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Numerical Solution Of The Convection Diffusion Equation With a Source Term Via The Spectral Method

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Abstract In this article, we present a numerical study of the convection-diffusion equation with a source term, formulated as a mixed initial-boundary value problem, using a spectral element method for the space discretization, which enables problem **(1.1)** in a finite regular set Λ to be solved as a set of ordinary differential equations. We prove the existence, uniqueness and the stability of the solution. Finally, we estimate the error between the exact and approximated discrete solutions, and illustrate the features of our method with numerical examples. To solve the discrete problem, we use the inverse matrix.

1 Introduction

The convection-diffusion equation is a combination of the diffusion and convection (advection) equations. It describes physical phenomena where particles, energy, or other physical quantities are transferred within a system due to two processes: diffusion and convection, see also [4, 13, 63].
Many problems in physics, mathematical physics, and various other fields of science can be modeled by the convection-diffusion equation. For numerous examples and additional applications, see [9, 18, 24, 33, 35, 45, 51]. Various numerical methods have been employed to solve the convection-diffusion equation. Appadu [10] solved the equation using both standard and non-standard finite difference schemes. El-Hawary and Abdel-Rahman [28] studied the numerical solution of the convection-diffusion equation (linear Burger's equation) using a spectral spline method. Boztosun and Charafi [29] explored the numerical solution of the linear advection-diffusion equation using mesh-free and mesh-dependent methods. Krukier et al. [37] presented a numerical solution of the steady convection-diffusion equation with dominant convection in a domain with two spatial variables, as also discussed in [39]. In [44], Porshokouhi et al. applied a homotopy perturbation method to solve the convection-diffusion equation. Chawla et al. [46] introduced extended one-step time-integration schemes for convection-diffusion equations. Olayiwola [47] used the variational iteration method to solve the convection-diffusion equation. Feng [54], employed an explicit finite difference method to solve the convection-diffusion equation. Temsah [57] presented a steady-state solution for the convection-diffusion equation using the El-Gendi method. El-Wakil et al. [58] solved the convection-diffusion equation using the Adomian decomposition method. For more details on the convection-diffusion equation, see also [3, 5, 25, 31, 32, 38, 43, 48, 55].

The primary objective of this work is the numerical analysis of the discretization of the convection-diffusion equation with a source term, formulated as a mixed initial-boundary value problem. We have developed a spectral method to improve the accuracy of the solutions while reducing computational complexity. By reducing the size of the resulting system to $(N-1)$

instead of $(N-1)^2$ as in previous works (e.g., Bernardi and Maday, 1992; Daug, 1996; Bernardi et al., 1999), we achieved a significant improvement in solution efficiency. Furthermore, this spectral method exhibits a faster rate of error reduction compared to traditional methods such as the Finite Difference Method (FDM) and the Finite Element Method (FEM), making it a highly efficient approach for solving two-dimensional partial differential equations. For a more detailed and comprehensive analysis of these methods, we refer to [12, 14, 15, 17, 19, 23, 62].

The paper is organized as follows: After this introduction, Section 2 is devoted to orthogonal polynomials and their main properties. The variational formulation of the problem is presented in Section 3. The discrete problem and the proof of existence and uniqueness of the solution are introduced in Section 4. Numerical experiments and error estimates are discussed in Section 5, and finally, we conclude in Section 6. In this paper, we consider the convection-diffusion equation with a source term:

$$\partial_t u(x,t) - a\partial_x^2 u(x,t) + b\partial_x u(x,t) + cu(x,t) = f(x,t)\ , \quad x \in \Lambda, t > 0, \tag{1.1}$$

with the initial condition:

$$u(x,0) = u_0(x), \quad x \in \Lambda, \tag{1.2}$$

and the Dirichlet boundary conditions:

$$u(x,t) = 0 \qquad\qquad x \in \partial\Lambda, t > 0, \tag{1.3}$$

where $\Lambda = (-1,1)$ is a finite regular set with boundary $\partial\Lambda$, $b\partial_x u(x,t)$ and $a\partial_x^2 u(x,t)$ represent the convection and diffusion terms, respectively, and $f(x,t)$ is the heat source term. The parameters a, b and c are positive constants, with $u(x,t)$ representing the temperature at point x at time t. The discretization involves both spatial and temporal variables. In the case of $b = 0$, we have studied this problem numerically and theoretically in [1], and also in [2].

Thus, the problem described in **(1.1)** becomes a problem of a single spatial variable. By using an orthogonal matrix, we reduce this problem to a system of ordinary differential equations.

In this paper, we investigate this problem under inhomogeneous boundary conditions. We consider the approximate solution in the polynomial space $\mathbb{P}_N^0(\Omega)$, which is spanned by the elements $l_n(x)l_m(t)$, where $1 \le m, n \le N-1$, and $l_n(x)$ and $l_m(t)$ are the Lagrange polynomials.

In this work, we construct an approximate solution to the inhomogeneous mixed initial-boundary value problem **(1.1, 1.2, 1.3)** in the form:

$$u_N(x,t) = \sum_{n=1}^{N-1} a_n(t) l_n(x), \tag{1.4}$$

where

$$a_n(t) = \sum_{m=1}^{N-1} u_{nm} l_m(t). \tag{1.5}$$

The Lagrangian interpolates $l_n(x), 1 \le n \le N-1$, are defined at the points $x_i \in \overline{\Lambda} = [-1,1]$, $0 \le i \le N$, These interpolants satisfy the property $l_n(\xi_j) = \delta_{nj}, 1 \le n, j \le N-1$, where δ_{nj} is the Kronecker delta, and the points $\xi_j, 0 \le j \le N$ are the collocation points on the Gauss-Lobatto Legendre grid. The grid made by $\xi_j, 0 \le j \le N$, is denoted by Λ_{N+1}. The choice of the form **(1.4)** for the solution, combined with certain techniques, gives a linear system which can be written in matrix form as: $\Gamma Da - Aa = \Gamma G$, where A is a square, positive-definite matrix, and Γ is a diagonal invertible matrix, and the operator $D = \frac{d}{dt}$. We write $a = Pv$, where P is an orthogonal matrix such that $P^{-1}\left(\Gamma^{-1}A\right)P = C$ is a diagonal matrix, This results in a system of $N-1$ ordinary differential equations. We can use Lagrange's method of undetermined parameters to solve for each component $v_i(t)$ of v, and finally, we obtain the expression for $a_n(t)$ which provides the desired approximate solution, see also [1, 6, 7, 8, 30, 36, 59, 60, 64].

