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On Rough Near Algebras

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Abstract This article introduces the idea of a new algebraic structure rough near algebra and provides few important examples. Fundamental features related to this approach are also explored. We also present the topics rough sub near algebra, rough ideals and examine their basic properties. Rough near algebra homomorphism is defined and proved some of the significant results using the concepts of rough sub near algebra and rough ideals.

1 Introduction

The issue of insufficient information has long been studied by mathematicians and researchers. It has recently risen in prominence for computer scientists as well, specifically in the area of artificial intelligence. The question of how to think and act with insufficient information can be approached from numerous perspectives. Without a doubt, fuzzy set theory of Zadeh [26] is the most productive one. Another novel mathematical approach to deal with uncertain and imprecise information system is the theory of rough sets and was introduced by Pawlak [19, 20]. Several scholars and professionals all over the world have been attracted to the rough set theory and making significant contributions to its progress and applications. Interesting uses of rough set theory were explored. Specifically in the fields of machine learning, knowledge acquisition, decision analysis, information discovery from databases, expert systems, inductive reasoning and pattern recognition, the rough set approach seem to be crucially essential to artificial intelligence and cognitive sciences. Rough set theory main benefit in data analysis is that it doesn't need any extra or preliminary data information. A pair comprising a universal set and an equivalence relation where the equivalence is an indiscernibility relation indicating imprecise knowledge of the universal set forms the basis of rough set theory. Upper and lower approximation sets can be considered as subsets of the universal set. Researchers worked to integrate algebraic structures and rough set theory. Biswas and Nanda [4] introduced the notion of rough groups and rough subgroups. Miao [17] improved the definitions of rough group and rough subgroup and proved their new properties. Davvaz [7, 8] established a connection between ring theory, module theory and rough sets. In addition to providing properties of the lower and upper approximations in a ring, he proposed the idea of a rough sub ring with respect to an ideal of a ring, which is an extended notion of a sub ring in a ring. Agusfrianto and Mahatma [1] gave the new definition of rough ring and rough sub ring along with examples and proved few related theorems. Almohammadi and Ozel [3] introduced the concept of rough vector spaces and some algebraic results on it are proved also introduced topological rough vector spaces by uniting the notions of rough vector space with topological space. Zhang [25] introduced the concepts rough module, rough sub module, rough quotient module in approximation spaces and gave some of their properties. The algebraic structure rough near ring has been presented by Marynirmala and Sivakumar [16] and several features pertaining to ideals were examined. Hoskova-Mayerova and Al-Tahan [11] introduced the notions of anti-fuzzy multi-sub near rings (multi-ideals) of near rings and studied their properties. Brown [6] proposed the idea of near algebra. An algebraic system with two binary operations that admits field as a right operator domain and satisfies all of the ring's axioms, with a possible exclusion of one distributive property, is referred to as a near algebra. He analysed a few types of near algebras and their structures. One conceivable application to physics is the study of near algebras; models of quantum mechanics have been developed where the operators constitute just a near algebra. For some related study, see [2, 5, 9, 10, 12, 13, 14, 18, 21, 23, 24]. The present investigation draws a connection between near algebra and rough set and introduce the notion of a new algebraic structure called rough near algebra. We investigate few basic properties of rough sub near algebra and rough ideal. Also define rough homomorphism and few properties related to rough sub near algebra and ideals are examined.

2 Preliminaries

We revisit some of the fundamental definitions of near algebra and rough sets from various sources in this part.

Definition 2.1. [22] A linear space X over a field F is called an *algebra* over the field F if multiplication is defined such that
(i) X forms a semi group under multiplication,
(ii) multiplication is distributive over addition
i.e. $(x+y)z = xz + yz$ and $x(y+z) = xy + xz \quad \forall x, y, z \in X$,
(iii) $t(xy) = (tx)y = x(ty) \ \ \forall t \in F$ and $x, y \in X$.

Definition 2.2. [6] A *(right) near algebra* N over a field F is a linear space N over F on which a multiplication is defined such that
(i) N forms a semi group under multiplication,
(ii) multiplication is right distributive over addition
i.e. $(x+y)z = xz + yz \quad \forall x, y, z \in N$,
(iii) $t(xy) = (tx)y \ \ \forall t \in F, x, y \in N$.

Definition 2.3. [19] A pair (U, R) of a non-empty set U and an equivalence relation R is said to be an *approximation space*.

Definition 2.4. [19] Let (U, R) be an approximation space and N be a subset of U. The sets
$\overline{N} = \overline{App}(N) = \{x \in U | [x]_R \cap N \neq \emptyset\}$ and $\underline{N} = \underline{App}(N) = \{x \in U | [x]_R \subseteq N\}$
are called *upper and lower approximations* respectively.
Here $[x]_R$ denotes the equivalence class containing x in R.

