^l(\varphi) = -1.45$, $F_3^U(\varphi) = -1.375$, for the truth membership, $F_3^l(\theta) = -1.45$, $F_3^U(\theta) = F_3^U(\varphi) + s_3\left(F_1^U(\varphi) - F_3^l(\varphi)\right) = -1.45 + s_3$ for the indeterminacy membership, $F_3^l(\varphi) = F_3^l(\varphi) + t_3\left(F_1^U(\varphi) - F_3^U(\varphi)\right) = -1.45 + t_3$, $F_3^U(\varphi) = -1.375$, for the falsity membership, where t_1 and s_3 are predetermined real numbers within the interval (0, 1). Based on these bounds, the membership function can be defined as follows:

$$\varphi_3(F_3(\bar{y})) = \begin{cases} 1, & if\ \ F_3(\bar{y}) < -1.45 \\ 1 - \dfrac{(-3y_1 - 1) + \dfrac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)} + 1.45}{0.075}, & if\ \ -1.45 \leq F_3(\bar{y}) \leq -1.375 \\ 0, & if\ F_3(\bar{y}) > -1.375 \end{cases}$$

$$\theta_3(F_3(\bar{y})) = \begin{cases} 1, & if\ \ F_3(\bar{y}) < -1.45 \\ 1 - \dfrac{(-3y_1 - 1) + \dfrac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)} + 1.45}{s_3}, & if\ \ -1.45 \leq F_3(\bar{y}) \leq -1.45 + s_3 \\ 0, & if\ F_3(\bar{y}) > -1.45 + s_3 \end{cases}$$

$$\phi_3(F_3(\bar{y})) = \begin{cases} 1, & if\ \ F_3(\bar{y}) > -1.375 \\ 1 - \dfrac{-1.375 - (-3y_1 - 1) + \dfrac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)}}{0.075 - t_3}, & if\ \ -1.45 + t_3 \leq F_3(\bar{y}) \leq -1.375 \\ 0, & if\ F_3(\bar{y}) < -1.45 + t_3 \end{cases}$$

The construction of an equivalent neutrosophic mathematical model for the proposed problem is as follows:

$$Max\ \zeta - \delta + \eta$$

S.t

$$y_1 \leq 6, y_2 \leq 6, 2y_1 + y_2 \leq 9, -2y_1 + y_2 \leq 5, y_1 - y_2 \leq 5 \text{ and } y_1, y_2 \geq 0.$$

$$(-y_1 - 1) + \frac{(-y_1 + 2y_2 - 5)}{(7y_1 + 3y_2 + 1)} - 0.5495\zeta \ \leq -1.237$$

$$(-2y_2 - 1) + \frac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)} - 15.3973\zeta \ \leq -15.67$$

$$(-3y_1 - 1) + \frac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)} + 0.075\zeta \ \leq -1.375$$

$$(-y_1 - 1) + \frac{(-y_1 + 2y_2 - 5)}{(7y_1 + 3y_2 + 1)} + s_1\eta - s_1 \leq -0.6875$$

$$(-2y_2 - 1) + \frac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)} + s_2\eta - s_2 \leq -0.2727$$

$$(-3y_1 - 1) + \frac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)} + s_3\eta - s_3 \leq -1.45$$

$$(-y_1 - 1) + \frac{(-y_1 + 2y_2 - 5)}{(7y_1 + 3y_2 + 1)} - (-0.5495 - t_1)\delta - t_1 \leq -0.6875$$

$$(-2y_2 - 1) + \frac{(2y_1 - 3y_2 - 5)}{(y_1 + 1)} - (-15.3973 - t_2)\delta - t_2 \leq -0.2727$$

$$(-3y_1 - 1) + \frac{(5y_1 + 2y_2 - 19)}{(-5y_1 + 20)} - (-0.075 - t_3)\delta - t_3 \leq -1.45$$

$$\zeta \geq \eta, \zeta \geq \delta, \zeta + \eta + \delta \leq 3, \zeta, \eta, \delta \in [0, 1], i = 1, 2, 3.$$

Now, solve the above numerical illustration using LINGO 20.0 optimization software and obtain the best compromise solutions. The compromise solutions are represented as $F_1 = -3.0172$, $F_2 = -6.9730$, $F_3 = -1.8735$.

Table 2. Compromise solutions of MOL+LFPP (Example 2).