2 Orthogonal polynomials

We work in the interval Λ and use the Legendre polynomials L_n,where $n \geq 0$. Each polynomial L_n has degree n and is orthogonal to the other polynomials in the space

$$L^2(\Lambda) = \left\{ \varphi : \Lambda \to \mathbb{R}, \text{measurable } / \int_{-1}^{1} \varphi^2(x)\, dx < +\infty \right\}. \tag{2.1}$$

and satisfies the following property

$$\int_{-1}^{1} L_n(x) L_m(x) dx = \frac{2}{2n+1} \delta_{nm}. \tag{2.2}$$

$$h_n'(x) = -n(n+1) L_n(x),\, h_n(x) = \left(1 - x^2\right) L_n^{'}(x),\, n \geq 0, \tag{2.3}$$

$$h_n(x) = \frac{n(n+1)}{2n+1} \left(L_{n-1}(x) - L_{n+1}(x)\right) \tag{2.4}$$

$$\int_{\Lambda} (h_n(x))^2\, dx = \frac{4\left[n(n+1)\right]^2}{(4n^2-1)(2n+3)}. \tag{2.5}$$

3 Variational Formulation

3.1 The spaces

The pivot space for the problem (**1.1**) is the space $L^2(\Lambda)$, and the variational space is the Sobolev space

$$H^1(\Lambda) = \left\{ v \in L^2(\Lambda) \;/\; \partial_x v \in L^2(\Lambda) \right\}, \tag{3.1}$$

with the corresponding norms defined as follows:

$$\|v\|^2_{L^2(\Lambda)} = \int_{\Lambda} v^2 dx,$$

$$\|v\|^2_{H^1(\Lambda)} = \int_{\Lambda} (v^2 + (\partial_x v)^2) dx. \tag{3.2}$$

3.2 The continuous problem

To introduce the variational formulation for the continuous problem (**1.1**), we define the subspace of the variational space with zero Dirichlet trace as

$$H_0^1(\Lambda) = \left\{ v \in H^1(\Lambda) \;/\; v = 0 \text{ on } \partial\Lambda \right\}. \tag{3.3}$$

We define the product in $L^2(\Lambda)$ as

$$(f, v) = \int_{\Lambda} f(x,t) v(x,t) dx. \tag{3.4}$$

The continuous problem (**1.1**) admits the following equivalent variational formulation:
Find $u \in H_0^1(\Lambda)$, such that

$$\forall v \in H_0^1(\Lambda),\, \Phi(u, v) = \langle f, v \rangle, \tag{3.5}$$

where

$$\Phi(u, v) = \int_{\Lambda} \left(\partial_t u - a\partial_x^2 u + b\partial_x u + cu\right) v dx, \tag{3.6}$$

and integrating by parts gives

$$\Phi(u, v) = \int_{\Lambda} (\partial_t uv + a\partial_x u \partial_x v + b\partial_x uv + cuv)\, dx. \tag{3.7}$$

4 Discrete space and form

Let N denote the discretization parameter for the problem (1.1), where in the spectral method, N represents the degree of the polynomials. The approximate space is generated by the finite-dimensional subspace of $L^2(\Lambda)$, and $\mathbb{P}_N^0(\Lambda)$ is the approximate subspace of $H_0^1(\Lambda)$, where

$$\mathbb{P}_N^0(\Lambda) = \{p_n \in \mathbb{P}_N(\Lambda) \ / \ p_n(1) = p_n(-1) = 0\},$$

and $\mathbb{P}_N(\Lambda)$ is the set of polynomials of degree less than or equal to N. Furthermore, we take into account the exact quadrature formula and introduce the bilinear form Φ_N as an approximation to the form Φ, and we approximate the scalar product $(.,.)$ for $(.,.)_N$, as discussed in [1, 6, 8, 30, 60, 64].

4.1 The Discrete problem

Firstly, we observe that the Lagrange polynomials $l_n(x)$, where $0 \leq n \leq N$, form a basis for $\mathbb{P}_N^0(\Lambda)$. The exact solution u of problem (1.1) is approximated by the solution u_N^I belonging to $\mathbb{P}_N^0(\Lambda)$, with $\left(u_N^I - u_{N0}\right) \in \mathbb{P}_N^0(\Lambda)$. The corresponding variational problem is:

$$\begin{cases} \text{find } u_N^I \in \mathbb{P}_N^0(\Lambda), \text{ s.t} \\ \forall v_N \in \mathbb{P}_N^0(\Lambda), \Phi_N(u_N^I, v_N) = (f_N, v_N)_N \end{cases}, \tag{4.1}$$

where

$$\Phi_N(u_N^I, v_N) = \sum_{k=0}^{N} \left(\partial_t u_N^I v_N + a\partial_x u_N^I \partial_x v_N + b\partial_x u_N^I v_N + c u_N^I v_N\right)(\xi_k, t)\rho_k, \tag{4.2}$$

and ξ_k, ρ_k for $0 \leq k \leq N$ are defined in proposition 4.1, and $u_N^I = u_N + u_{N0}$, with $u_N \in \mathbb{P}_N^0(\Lambda)$. The problem (4.1) is equivalent to the following problem: Find $u_N^I \in \mathbb{P}_N^0(\Lambda)$ with $u_N = u_N^I - u_{N0} \in \mathbb{P}_N^0(\Lambda)$ such that, $\forall v_N \in \mathbb{P}_N^0(\Lambda)$

$$\Phi_N(u_N, v_N) = \Theta_N(u_{N0}, v_N), \tag{4.3}$$

where

$$\Theta_N(u_{N0}, v_N) = (f_N, v_N)_N - \Phi_N(u_{N0}, v_N). \tag{4.4}$$

4.2 Existence and uniqueness of solution

Quadrature formula

Proposition 4.1. *There exists a unique set of $N-1$ nodes ξ_j, $1 \leq j \leq N-1$, in Λ, with the conditions $\xi_0 = -1$ and $\xi_N = 1$, as well as $N+1$ positive weights $\rho_j, 0 \leq j \leq N$, such that the following exactness property holds:*

$$\forall \varphi \in \mathbb{P}_{2N-1}(\Lambda), \int_{-1}^{1} \varphi(x)\, dx = \sum_{j=0}^{N} \varphi(\xi_j)\, \rho_j. \tag{4.5}$$

Here, ξ_j for $1 \leq j \leq N-1$ are the roots of the polynomial $L_N^{'}$. and the weights ρ_j are given by:

$$\begin{cases} \rho_0 = \rho_N = \frac{2}{N(N+1)} \\ \rho_j = \frac{\rho_0}{L_N^2(\xi_j)}, 1 \leq j \leq N-1 \end{cases}. \tag{4.6}$$