Definition 2.5. [19] The *boundary* of a set N is defined as $Bn(N) = \overline{N} \setminus \underline{N}$.

Remark 2.6. [19] Let U be an approximation space and $M, N \subseteq U$. We have
(i) $\underline{N} \subseteq N \subseteq \overline{N}$,
(ii) $\underline{U} = U = \overline{U}$,
(iii) $\underline{\emptyset} = \emptyset = \overline{\emptyset}$,
(iv) $\overline{M \cup N} = \overline{M} \cup \overline{N}$,
(v) $\underline{M \cap N} = \underline{M} \cap \underline{N}$,
(vi) $\overline{M \cap N} \subseteq \overline{M} \cap \overline{N}$,
(vii) $\underline{M \cup N} \subseteq \underline{M} \cup \underline{N}$,
(viii) $\overline{M} \subseteq \overline{N}$ and $\underline{M} \subseteq \underline{N}$ whenever $M \subseteq N$.

Definition 2.7. [19] Let (U, R) be an approximation space and N be a subset of U. Then N is called a *rough set* in (U, R) if and only if $Bn(N) \neq \emptyset$.

Definition 2.8. [17] Let (U, R) be an approximation space and "multiplication" be a binary operation defined on U. A subset G of U is called a *rough group* if the following four properties are satisfied:
(i) $xy \in \overline{G}$ for all $x, y \in G$,
(ii) $x(y+z) = xy + xz$ for all $x, y, z \in G$,
(iii) For every $x \in G$ there exists $e \in \overline{G}$ such that $xe = ex = x$ and
(iv) For every $x \in G$, there exists $y \in G$ such that $xy = yx = e$.
Here e is called as *rough identity* and y is known as *rough inverse* of x and denoted with x^{-1}.

Remark 2.9. In the above definition, if G satisfies conditions (i) and (ii), then G is known as *rough semi group*.

Definition 2.10. [17] If G is a rough group and $H \subseteq G$, then H is said to be a *rough sub group* of G if it is itself a rough group induced with operation of G.

Theorem 2.11. [17] *A subset H of a rough group G is a rough sub group of G if and only if*
(i) $xy \in \overline{H}$ for all $x, y \in H$,
(ii) $x^{-1} \in H$ for all $x \in H$.

Definition 2.12. [3] Let (U, R) be an approximation space. A subset V of U is called a *rough vector space* over a field F, if V is a rough group under the binary operation "$+$" and the commutative property with respect to "$+$" holds in $\overline{V}$. Moreover, for all $t \in F$ and $x \in V$ there is an element $tx \in \overline{V}$ such that the following axioms are satisfied for all $x, y \in V$ and all $a, b \in F$,
(i) $a(x + y) = ax + ay$,
(ii) $(a + b)x = ax + bx$,
(iii) $a(bx) = (ab)x$,
(iv) $1x = x$ where $1 \in F$.

Definition 2.13. [3] Let (U, R) be an approximation space and $V \subseteq U$ be a rough vector space over the field F. A non-empty subset S of V is called its *rough subspace* over F whenever it is a rough vector space with respect to the same operations those defined on V.

Remark 2.14. [3] A subset S of a rough vector space V over a field F is a rough subspace if and only if S is a rough subgroup and $tx \in \overline{S}$ for all $t \in F, x \in S$.

3 Rough Near algebras

In this section we introduce the notions of rough near algebra, rough sub near algebra and investigate their characteristics.

Definition 3.1. Let (U, R) be an approximation space and $N \subseteq U$. Rough vector space N over a field F is called *rough algebra* over the field F if multiplication is defined such that
(i) N is a rough semi group under multiplication,
(ii) Multiplication is distributive over addition in $\overline{N}$
i.e., $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$ for all $x, y, z \in N$,
(iii) $t(xy) = (tx)y = x(ty)$ for all $t \in F, x, y \in N$.

Example 3.2. Let $U = \mathbb{R}$ be a set of real numbers with the operation usual addition and the classification of U is C_1, C_2 where C_1 is the set of non-negative real numbers and $C_2 = \mathbb{R} \setminus C_1$. Let $N = \mathbb{R} \setminus \{0\}$. Then $\overline{N} = \mathbb{R}$. Consider the field $F = \mathbb{R}$. For $x, y \in N$, define ordinary addition $(+)$. Here $0 \in \overline{N}$ is considered as the rough identity. For all $t \in F$ and $x \in N$ define scalar multiplication tx as the usual multiplication of real numbers. Direct verification shows that N is a rough vector space over F.