Approach/ Methods	y	y_2	F_1	F_2	F_3
FGPA-I	0	0.302	-3.306	-7.51	-1.92
FGPA-II	0	0	-6.0	-6.0	-1.95
WSA	5	1	-7.031	-2.1321	-1.972
NGPA	5	1	-3.0172	-6.9730	-1.8735

Table 2 presents the compromise solutions for the MOL+LFPP based on example 2, showcasing the solution points (y_1, y_2) and their corresponding objective values (F_1, F_2, F_3) obtained using the different approaches: FGPA-I, FGPA-II, WSA, and NGPA. The results highlight the varying performance and trade-offs among the methods in achieving the optimal solution for the given objectives.

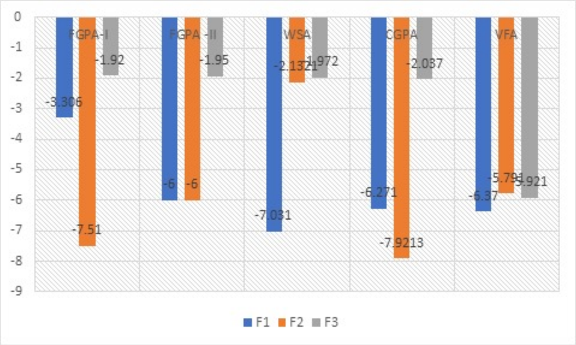

Figure 2. Shows the compromise solution using different approaches.
Figure 2 illustrates the compromise solutions using different approaches for the MOL+LFPP. The NGPA provides a better solution than the other approaches, demonstrating its effectiveness in optimizing the objectives.

6 Conclusion

In this study, we developed and analyzed a mathematical model for the multi-objective linear plus linear fractional programming problem (MOL+LFPP). By employing three distinct approaches—Fuzzy Programming Approach (FPA), Weighted Sum Approach (WSA), and Neutrosophic Goal Programming Approach (NGPA)—we generated sets of efficient solutions for the proposed problem. Numerical illustrations solved using LINGO 20.0 optimization software demonstrated the feasibility and effectiveness of these approaches.
The results reveal that NGPA consistently outperforms the other methods, providing superior compromise solutions with the highest values across all objective functions. Specifically, in Example 1 and Example 2, NGPA achieved objective values of $(-6.3123, 3.025, 7.912)$ and $(-3.0172, -6.9730, -1.8735)$, respectively. These outcomes underscore NGPA's robustness and reliability in handling the complexities of MOL+LFPP, making it a particularly effective approach for such problems.
Furthermore, the study underscores the adaptability of NGPA in dealing with multi-objective optimization scenarios, where traditional methods may fall short in balancing conflicting objectives. The findings suggest that NGPA provides better objective values and a more comprehensive framework for decision-makers seeking optimal solutions in complex, multi-objective environments. Future research could explore the integration of NGPA with other advanced techniques to enhance its application in broader optimization contexts.

6.1 Managerial and Practical Implications

The findings of this research have significant implications for managers and practitioners involved in multi-objective decision-making scenarios. The proposed methodologies provide robust frameworks for handling complex optimization problems with linear and fractional objectives [73, 74]. Managers can leverage these approaches to make more informed decisions that simultaneously consider multiple, often competing, objectives [75]. The ability to find compromise solutions that optimize various objectives can lead to more efficient and effective resource allocation and operational strategies [76, 77]. The use of fuzzy weighted sum and neutrosophic programming approaches allows for incorporating uncertainty and vagueness in real-world scenarios, enhancing the adaptability of the decision-making process.

6.2 Limitations and Future Research

While this study presents valuable insights, it also has certain limitations that pave the way for future research. The current study is limited to examples with relatively few objectives and constraints. Future research could explore the scalability of these approaches to larger and more complex MOL+LFPP instances. The numerical examples used in this paper are hypothetical, so applying these methodologies to real-world data would provide more practical insights and validate the robustness of the proposed solutions [86, 88]. Further comparative studies with other multi-objective optimization approaches could enrich the understanding of the relative strengths and weaknesses of FPA, WSA, and NGPA. Investigating the performance of these methodologies in dynamic and time-varying environments could enhance their applicability in rapidly changing industries [78]. Additionally, exploring hybrid methodologies that combine elements of fuzzy, weighted sum, and neutrosophic approaches with other optimization techniques could lead to more versatile and powerful solution frameworks. By addressing these limitations, future research can build on the foundation laid by this study to further advance the field of multi-objective optimization in both theoretical and practical dimensions.

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