Proof. See [2, 14, 15]. □

Definition 4.2. We define the discrete product for all polynomials v_N and u_N in $\mathbb{P}_N^0(\Lambda)$ as:

$$(u_N, v_N)_N = \sum_{k=0}^{N} u_N(\xi_k, t) v_N(\xi_k, t)\rho_k. \tag{4.7}$$

Lemma 4.3. *The polynomial* $h_{N-1} \in \mathbb{P}_N^0(\Lambda)$ *verifies the double inequality:*

$$\|h_{N-1}\|_{L^2(\Lambda)}^2 \leq (h_{N-1}, h_{N-1})_N \leq \frac{3}{2}\|h_{N-1}\|_{L^2(\Lambda)}^2. \tag{4.8}$$

Proof. See [2, 8]. □

Proposition 4.4. *For all polynomials* $h_n \in \mathbb{P}_n^0(\Lambda)$*, the following inequalities hold:*

$$n\|h_n\|_{L^2(\Lambda)} \leq \left\|h_n^{'}\right\|_{L^2(\Lambda)} \leq 3n\|h_n\|_{L^2(\Lambda)}. \tag{4.9}$$

Proof. See [8]. □

Also, the Lagrange polynomials $l_j(x)$ for $j = \overline{1, N-1}$ can be written in the following form

$$l_j(x) = \sum_{k=0}^{N-1} \gamma_{kj} h_k(x),$$

and using **(2.3),** we get

$$l_j(x) = \sum_{k=0}^{N-1} \lambda_{kj} L_k(x). \tag{4.10}$$

Proposition 4.5. *The set of polynomials* $\{L_n(\zeta)\}$*, for* $n = 0, ..., N$*, forms a basis for the polynomial space* $\mathbb{P}_N(\Lambda)$*. Therefore, any polynomial* $\varphi_N \in \mathbb{P}_N(\Lambda)$ *can be written as* $\varphi_N(\zeta) = \sum_{n=0}^{N} \alpha_n L_n(\zeta)$*. Furthermore, we have the following inequality:*

$$c_1 \log(2N+1) \leq \|\varphi_N\|_{L^2(\Lambda)}^2 \leq c_2 \log(\exp(2)(2N+1)), \tag{4.11}$$

where $(c_1, c_2) = \left(\min(\alpha_n^2), \max(\alpha_n^2)\right)$.

Proof. See [2, 8]. □

Proposition 4.6. *For a positive integer* m*, the Sobolev space* $H^m(\Lambda)$ *is defined as:*

$$H^m(\Lambda) = \left\{\varphi \in L^2(\Lambda) : 1 \leq k \leq m, \frac{d^k}{dx^k}\varphi \in L^2(\Lambda)\right\}, \tag{4.12}$$

with the norm:

$$\|\varphi\|_{H^m(\Lambda)}^2 = \int_\Lambda \sum_{k=0}^{m} \left(\frac{d^k}{dx^k}\varphi\right)^2 (x)\, dx. \tag{4.13}$$

Proposition 4.7. *The bilinear form* $\Phi_N(.,.)$ *in equation **(4.3)** satisfies the following properties of continuity:*

$$\forall u_N \in \mathbb{P}_N^0(\Lambda),\ \ \forall v_N \in \mathbb{P}_N^0(\Lambda),\ \ |\Phi_N(u_N, v_N)| \leq \frac{3}{2}\max(a+b, c+C_4)\left(||u_N||_{H_0^1(\Lambda)} \cdot ||v_N||_{H_0^1(\Lambda)}\right), \tag{4.14}$$

and of ellipticity:

$$\forall u_N \in \mathbb{P}_N^0(\Lambda),\ \ |\Phi_N(u_N, u_N)| \geq \min(a, c+C_3)\left(||u_N||_{H_0^1(\Lambda)}^2\right). \tag{4.15}$$

Proof. The continuity: The bilinear form Φ_N is expressed as:

$$\Phi_N(u_N, v_N) = \sum_{k=0}^{N} \partial_t u_N(\xi_k, t) v_N(\xi_k, t)\rho_k + a\sum_{k=0}^{N} \partial_x u_N(\xi_k, t)\partial_x v_N(\xi_k, t)\rho_k$$
$$+b\sum_{k=0}^{N} \partial_x u_N(\xi_k, t) v_N(\xi_k, t)\rho_k + c\sum_{k=0}^{N} u_N(\xi_k, t) v_N(\xi_k, t)\rho_k.$$

We assume that the solution and its derivatives are bounded, so there exist two positive constants C_3 and C_4 such that

$$C_3 \left|u_N(\xi_k,t)\right| \leq \left|\partial_t u_N(\xi_k,t)\right| \leq C_4 \left|u_N(\xi_k,t)\right|. \tag{4.16}$$

Using lemma (4.3), the exact quadrature formula, and the Cauchy-Schwarz inequality, we can derive the desired results, see also (Bernardi and Maday [16], Boutaghou and Nouri [8]).

The ellipticity: The bilinear form Φ_N is written as:

$$\begin{aligned}\Phi_N(u_N,u_N) &= \sum_{k=0}^{N}\partial_t u_N(\xi_k,t)u_N(\xi_k,t)\rho_k + a\sum_{k=0}^{N}\partial_x u_N(\xi_k,t)\partial_x u_N(\xi_k,t)\rho_k + b\sum_{k=0}^{N}\partial_x u_N(\xi_k,t)u_N(\xi_k,t\\ &+c\sum_{k=0}^{N}u_N(\xi_k,t)u_N(\xi_k,t)\rho_k.\end{aligned}$$

Using the exact quadrature formula, we rewrite the expression as:

$$\begin{aligned}\Phi_N(u_N,u_N) &= \sum_{k=0}^{N}\partial_t u_N(\xi_k,t)u_N(\xi_k,t)\rho_k + a\int_{-1}^{1}\partial_x u_N(x,t)\partial_x u_N(x,t)dx + b\sum_{k=0}^{N}\partial_x u_N(\xi_k,t)u_N(\xi_k,t)\\ &+c\sum_{k=0}^{N}u_N(\xi_k,t)u_N(\xi_k,t)\rho_k,\end{aligned}$$

From inequality (4.16) and the orthogonality properties, we obtain:

$|\Phi_N(u_N,u_N)| \geq C_3\sum_{k=0}^{N}u_N(\xi_k,t)u_N(\xi_k,t)\rho_k + a\int_{-1}^{1}\partial_x u_N(x,t)\partial_x u_N(x,t)dx + c\sum_{k=0}^{N}u_N(\xi_k,t)u_N(\xi_k,t)\rho_k.$

Using inequality (4.8) we write:

$$|\Phi_N(u_N,u_N)| \geq \min(a, c+C_3)\left(||u_N||^2_{H_0^1(\Lambda)}\right),$$

which yields the desired result. □

Proposition 4.8. *(The inequality of stability) For any continuous function $g = u_0$ on Λ, the problem (4.3) has a unique solution u_N in $\mathbb{P}_N^0(\Lambda)$, and this solution verifies the inequality of stability:*

$$\|u_N(x,t)\|_{H_0^1(\Lambda)} \leq \gamma\left(\|f_N(x,t)\|_{L^2(\Lambda)} + \|g_N(x)\|_{L^2(\Lambda)}\right), \tag{4.17}$$

where γ is a positive constant.