For $x, y \in N$ define multiplication $(*)$ as $x*y = \frac{xy}{2}$. Let $x, y, z \in N$. Then (i) $x*y = \frac{xy}{2} \in \overline{N}$ and (ii) $(x * y) * z = \frac{xy}{2} * z = \frac{\frac{xy}{2}z}{2} = \frac{(xy)z}{4} = \frac{x\frac{yz}{2}}{2} = x * \frac{yz}{2} = x * (y * z)$. Therefore $(N, *)$ is a rough semi group. Now $x * (y + z) = \frac{x(y+z)}{2} = \frac{xy+xz}{2} = \frac{xy}{2} + \frac{xz}{2} = (x * y) + (x * z)$. Similarly, $(x + y) * z = (x * z) + (y * z)$. Hence multiplication is distributive over addition. Also $(tx) * y = \frac{(tx)y}{2} = \frac{t(xy)}{2} = t(\frac{xy}{2}) = t(x * y)$ and $t(x * y) = x * (ty)$. Therefore, N is a rough algebra over $\mathbb{R}$.

Definition 3.3. Let (U, R) be an approximation space and $N \subseteq U$. A rough vector space N over a field F is called a *rough (right) near algebra* over the field F if defined such that
(i) N is a rough semi group under multiplication,
(ii) Multiplication is right distributive over addition in $\overline{N}$
i.e., $(x + y)z = xz + yz$ for all $x, y, z \in N$,
(iii) $t(xy) = (tx)y$ for all $t \in F$ and $x, y \in N$.

Example 3.4. Let $X = U = \{0, x, y, z\}$ be a set with two binary operations "$+$" and "." which are given below:

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$.$	0	x	y	z
0	0	0	0	0
x	0	y	0	y
y	0	0	0	0
z	0	y	0	y

Consider the field $\mathbb{Z}_2$. Let a scalar multiplication on X be defined by $0x = 0, 1x = x$ for each $x \in X; 0, 1 \in \mathbb{Z}_2$. Define a relation R on X as "xRy iff x and y are either zero's or non-zero's". It is easy to verify that, R is reflexive, symmetric and transitive. Hence R is an equivalence relation, and the classification of X is $\{C_1, C_2\}$, where $C_1 = \{0\}$ and $C_2 = \{x, y, z\}$. Let $N = \{0, x, y\}$. Then $\overline{N} = X$. Here $0 \in \overline{N}$ is considered as the rough identity and $-0 = 0, -x = x, -y = y$. Remaining conditions of rough (right) near algebra can be verified easily. Hence N is a rough (right) near algebra over the field $\mathbb{Z}_2$.

Example 3.5. Let $X = \mathbb{Z}_5 \times \mathbb{Z}_5 = \{(x, y)|x, y \in \mathbb{Z}_5\}$ be a set with addition modulo 5. Consider the field $\mathbb{Z}_5$. Let a scalar multiplication on X be defined by $t(x, y) = (tx, ty)$ for $t \in \mathbb{Z}_5$. Define the multiplication on X as $(x, y)(x', y') = (x, y)$.

Let $N = \{(3, 1), (2, 4), (2, 1), (3, 4), (2, 2), (3, 3), (1, 0), (4, 0)\} \subseteq X$. The classification of X is $\{C_1, C_2\}$, where C_1 is the collection of all ordered pairs with at least one zero component and $C_2 = X \setminus C_1$. Then $\overline{N} = X$. Here $(0, 0) \in \overline{N}$ is considered as rough identity and $(3, 1)^{-1} = (2, 4), (1, 0)^{-1} = (4, 0), (2, 1)^{-1} = (3, 4), (2, 2)^{-1} = (3, 3)$. Direct verification shows that N is rough (right) near algebra over $\mathbb{Z}_5$.

Definition 3.6. A rough vector space N over a field F is called a *rough (left) near algebra* over a field F if multiplication is defined such that
(i) N is a rough semi group under multiplication,
(ii) Multiplication is left distributive over addition in $\overline{N}$
i.e., $x(y + z) = xy + xz$ for all $x, y, z \in N$,
(iii) $t(xy) = x(ty)$ for all $t \in F$ and $x, y \in N$.

Example 3.7. Consider the example 3.5. Define $(x, y)(x', y') = (x', y')$ instead of $(x, y)(x', y') = (x, y)$. Then N is a rough (left) near algebra over the field $\mathbb{Z}_5$.

Throughout this paper we confine ourselves to rough (right) near algebra and simply call it as a rough near algebra.