Proof. From the variational formulation (4.3), we can write:

$$\Phi_N(u_N,u_N) = (f_N,u_N)_N - \Phi_N(g_N,u_N) \leq |(f_N,u_N)_N| + |\Phi_N(g_N,u_N)|. \tag{4.18}$$

Using inequality (4.8) and the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}|(f_N,u_N)_N| + |\Phi_N(g_N,u_N)| &\leq \frac{3}{2}\|f_N(x,t)\|_{L^2(\Lambda)}.\|u_N(x,t)\|_{L^2(\Lambda)} + a\|\partial_x g_N(x)\|_{L^2(\Lambda)}.\|\partial_x u_N(x,t)\|_{L^2(}\\ &+b\|\partial_x g_N(x)\|_{L^2(\Lambda)}.\|u_N(x,t)\|_{L^2(\Lambda)} + \frac{3c}{2}\|g_N(x)\|_{L^2(\Lambda)}.\|u_N(x,t)\|_{L^2(\Lambda)}\end{aligned}$$

The quantities $\|\partial_x g_N(x)\|_{L^2(\Lambda)}$ and $\|\partial_x u_N(x,t)\|_{L^2(\Lambda)}$ are bounded. Therefore, there exists a positive constant γ such that:

$\Phi_N(u_N,u_N) \leq |(f_N,u_N)_N| + |\Phi_N(g_N,u_N)| \leq \gamma\left(\|f_N(x,t)\|_{L^2(\Lambda)} + \|g_N(x)\|_{L^2(\Lambda)}\right)\|u_N(x,t)\|_{H_0^1(\Lambda)},$

Finally, using the ellipticity inequality (4.15), yields the desired result. □

5 Numerical experiment

At the points ξ_k, $1 \leq k \leq N-1$ the problem (1.1, 1.2, 1.3) is transformed into a system of equations:

$$\begin{cases} \sum_{n=1}^{N-1} l_n(\xi_k) a'_n(t) + \left[c l_n(\xi_k) + b l'_n(\xi_k) - a l'_n(\xi_k) \right] a_n(t) = \sum_{n=1}^{N-1} f_n(t) l_n(\xi_k) + a u''_{N0}(\xi_k) - b u'_{N0}(\xi_k) - c u_{N0}(\\ u_N(\xi_k, t) = 0, \qquad \text{on } \partial\Lambda \cap \Lambda_{N+1} \\ u_N(x, 0) = u_{N0}(x) \\ f(x,t) = \sum_{n=1}^{N-1} f_n(t) l_n(x), \; f_n(t) = \sum_{j=1}^{N-1} f_{jn} l_j(t), \; f_{jn} = f(\xi_j, t_n) \end{cases} \tag{5.1}$$

Since the functions

$$c l_n(x) + b l'_n(x) - a l''_n(x), \; 1 \leq n \leq N-1,$$

are polynomials with degree N, we multiply both sides by $l_m(\xi_k)\rho_k$ and applying the sum, by using the quadrature formula, when m varies from 1 to $N-1$, we obtain a linear system, then we can write this system in a matrix form:

$$\Gamma D a - A a = \Gamma G. \tag{5.2}$$

Where A is a square, positive-definite matrix of order $N-1$, with elements:

$$\alpha_{mn} = (-c l_n(\xi_m) - b l'_n(\xi_m) + a l''_n(\xi_m)) l_m(\xi_k)\rho_m, \; n = \overline{1, N-1}, m = \overline{1, N-1}.$$

Γ is a diagonal invertible matrix with elements:

$$\gamma_{mn} = \begin{cases} \rho_m \;, n = m \\ 0, \;\; n \neq m \end{cases} \quad , \; m, n = \overline{1, N-1},$$

G is a known vector:

$$\begin{aligned} G \;\; = \;\; & (f_1(t) + a u''_{N0}(\xi_1) - b u'_{N0}(\xi_1) - c u_{N0}(\xi_1), f_2(t) + a u''_{N0}(\xi_2) - b u'_{N0}(\xi_2) - c u_{N0}(\xi_2) \\ & , ..., f_{N-1}(t) + a u''_{N0}(\xi_{N-1}) - b u'_{N0}(\xi_{N-1}) - c u_{N0}(\xi_{N-1}))^t, \end{aligned}$$

$a(t)$ is the unknown vector of coefficients:

$$a(t) = (a_1(t), a_2(t), a_3(t), ..., a_{N-2}(t), a_{N-1}(t))^t,$$

the operator,

$$D = \frac{d}{dt}.$$

We now multiply equation (5.2) by the inverse matrix Γ^{-1} to obtain:

$$Da - \Gamma^{-1} A a = G \, . \tag{5.3}$$

The matrix $\Gamma^{-1}A$ has positive eigenvalues, and there exists an orthogonal matrix P such that,

$$P^{-1}\left(\Gamma^{-1}A\right)P = C,$$

where C is a diagonal matrix with eigenvalues $\lambda_i = \alpha_{ii}$, for $i = \overline{1, N-1}$ of the matrix $\Gamma^{-1}A$, if we consider the vector v such that

$$a = Pv,$$

then the system (5.3) becomes

$$PDv - (\Gamma^{-1}A)Pv = G. \tag{5.4}$$

Multiplying both sides by P^{-1} results in:

$$Dv - Cv = P^{-1}G. \tag{5.5}$$

This is a system of $N-1$ linear ordinary differential equations:

$$v_k'(t) - \lambda_k v_k(t) = h_k(t), \tag{5.6}$$

where

$$h_k(t) = \sum_{j=1}^{N-1} p^{-1}(k,j)\left(f_j(t) + a u''_{N0}(\xi_k) - b u'_{N0}(\xi_k) - c u_{N0}(\xi_k)\right), 1 \le k \le N-1, \tag{5.7}$$

$p^{-1}(k,j)$ are the elements of the inverse matrix P^{-1}. To solve the equations (5.6) we use Lagrange's method [64], we may write the solution in the closed form:

$$v_k(t) = e^{\lambda_k t}\left(\int_0^t e^{-\lambda_k s} h_k(s)ds + d_k\right), \tag{5.8}$$

where d_k is a constant to be determined from the boundary conditions. Thus, equation (5.8) can be written as:

$$v_k(t) = e^{\lambda_k t}\left(\int_0^t e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_{N0}(\xi_k)\right). \tag{5.9}$$

Finally, we obtain the functions,

$$a_n(t) = \sum_{j=1}^{N-1} p_{nj} v_j(t), \tag{5.10}$$

where p_{nj}, $1 \le n, j \le N-1$ are the elements of the matrix P, and the approximation solution is:

$$u(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} p_{nj}\left(\int_0^t e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_0(\xi_k)\right) e^{\lambda_k t} l_n(x).$$

For the time interval $t \in [0,T]$, the solution is written as:

$$u(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} u_{nj} l_n(x) l_j(t), \quad a_n(t) = \sum_{j=1}^{N-1} u_{nj} l_j(t), \tag{5.11}$$

where the coefficients u_{nj} are determined by:

$$u_{nj} = \sum_{j=1}^{N-1} p_{nj}\left(\int_0^{t_j} e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_{N0}(\xi_k)\right) e^{\lambda_k t_j}.$$

Thus, the approximate solution is:

$$u_N(x,t) = \sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(\sum_{j=1}^{N-1} p_{nj}\left(\int_0^{t_j} e^{-\lambda_k s} h_k(s)ds + \sum_{j=1}^{N-1} p_{kj}^{-1} u_{N0}(\xi_k)\right) e^{\lambda_k t_j}\right) l_n(x) l_m(t) + \phi(x),$$

where $\phi(x) = \sum_{n=1}^{N-1} u_{N0}(\xi_n) l_n(x)$.

By using (5.7), we get:

$$\begin{aligned} u_N(x,t) &= \sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(\sum_{j=1}^{N-1} p_{nj}\left(\int_0^{t_j} e^{-\lambda_k(s-t_j)} \sum_{j=1}^{N-1} p^{-1}(k,j)\left(f_j(s) + a u''_{N0}(\xi_k) - b u'_{N0}(\xi_k) - c u_{N0}(\xi_k)\right)\right.\right. \\ &\quad \left.\left. + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} u_0(\xi_k)\right) e^{\lambda_k t_j}\right) l_n(x) l_m(t) + \phi(x), \end{aligned}$$

5.1 Numerical integration

The function
$$q_k(s) = e^{-\lambda_k(s-t)}h_k(s), \tag{5.12}$$
appears in the integral. We approximate this integral numerically since it may not have an explicit primitive. You can use polynomial interpolation to approximate this integral. The Lagrange polynomial interpolation for $q_k(s)$ is given by:
$$q_{Nj}(s) = \sum_{n=0}^{N} q_j(t_n)l_j(s),$$
where $t_n, 0 \leq n \leq N$, are the collocation points defined by $t_n = \frac{T}{2}(\xi_n + 1)$ and ξ_n are the collocation points on the Gauss-Lobatto Legendre grid, then the approximation of the integral (5.9)
$$v_{Nj}(t) = \int_0^t q_{Nj}(s)ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} u_{N0}(\xi_k)\right) e^{\lambda_k t},$$
then we obtain
$$b_n(t) = \sum_{j=1}^{N-1} p_{nj}(t_n)v_{Nj}(t),$$
where p_{nj}, $1 \leq n, j \leq N-1$ are the elements of the matrix P , using (1.4) we get the approximate solution
$$u_N(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} p_{nj}v_{Nj}(t)l_n(x).$$

5.2 Error estimation

Definition 5.1. The polynomial space $\mathbb{P}_N^0(\Lambda)$ is dense in the space of continuous functions on Λ, and hence in $H_0^1(\Lambda)$ Therefore, any function $u \in H_0^1(\Lambda)$ admits the expansion
$$u(x,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \alpha(k,l)h_k(x)t_l(t). \tag{5.13}$$

We know that
$$t_n(t) = \frac{n(n+1)}{2(2n+1)}\left(p_{n-1}(t) - p_{n+1}(t)\right), \tag{5.14}$$
where
$$p_n(t) = L_n(\frac{2}{T}t - 1), n \geq 0. \tag{5.15}$$
Using equation (5.14), we can write
$$u(x,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \gamma(k,l)h_k(x)p_l(t). \tag{5.16}$$

Proposition 5.2. *The following estimate holds between the exact solution $u \in H_0^1(\Lambda)$ and the approximate solution $u_N \in \mathbb{P}_N^0(\Lambda)$:*
$$\|u - u_N\|_{L^2(\Lambda)} \leq 3CN^{-1}\left(\|(u_0 - u_{N0})\|_{L^2(\Lambda)} + \|f - f_N\|_{L^2(\Lambda)}\right), \tag{5.17}$$
where C is a real positive constant.

Proof. Using the ellipticity condition (4.15) and (4.9), we can write,
$$\begin{aligned} N^2\|u - u_N\|^2_{L^2(\Lambda)} &\leq \Phi(u - u_N, u - u_N) = (f - f_N, u - u_N)_N - \Phi(u_0 - u_{N0}, u - u_N), \\ &\leq C\left(\left|\int_\Lambda (f - f_N)(u - u_N)\,dx\right| + |\Phi(u_0 - u_{N0}, u - u_N)|\right). \end{aligned} \tag{5.18}$$

Where C is a real positive constant, using the Cauchy-Schwarz inequality, we find

$$\left|\int_{\Lambda}(f-f_N)(u-u_N)\,dx\right| \leq \|f-f_N\|_{L^2(\Lambda)}\|u-u_N\|_{L^2(\Lambda)}, \tag{5.19}$$

By applying the triangle inequality, we obtain

$$\begin{aligned}|\Phi(u_0-u_{N0},u-u_N)| \leq\ & \left|a\int_{\Lambda}\partial_x(u_0-u_{N0})\,\partial_x(u-u_N)\,dx\right| + \left|\int_{\Lambda}\partial_t(u_0-u_{N0})(u-u_N)\,dx\right| \\ & + \left|b\int_{\Lambda}\partial_x(u_0-u_{N0})(u-u_N)\,dx\right| + \left|c\int_{\Lambda}(u_0-u_{N0})(u-u_N)\,dx\right|.\end{aligned}$$