Theorem 3.8. *Let N, N' be two rough near algebras over a field F. Then the direct product $N \times N' = \{(x, x')|x \in N, x' \in N'\}$ is a rough near algebra over the field F with the sufficient condition $\overline{N} \times \overline{N'} \subseteq \overline{N \times N'}$, where addition, scalar multiplication and vector multiplication are given by for any $(x, x'), (y, y') \in N \times N'$ and $t \in F$, $(x, x') + (y, y') = (x + y, x' + y'), t(x, x') = (tx, tx'), (x, x')(y, y') = (xy, x'y')$.*

Proof. Let $(x, x'), (y, y'), (z, z') \in N \times N'$ and $t, k \in F$.
(i) $(x, x') + (y, y') = (x + x', y + y') \in \overline{N} \times \overline{N'} \subseteq \overline{N \times N'}$.
(ii)

$$\begin{aligned} ((x, x') + (y, y')) + (z, z') &= (x + y, x' + y') + (z, z') \\ &= ((x + y) + z, (x' + y') + z') \\ &= (x + (y + z), x' + (y' + z')) \\ &= (x, x') + (y + z, y' + z') \\ &= (x, x') + ((y, y') + (z, z')). \end{aligned}$$

(iii) Let $(x, x') \in N \times N'$. Then $x \in N, x' \in N'$. We can find $e \in \overline{N}, e' \in \overline{N'}$ such that $x + e = x, x' + e' = x'$. Now $(x, x') + (e, e') = (x + e, x' + e') = (x, x')$. Hence $(e, e') \in \overline{N} \times \overline{N'} \subseteq \overline{N \times N'}$ is the rough identity.

(iv) Let $(x, x') \in N \times N'$. Then $x \in N, x' \in N'$. We can find $y \in N, y' \in N'$ such that $x + y = e, x' + y' = e'$. Now $(x, x') + (y, y') = (x + y, x' + y') = (e, e')$. Hence (y, y') is the rough inverse of (x, x').

(v) $(x, x') + (y, y') = (x + y, x' + y') = (y + x, y' + x') = (y, y') + (x, x')$.

(vi) By the definition of scalar multiplication, we have $t(x,x') = (tx,tx') \in \overline{N} \times \overline{N'} \subseteq \overline{N \times N'}$ for any $t \in F, (x,x') \in N \times N'$.

(vii)

$$\begin{aligned} t((x,x')+(y,y')) &= t(x+y,x'+y') \\ &= (t(x+y),t(x'+y')) \\ &= (tx+ty,tx'+ty') \\ &= (tx,tx')+(ty,ty') \\ &= t(x,x')+t(y,y'). \end{aligned}$$

(viii)

$$\begin{aligned} (t+k)(x,x') &= ((t+k)x,(t+k)x') \\ &= (tx+kx,tx'+kx') \\ &= (tx,tx')+(kx,kx') \\ &= t(x,x')+k(x,x'). \end{aligned}$$

(ix) $t(k(x,x')) = t(kx,kx') = (t(kx),t(kx')) = ((tk)x,(tk)x') = tk(x,x')$.

(x) For $1 \in F, 1(x,x') = (1x,1x') = (x,x')$.
Thus $N \times N'$ is a rough vector space over the field F.
Now $(x,x')(y,y') = (xy,x'y') \in \overline{N} \times \overline{N'} \subseteq \overline{N \times N'}$. Also,

$$\begin{aligned} (x,x')((y,y')(z,z')) &= (x,x')(yz,y'z') \\ &= (x(yz),x'(y'z')) \\ &= ((xy)z,(x'y')z') \\ &= (xy,x'y')(z,z') \\ &= ((x,x')(y,y'))(z,z'). \end{aligned}$$

Therefore, $N \times N'$ is a rough semi group. Now

$$\begin{aligned} ((x,x')+(y,y'))(z,z') &= (x+y,x'+y')(z,z') \\ &= ((x+y)z,(x'+y')z') \\ &= (xz+yz,x'z'+y'z') \\ &= (xz,x'z')+(yz,y'z') \\ &= (x,x')(z,z')+(y,z)(y',z'). \end{aligned}$$

Thus multiplication is right distributive over addition. Also

$$\begin{aligned} (t(x,x'))(y,y') &= (tx,tx')(y,y') \\ &= ((tx)y,(tx')y') \\ &= (t(xy),t(x'y')) \\ &= t(xy,x'y') \\ &= t((x,x')(y,y')) \end{aligned}$$

Hence $N \times N'$ is a rough near algebra over the field F. □

Definition 3.9. Let N be a rough near algebra over a field F. For $p,q,r \in N$, the element $p(q+r)-pq-pr$ is called the *distributor* of q and r with respect to p and is denoted by $[p,q,r]$.