Since u_0 is independent of t, we have

$$\int_{\Lambda}\partial_t(u_0-u_{N0})(u-u_N)\,dx = 0,$$

Thus, by the Cauchy-Schwarz inequality, we find

$$\left|c\int_{\Lambda}(u_0-u_{N0})(u-u_N)\,dx\right| \leq c\|(u_0-u_{N0})\|_{L^2(\Lambda)}\|(u-u_N)\|_{L^2(\Lambda)}, \tag{5.20}$$

and

$$\left|a\int_{\Lambda}\partial_x(u_0-u_{N0})\,\partial_x(u-u_N)\,dx\right| \leq a\|\partial_x(u_0-u_{N0})\|_{L^2(\Lambda)}\|\partial_x(u-u_N)\|_{L^2(\Lambda)}, \tag{5.21}$$

and

$$\left|b\int_{\Lambda}\partial_x(u_0-u_{N0})(u-u_N)\,dx\right| \leq b\|\partial_x(u_0-u_{N0})\|_{L^2(\Lambda)}\|(u-u_N)\|_{L^2(\Lambda)}, \tag{5.22}$$

using **(5.19), (5.20), (5.21), (5.22)** and **(4.9),** we get

$$N^2\|u-u_N\|^2_{L^2(\Lambda)} \leq 3CN\left(\|(u_0-u_{N0})\|_{L^2(\Lambda)} + \|f-f_N\|_{L^2(\Lambda)}\right)\|(u-u_N)\|_{L^2(\Lambda)}.$$

Finally, we obtain the desired result. □

5.3 Condition number

Definition 5.3. The condition number of an $n\times n$ non-singular matrix A is defined as:

$$k_P(A) = \|A\|_P\left\|A^{-1}\right\|_P, \tag{5.23}$$

where $\|A\|_P$ is the spectral norm of A, given by: $\rho = (A^tA)^{\frac{1}{2}}$.

Remark 5.4. The condition number of a matrix A gives a measure of how sensitive systems of equations, with coefficients matrix A, are to small perturbations such as those caused by rounding. Then if the condition number of a matrix is large, the effect of rounding error in the solution process may be serious [64].

To compute the condition number of different order of these matrix we use the spectral norm, and all operations are made by the Maple, using [22].

5.4 Figure illustration

We consider the exact explicit solution given by: $u(x,t) = -\exp(-0.02\pi^2t)\sin(\pi x)$,with $a = b = 1$, with the initial condition: $u(x,0) = u_0(x) = -\sin(\pi x)$ and the source term: $f(x,t) = \big((-0.98\pi^2-1)\sin(\pi x) - \pi\cos(\pi x)\big)\exp(-0.02\pi^2t)$.

The Figures 1 and 2 present the behavior of the condition number and the error, with N varying from 3 to 12. We plot $(N, log(k_P(A)))$. In Figure 3, we show the behavior of the functions $a_n(t)$ as n varies from 3 to 12. Figures 4 and 5, display the true and the approximate solutions u and u_N, respectively, for $N = 12$.

Remark 5.5. This Figure shows that the error decreases rapidly when N increass. Here we plot $\left(N, \|u - u_N\|_{L^2(\Lambda)}\right)$.

Remark 5.6. In Table 1, the results demonstrate that the computational method used achieves a very rapid convergence in solution accuracy as N increases. This rapid decrease in error makes the method highly efficient in providing accurate solutions in a short amount of time and the high experimental order of convergence EOC, particularly for smaller values of N, indicates a substantial improvement in accuracy.

It is important to note that the large values of the EOC for smaller N suggest a significant acceleration in the accuracy of the solution, emphasizing the effectiveness of the spectral method employed.

N	$e_N = \|u - u_N\|_{L^2(\Lambda)}$	$\|(u_0 - u_{N0})\|_{L^2(\Lambda)} + \|f - f_N\|_{L^2(\Lambda)}$	$EOC(e_N, e_{N+2})$
4	1.92×10^{-1}	10.10	6.79
6	1.23×10^{-2}	1.70	11.57
8	4.4×10^{-4}	1.23×10^{-1}	16.91
10	1.01×10^{-5}	4.91×10^{-3}	11.41
12	1.39×10^{-6}	2.22×10^{-4}	−

Table1: The behavior of the error and the experimental order of convergence EOC

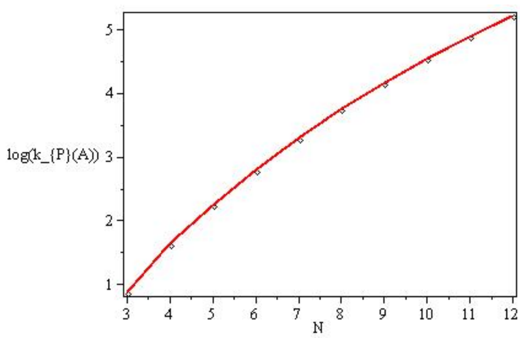

Figure 1. The behavior of the condition number when N vary from 3 to 12

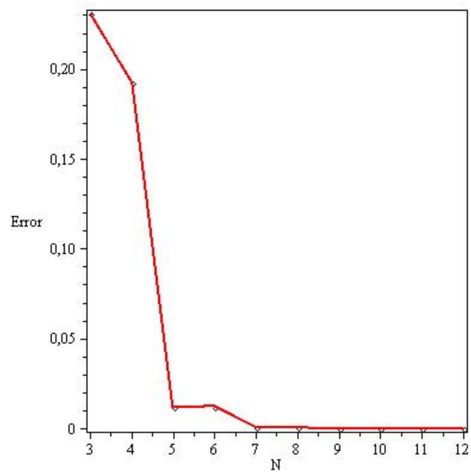

Figure 2. The behavior of the error when N vary from 3 to 12

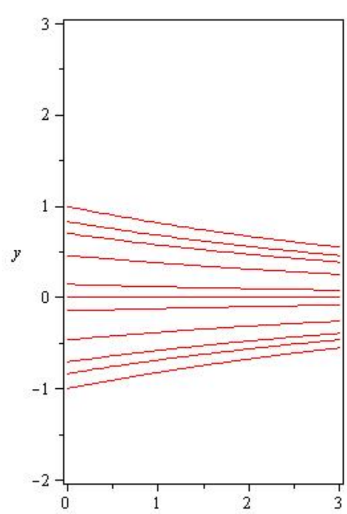

Figure 3. Plots of the functions $a_n(t)$, n vary from 3 to 12

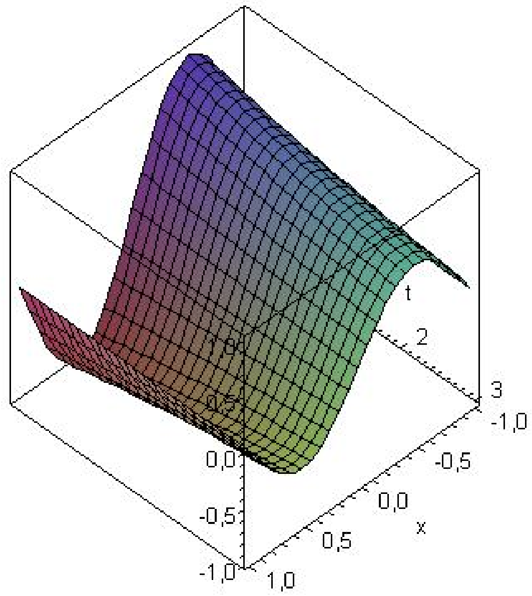

Figure 4. The true solution $u(x,t)$

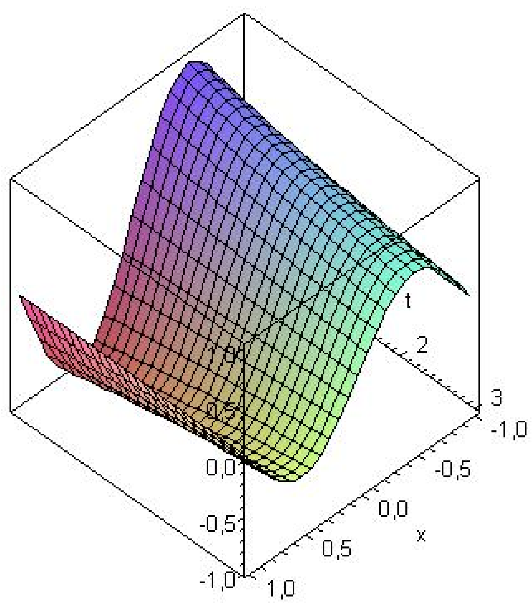

Figure 5. The approximation solution $u_N(x,t)$ when $N = 12$

6 Conclusion

The primary objective of this work was to reduce the two-dimensional problem to a one-dimensional domain by using an orthogonal matrix. As a result, the linear systems **(5.2)**,**(5.3)**,**(5.4)** and **(5.5)** are of size $(N-1)$, whereas in other approaches, the matrix size is $(N-1)^2$. This reduction significantly simplifies the computational complexity, providing a more efficient way to solve the problem.

References

[1] A. Lateli, A. Boutaghou and T. Hamaizia, Spectral method for the diffusion equation with a source term, International Journal of Nonlinear Analysis and Applications 13, 1343-1355 (2022).

[2] A. Lateli, T. Hamaizia, A. Boutagou, Autour des méthodes spectrales pour la résolution des équations aux dérivées partielles, Université Frères Mentouri-Constantine 1, (2022).

[3] A. Fallahzadeh, K. Shakibi, A method to solve Convection-Diffusion equation based on homotopy analysis method. Journal of interpolation and approximation in scientific computing 2015, 1-8 (2015).

[4] A. D. POLYANIN, Handbook of linear partial differential equations for engineers and scientists, Chapman and hall/crc, 2001.

[5] A. Mohammadi, M. Manteghian, A. Mohammadi, Numerical solution of the one-dimensional advection-diffusion equation using simultaneously temporal and spatial weighted parameters. Australian Journal of Basic and Applied Sciences 5, 1536-1543 (2011).

[6] A. Boutaghou, Spectral Method for the Nonhomogeneous Wave Equation with Axial Symmetry. Palestine Journal of Mathematics 12, 94–102 (2023).

[7] A. Boutaghou, Spectral Method for Mixed Initial-Boundary Value Problem. Paletine Journal of Mathematics 4, 12-20 (2015).

[8] A. Boutaghou, F.Z. Nouri., On Finite Spectral Method for Axi-symmetric Elliptic problem, accepted on SAS International Publications, Journal of Analysis and Applications, vol. 4. (2006), N: 3,pp 149-168.

[9] A. Daga, V. Pradhan, Analytical solution of advection diffusion equation in homogeneous medium. PRATIBHA: International Journal of Science, Spirituality, Business and Technology (IJSSBT) 2, p2277-7261 (2013).

[10] A. R. Appadu, Numerical solution of the 1D advection-diffusion equation using standard and nonstandard finite difference schemes. Journal of Applied Mathematics 2013, (2013).

[11] A. Quarteroni, Riccardo Sacco et Fausto Saleri., Méthodes Numériques, Algorithmes, Analyse et applications, Springer-Verlag Italia, Milano 2007.

[12] B. Mercier., stabilité et convergence des méthodes spectrales polynomiales: Application à l'équation d'avection. R.A.I.R.O. Anal. numér. 16 (1982) 97-100.

[13] C. E. Baukal Jr, V. Gershtein, X. J. Li, Computational fluid dynamics in industrial combustion. (CRC press, 2000).

[14] C. Bernardi, Y. Maday., Approximations spectrales de problèmes aux limites elliptiques. Mathématiques & applications (Springer-verl., Paris Berlin Heidelberg [etc.], 1992), pp. 242.

[15] C. Bernardi, M. Dauge, Y. Maday, M. Azaïez., Spectral methods for axisymmetric domains numerical algorithms and tests due to Mejdi Azaïez. Series in applied mathematics (Gauthier-Villars, Paris Amsterdam Lausanne [etc.], 1999), pp. V-345.

[16] C. Bernardi, Y. Maday, Spectral, spectral element and mortar element methods, in Theory and Numerics of Differential Equations: Durham 2000. (Springer, 2001), pp. 1-57.

[17] C. Canuto, Spectral methods in fluid dynamics. Springer series in computational physics (Springer-Verlag, Berlin ; New York, ed. Corr. 2nd print., 1988), pp. xiv, 567 p.

[18] D. Anderson, J. C. Tannehill, R. H. Pletcher, R. Munipalli, V. Shankar, Computational fluid mechanics and heat transfer. (CRC press, 2020).

[19] D. Gottlieb., The stability of pseudospectral Chebyshev methods. Math Comput. 36 (1981), 107-118.

[20] D. Gottlieb, S. A. Orszag., Numerical analysis of spectral methods: theory and applications. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1977. CBMS-NSF Regional Conference Series in Applied Mathematics, No. 26.

[21] D.J.Acheson., Elementary Fluid Dynamics, Oxford University Press, New York 1990.

[22] D. Richards., Advanced mathematical methods with Maple. (Cambridge University Press, Cambridge, UK; New York, 2002).

[23] E. P. Stephan. M. Suri., On the convergences of the p-version of the boundary element Galerkin method.Math. Compt. 52, 31-48 (1989).

[24] E. Veling, in Impact of human activity on groundwater dynamics. Proceedings of a symposium held during the Sixth IAHS Scientific Assembly, Maastricht, Netherlands, 18-27 July 2001. (IAHS Press, 2001), pp. 271-276.

[25] F. S. Bazán, Chebyshev pseudospectral method for computing numerical solution of convection–diffusion equation. Applied Mathematics and Computation 200, 537-546 (2008).

[26] G. Allaire, Analyse numérique et optimisation: une introduction à la modélisation mathématique et à la simulation numérique. (Editions Ecole Polytechnique, 2005).

[27] H. J. Weber, G. B. Arfken, Essential Mathematical Methods for Physicists, Elsevier, Academic Press, San Diego, 2004.