Theorem 3.10. *Let N be a rough near algebra over a field F and $p,q,r,s \in N, t \in F$. Then we have the following distributor identities:*
(i) $[p,q,r] = [p,r,q]$,
(ii) $[p+s,q,r] = [p,q,r]+[s,q,r]$,
(iii) $t[p,q,r] = [tp,q,r]$,
(iv) $[p,q,r]s = [p,qs,rs]$.

Proof. Let $p, q, r, s \in N$ and $t \in F$. Then
(i) $[p, q, r] = p(q + r) - pq - pr = p(r + q) - pr - pq = [p, r, q]$,
(ii)

$$\begin{aligned}[p + s, q, r] &= (p + s)(q + r) - (p + s)q - (p + s)r \\ &= p(q + r) + s(q + r) - pq - sq - pr - sr \\ &= p(q + r) - pq - pr + s(q + r) - sq - sr \\ &= [p, q, r] + [s, q, r],\end{aligned}$$

(iii)

$$\begin{aligned}[tp, q, r] &= (tp)(q + r) - (tp)q - (tp)r \\ &= t(p(q + r)) - t(pq) - t(pr) \\ &= t(p(q + r) - pq - pr) \\ &= t[p, q, r],\end{aligned}$$

(iv)

$$\begin{aligned}[p, qs, rs] &= p(qs + rs) - p(qs) - p(rs) \\ &= p(q + r)s - (pq)s - (pr)s \\ &= (p(q + r) - pq - pr)s \\ &= [p, q, r]s.\end{aligned}$$

□

Definition 3.11. Let (U, R) be an approximation space and $N \subseteq U$ be a rough near algebra over a field F. A non-empty subset S of N is called its rough sub near algebra of N over the same field F, whenever S itself a rough near algebra with respect to the same operations those defined on N.

Theorem 3.12. *Let S be a non-empty subset of a rough near algebra N over a field F. Then S is a rough sub near algebra of N over the field F if and only if*
(i) $x - y \in \overline{S}$,
(ii) $tx \in \overline{S}$,
(iii) $xy \in \overline{S}$ for any $x, y \in S, t \in F$.

Proof. Suppose that S is a rough sub near algebra of N. Then $x - y, tx, xy \in \overline{S}$ for any $x, y \in S, t \in F$. On the contrary, based on the assumptions $(i), (ii)$ and (iii), it is clear that, S is a rough subspace and S is a rough semi group. Let $x, y, z \in S$. Then $x, y, z \in N$. Also $(x+y)z = xz+yz$ for all $x, y, z \in N$ and $t(xy) = (tx)y$ for $t \in F, x, y \in N$. Hence S is a rough sub near algebra of N. □

Example 3.13. Let $X = \mathbb{Z}_5 \times \mathbb{Z}_5 = \{(x, y)|x, y \in \mathbb{Z}_5\}$ and $N = \{(3, 1), (2, 4), (2, 1), (3, 4), (2, 2), (3, 3), (1, 0), (4, 0), (3, 0), (2, 0)\} \subseteq X$. The classification of X is $\{C_1, C_2\}$, where C_1 is the collection of all ordered pairs with at least one zero component and $C_2 = X \setminus C_1$. Then $\overline{N} = X$. Direct verification shows that, N is a rough near algebra over $\mathbb{Z}_5$ (see example 3.5).
Let $S_1 = \{(2, 2), (3, 3), (1, 0), (4, 0)\}$ and $S_2 = \{(2, 2), (3, 3), (2, 0), (3, 0)\}$. Then S_1, S_2 are two rough sub near algebras of N. Now $S_1 \cap S_2 = \{(2, 2), (3, 3)\}$ and $\overline{S_1 \cap S_2} = C_2$ has no rough identity. Hence $S_1 \cap S_2$ is not a rough sub near algebra.

We present the condition that a rough sub near algebra is formed by the intersection of two rough sub near algebras in the subsequent theorem.

Theorem 3.14. *Let S_1 and S_2 be rough sub near algebras of a rough near algebra N over a field F. Then $S_1 \cap S_2$ is a rough sub near algebra of N over a field F when $\overline{S_1} \cap \overline{S_2} \subseteq \overline{S_1 \cap S_2}$.*

Proof. Let $x, y \in S_1 \cap S_2$ and $t \in F$. Then $x, y \in S_1$ and $x, y \in S_2$. Thus $x - y \in \overline{S_1}$, $tx \in \overline{S_1}$, $xy \in \overline{S_1}$ and $x - y \in \overline{S_2}$, $tx \in \overline{S_2}$, $xy \in \overline{S_2}$. Therefore, $x - y \in \overline{S_1} \cap \overline{S_2}$, $tx \in \overline{S_1} \cap \overline{S_2}$, $xy \in \overline{S_1} \cap \overline{S_2}$. Since $\overline{S_1} \cap \overline{S_2} \subseteq \overline{S_1 \cap S_2}$, $x - y \in \overline{S_1 \cap S_2}$, $tx \in \overline{S_1 \cap S_2}$, $xy \in \overline{S_1 \cap S_2}$. Hence $S_1 \cap S_2$ is a rough sub near algebra of N. □

4 Rough Ideals

In this section we define and explore the concepts of rough ideals and rough near algebra homomorphism, examining their properties and characteristics.