[28] H. El-Hawary, E. Abdel-Rahman, Numerical solution of the generalized Burger's equation via spectral/spline methods. Applied mathematics and computation 170, 267-279 (2005).

[29] I. Boztosun, A. Charafi, An analysis of the linear advection–diffusion equation using mesh-free and mesh-dependent methods. Engineering Analysis with Boundary Elements 26, 889-895 (2002).

[30] J. De Frutos, R. Munoz-Sola., 'Chebyshev Pseudospectral Collocation for Parabolic Problems with Non-constant Coefficients', in Proceedings of the third international conference on spectral and high order methods (Citeseer, 1996), pp. 101-107.

[31] J. Crank, P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. Advances in Computational Mathematics 6, 207-226 (1996).

[32] J. Zhang, Preconditioned iterative methods and finite difference schemes for convection–diffusion. Applied mathematics and computation 109, 11-30 (2000).

[33] J. Parlange, Water transport in soils. Annual review of fluid mechanics 12, 77-102 (1980).

[34] L. Pujo-Menjouet., Introduction aux équations différentielles, pujo@math.univ-lyon1.fr, Université Claude Bernard, Lyon I, 43, boulevard 11 novembre 1918, 69622 Villeurbanne cedex, France.

[35] L. C. Evans, Partial differential equations, American Mathematical Society, 2022.

[36] L. Adibmanesh, J. Rashidinia, The application of Sinc and B-Spline functions to numerical solution of the time-fractional convection-diffusion equations, Palestine Journal of Mathematics, 12 (2023).

[37] L. A. Krukier, O. Pichugina, B. L. Krukier, Numerical solution of the steady convection-diffusion equation with dominant convection. Procedia computer science 18, 2095-2100 (2013).

[38] L. A. Krukier, T. Martinova, B. L. Krukier, O. Pichugina, Special iterative methods for solution of the steady Convection-Diffusion-Reaction equation with dominant convection. Procedia Computer Science 51, 1239-1248 (2015).

[39] M. Assabaai, NUMERICAL SOLUTION OF CONVECTION-DIFFUSION EQUATION BY CHEBYSHEV SPECTRAL METHOD VIA LIE GROUP METHOD. Yanbu Journal of Engineering and Science 16, 57-63 (2021).

[40] M. Crouzeix, A. L. Mignot., Analyse numérique des équations differentilles,(Masson, Paris, 1989).

[41] M. Dauge, Spectral-fourier method for axi-symmetric problems, Proc. Third Int. Conf. Spect. High Order Meth, 1996, pp. 55-62

[42] M. Gisclon, A propos de l'équation de la chaleur et de l'analyse de Fourier. Le journal de maths des élèves 1 (4),(1998), 190-197.

[43] M. Dehghan, On the numerical solution of the one-dimensional convection-diffusion equation. Mathematical Problems in Engineering 2005, 61-74 (2005).

[44] M. G. Porshokouhi, B. Ghanbari, M. Gholami, M. Rashidi, Approximate solution of convection-diffusion equation by the homotopy perturbation method. Gen 1, 108-114 (2010).

[45] M. P. Chernesky, On preconditioned Krylov subspace methods for discrete convection–diffusion problems. Numerical Methods for Partial Differential Equations: An International Journal 13, 321-330 (1997).

[46] M. Chawla, M. Al-Zanaidi, M. Al-Aslab, Extended one-step time-integration schemes for convection-diffusion equations. Computers & Mathematics with Applications 39, 71-84 (2000).

[47] M. Olayiwola, Application of variational iteration method to the solution of convection-diffusion equation. Journal of Applied & Computational Mathematics 5, 2-5 (2016).

[48] M. Ghasemia, K. M. TAVASSOLI, Application of He's homotopy perturbation method to solve a diffusion-convection problem. 171-186 (2010).

[49] M. Sibony, J.-C. Mardon, Approximations et équations différentielles.(Hermann, Paris, 1982).

[50] M. Dauge., Spectral-Fourier Method for axi-symmetric Problems, ICOSAHOM 95: Proceedings of the third International Conference on Spectral and High Order Methods, Houston Journal of mathematics, University of Houston, 1996.

[51] N. Kumar, Unsteady flow against dispersion in finite porous media. Journal of Hydrology 63, 345-358 (1983).

[52] P. G. Ciarlet., Introduction à l'analyse numérique matricielle et à l'optimisation. Collection Mathématiques appliquées pour la maîtrise (Masson, Paris New York Barcelone, 1982), pp. XII-279.

[53] P. J. Davis, P. Rabinowitz., Methods of numerical integration. Computer science and applied mathematics (Academic Press, Orlando, ed. 2nd, 1984), pp. xiv, 612 p.

[54] Q. Feng, in Proceedings of the World Congress on Engineering. (2009), vol. 2, pp. 1094-1097.

[55] R. C. Chin, T. A. Manteuffel, J. de Pillis, ADI as a preconditioning for solving the convection-diffusion equation. SIAM journal on scientific and statistical computing 5, 281-299 (1984).

[56] R. Dautray, J. L. Lions., Analyse mathématique et calcul numérique pour les sciences et les techniques, mason Paris, tome 3, 1987.

[57] R. Temsah, Numerical solutions for convection–diffusion equation using El-Gendi method. Communications in Nonlinear Science and Numerical Simulation 14, 760-769 (2009).

[58] S. El-Wakil, M. Abdou, A. Elhanbaly, Adomian decomposition method for solving the diffusion–convection–reaction equations. Applied mathematics and computation 177, 729-736 (2006).

[59] S. Lal, GENERALIZED LEGENDRE WAVELET METHOD AND ITS APPLICATIONS IN APPROXIMATION OF FUNCTIONS OF BOUNDED DERIVATIVES. Palestine Journal of Mathematics 8, (2019).

[60] S. Larsson, V. Thomée., Partial differential equations with numerical methods. Texts in applied mathematics, (Springer, Berlin ; New York, 2003), pp. ix, 259 p.

[61] S. A, Orszag., Numerical simulation of incompressible flows within simple boundaries in Galerkin (spectral) representations, Studies in Appl. Math., 50, p. 293–327, 1971.

[62] S. A, Orszag., Spectral methods for problems in complex geometries , J. Comput. Phys. 37 (1980), 70-92.

[63] S. Chandrasekhar, S. Chandrasekhar, Selected Papers, Volume 3: Stochastic, Statistical, and Hydromagnetic Problems in Physics and Astronomy. (University of Chicago Press, 1989), vol. 3.

[64] T. N. Phillips, A.R. Davies., On semi-infinite spectral elements for Poisson problems with re-entrant boundary singularities, Journal of Computationl and Apllied Mathematics 21,(1988), p173-188.

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