Definition 4.1. Let (U, R) be an approximation space and $N \subseteq U$ be a rough near algebra over a field F. A non-empty subset I of N is called a *rough ideal* of N if it satisfies the following conditions:
(i) I is a rough subspace of the rough vector space N,
(ii) $ix \in \overline{I}$ for any $i \in I$ and $x \in N$,
(iii) $y(x+i) - yx \in \overline{I}$ for any $i \in I$ and $x, y \in N$.
If I satisfies (i) and (ii), then I is called a *rough right ideal* of N.
If I satisfies (i) and (iii), then I is called a *rough left ideal* of N.

Theorem 4.2. *Let I_1 and I_2 be rough ideals of a near algebra N over a field F. Then $I_1 \cap I_2$ is also a rough ideal of N over the field F whenever $\overline{I_1} \cap \overline{I_2} \subseteq \overline{I_1 \cap I_2}$.*

Proof. Since I_1, I_2 are two rough ideals of a near algebra N over a field F, then I_1, I_2 are two rough subspaces of the rough vector space N. Hence $I_1 \cap I_2$ is also a rough subspace of the rough vector space N. Let $i \in I_1 \cap I_2$ and $x, y \in N$. Then $i \in I_1, i \in I_2$, $ix \in \overline{I_1}$, $ix \in \overline{I_2}$. Therefore, $ix \in \overline{I_1} \cap \overline{I_2}$. Which implies $ix \in \overline{I_1 \cap I_2}$. Hence $I_1 \cap I_2$ is a rough right ideal of N. Now $y(x+i) - yx \in \overline{I_1}$ and $y(x+i) - yx \in \overline{I_2}$. Therefore, $y(x+i) - yx \in \overline{I_1} \cap \overline{I_2}$ which implies $y(x+i) - yx \in \overline{I_1 \cap I_2}$. Hence $I_1 \cap I_2$ is a rough left ideal of N. Therefore, $I_1 \cap I_2$ is a rough ideal of N. □

Theorem 4.3. *Let N be a rough near algebra over a field F. If I_1, I_2 are two rough right ideals of N over a field F, then $I_1 + I_2$ is also a rough right ideal of N over the field F whenever $\overline{I_1} + \overline{I_2} \subseteq \overline{I_1 + I_2}$.*

Proof. Let $i, j \in I_1 + I_2$. We know that $I_1 + I_2 = \{i_1 + i_2 | i_1 \in I_1, i_2 \in I_2\}$. Then $i = i_1 + i_2$ and $j = j_1 + j_2$ for some $i_1, j_1 \in I_1, i_2, j_2 \in I_2$. For any $t, k \in F, ti - kj = t(i_1 + i_2) - k(j_1 + j_2) = (ti_1 + ti_2) - (kj_1 + kj_2) = (ti_1 - kj_1) + (ti_2 - kj_2) \in \overline{I_1} + \overline{I_2} \subseteq \overline{I_1 + I_2}$. Hence $I_1 + I_2$ is a rough subspace of N. Let $x \in N$. Then $ix = (i_1 + i_2)x = (i_1x + i_2x) \in \overline{I_1} + \overline{I_2} \subseteq \overline{I_1 + I_2}$. Hence $I_1 + I_2$ is a rough right ideal of N. □

Definition 4.4. Let N, N' be two rough near algebras on an approximation space U over a field F. A mapping $f : \overline{N} \to \overline{N'}$ satisfying
(i) $f(x+y) = f(x) + f(y)$,
(ii) $f(tx) = tf(x)$,
(iii) $f(xy) = f(x)f(y)$ for all $x, y \in \overline{N}, t \in F$ is called as a *rough near algebra homomorphism* from N to N'.

A rough near algebra homomorphism $f : \overline{N} \to \overline{N'}$ is called a *rough near algebra epimorphism* if the mapping is onto. A rough near algebra homomorphism $f : \overline{N} \to \overline{N'}$ is called a *rough near algebra monomorphism* if the mapping is one to one. A rough near algebra homomorphism $f : \overline{N} \to \overline{N'}$ is called a *rough near algebra isomorphism* if the mapping is one-one and onto.

Proposition 4.5. *If $f : \overline{N} \to \overline{N'}$ is a rough near algebra homomorphism then*
(i) $f(0) = 0'$ where 0 and 0' are rough identities of N and N' respectively,
(ii) $f(x - y) = f(x) - f(y)$ for all $x, y \in N$.

Theorem 4.6. *Let $f : \overline{N} \to \overline{N'}$ be a rough near algebra homomorphism. Then the homomorphic image $f(N)$ of N is a rough sub near algebra of N' over a field F whenever $f(\overline{N}) \subseteq \overline{f(N)}$.*

Proof. We have $f(N) = \{f(x)|$ for some $x \in N\}$. Let $f(x), f(y) \in f(N)$ for some $x, y \in N$. Since N is a rough near algebra, $x - y, tx, xy \in \overline{N}$ for $t \in F$. Now,
(i) $f(x) - f(y) = f(x - y) \in f(\overline{N}) \subseteq \overline{f(N)}$,
(ii) $tf(x) = f(tx) \in f(\overline{N}) \subseteq \overline{f(N)}$,
(iii) $f(x)f(y) = f(xy) \in f(\overline{N}) \subseteq \overline{f(N)}$.
Therefore, $f(N)$ is a rough sub near algebra of N'. □

Theorem 4.7. *Let N, N' be two rough near algebras over a field F and $f : \overline{N} \to \overline{N'}$ be a rough near algebra homomorphism. If I is a rough ideal of N, then $f(I)$ is a rough ideal in N' whenever $f(\overline{I}) \subseteq \overline{f(I)}$.*

Proof. Let $f(i), f(j) \in f(I)$ for some $i, j \in I$. Since I is a rough ideal of N, $i - j, ti \in \overline{I}$ for some $t \in F$. Now $f(i) - f(j) = f(i - j) \in f(\overline{I}) \subseteq \overline{f(I)}$ and $tf(i) = f(ti) \in f(\overline{I}) \subseteq \overline{f(I)}$. Hence $f(I)$ is a rough subspace of the rough vector space N'. Since $0 \in \overline{I}, f(0) \in f(\overline{I}) \subseteq \overline{f(I)}$. Let $f(i) \in f(I)$ and $f(x) \in N'$ for $i \in I$ and $x \in N$. Since I is a rough right ideal of $N, ix \in \overline{I}$. Now, $f(i)f(x) = f(ix) \in f(\overline{I}) \subseteq \overline{f(I)}$. Hence $f(I)$ is a rough right ideal of N'. Let $f(x), f(y) \in N'$ and $f(i) \in f(I)$ for some $x, y \in N$ and $i \in I$. Since I is a rough left ideal of N, $y(x + i) - yx \in \overline{I}$. Now $f(y)(f(x) + f(i))) - f(y)f(x) = f(y)f(x + i) - f(yx) = f(y(x+i)) - f(yx) = f(y(x+i) - yx) \in f(\overline{I}) \subseteq \overline{f(I)}$. Therefore, $f(I)$ is a rough left ideal of N' and hence $f(I)$ is a rough ideal in N'. □

Theorem 4.8. *Let $f : \overline{N} \to \overline{N'}$ be a rough near algebra homomorphism. If I is a rough ideal of N' then $f^{-1}(I)$ is a rough ideal of N whenever $f^{-1}(\overline{I}) \subseteq \overline{f^{-1}(I)}$.*

Proof. Let $i, j \in f^{-1}(I)$. Then $f(i), f(j) \in I$ and $f(i - j) = f(i) - f(j) \in \overline{I}$. This implies $i - j \in f^{-1}(\overline{I}) \subseteq \overline{f^{-1}(I)}$. Also $f(ti) = tf(i) \in \overline{I}$ then $ti \in f^{-1}(\overline{I}) \subseteq \overline{f^{-1}(I)}$. Hence $f^{-1}(I)$ is a rough subspace of N. Let $i \in f^{-1}(I), x \in N$. Then $f(i) \in I, f(x) \in N'$. Since I is a rough right ideal of N', $f(i)f(x) \in \overline{I}$. This implies $f(ix) \in \overline{I}$ and hence $ix \in f^{-1}(\overline{I}) \subseteq \overline{f^{-1}(I)}$. Therefore, $f^{-1}(I)$ is a rough right ideal of N'.

Let $i \in f^{-1}(I), x, y \in N$. Then $f(i) \in I, f(x), f(y) \in N'$. Since I is a rough left ideal of N', $f(y)(f(x) + f(i)) - f(y)f(x) \in \overline{I}$. This implies $f(y)(f(x+i)) - f(yx) \in \overline{I}$. This implies $f(y)(f(x+i)) - f(yx) \in \overline{I}$. Thus $f(y(x+i) - yx) \in \overline{I}$. Therefore, $y(x+i) - yx \in f^{-1}(\overline{I}) \subseteq \overline{f^{-1}(I)}$. Hence $f^{-1}(I)$ is a rough left ideal of N'. Therefore, $f^{-1}(I)$ is a rough ideal of N. □

Definition 4.9. Let $f : \overline{N} \to \overline{N'}$ be a rough near algebra homomorphism. Then the set $Kerf = \{x \in N | f(x) = e'\}$,where e' is the rough identity of $\overline{N'}$ is called the *rough kernel* of f.

Definition 4.10. Let N be a rough near algebra on an approximation space U over the field F and I be a rough ideal of N. Then the set $x + \overline{I} = \{x + i | i \in \overline{I}\}$ is called the *rough coset* of I in N containing x, where x is a fixed element in $\overline{N}$.

Theorem 4.11. *Let I be a rough ideal of a rough near algebra N and $x, y \in \overline{N}$ then*
(i) $x \in x + \overline{I}$,
(ii) $x + \overline{I} = \overline{I}$ iff $x \in \overline{I}$,
(iii) $x + \overline{I} = y + \overline{I}$ iff $x \in y + \overline{I}$,
(iv) $x + \overline{I} = y + \overline{I}$ or $(x + \overline{I}) \cap (y + \overline{I}) = \emptyset$,
(v) $x + \overline{I} = y + \overline{I}$ iff $x - y \in \overline{I}$.

Proof. (i) Since $0 \in \overline{I}, x = x + 0 \in x + \overline{I}$.
(ii) Suppose $x + \overline{I} = \overline{I}$. By (i), $x \in x + \overline{I}$. Thus $x \in \overline{I}$. Conversely, assume that $x \in \overline{I}$. Since $\overline{I}$ is a rough near algebra, $x + \overline{I} = \overline{I}$.
(iii) Suppose $x + \overline{I} = y + \overline{I}$. By (i), $x \in x + \overline{I}$. Then $x \in y + \overline{I}$. Conversely, assume that $x \in y + \overline{I}$. Then $x = y + i$ for some $i \in \overline{I}$. Hence $x + \overline{I} = y + i + \overline{I} = y + \overline{I}$
(iv) Assume that $(x + \overline{I}) \cap (y + \overline{I}) \neq \emptyset$. Hence we can find an element $z \in (x + \overline{I}) \cap (y + \overline{I})$. Then $z \in (x + \overline{I})$ and $z \in (y + \overline{I})$. By (iii), $z + \overline{I} = x + \overline{I} = y + \overline{I}$.
(v) $x + \overline{I} = y + \overline{I}$ iff $x - y + \overline{I} = y - y + \overline{I}$ iff $x - y + \overline{I} = 0 + \overline{I}$ iff $x - y \in \overline{I}$. □

5 Conclusion

We introduced the algebraic structure rough near algebra. Also analyzed and investigated the characteristics of rough sub near algebra and rough ideal of a near algebra. Properties pertaining to rough sub near algebra and rough ideals are also explored with the help of rough near algebra homomorphism. Some of the important theorems proved in this article are
(i) Let S_1 and S_2 be two rough sub near algebras of a rough near algebra N over a field F. Then $S_1 \cap S_2$ is a rough sub near algebra of N over a field F whenever $\overline{S_1} \cap \overline{S_2} \subseteq \overline{S_1 \cap S_2}$.

(ii) Let I_1 and I_2 be two rough ideals of a near algebra N over a field F. Then $I_1 \cap I_2$ is also a rough ideal of N over the field F whenever $\overline{I_1} \cap \overline{I_2} \subseteq \overline{I_1 \cap I_2}$.
(iii) Let N be a rough near algebra over a field F. If I_1, I_2 are two rough right ideals of N over a field F, then $I_1 + I_2$ is also a rough right ideal of N over the field F whenever $\overline{I_1} + \overline{I_2} \subseteq \overline{I_1 + I_2}$.
(iv) Let N and N' be two rough near algebras over a field F and $f : \overline{N} \to \overline{N'}$ be a rough near algebra homomorphism. If I is a rough ideal of N, then $f(I)$ is a rough ideal in N' whenever $f(\overline{I}) \subseteq \overline{f(I)}$.
(v) Let $f : \overline{N} \to \overline{N'}$ be a rough near algebra homomorphism. If I is a rough ideal of N' then $f^{-1}(I)$ is a rough ideal of N whenever $f^{-1}(\overline{I}) \subseteq \overline{f^{-1}(I)}$.

Similar to this, rough set theory can be applied to various algebraic structures, and its characteristics can be investigated.

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