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Null Homology and Stable Currents in Bi-warped Product Submanifolds of an Odd-Dimensional Unit Sphere

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Abstract The aim of this paper is to investigate the topological characteristics of compact bi-warped product submanifolds of an odd-dimensional sphere. Furthermore, we aim to establish specific constraints on the warping functions f_1 and f_2, Dirichlet energy functions $E(f_1)$ and $E(f_2)$, as well as the first non-zero eigenvalues λ_1 and λ_2, in order to demonstrate the absence of stable currents for these submanifolds and establish their homology groups as zero.

1 Introduction

In 1969, Bishop and O'Neill [4] pioneered the concept of warped product manifolds with the aim of generating instances of Riemannian manifolds featuring negative or non-positive curvature. Indeed, the warped product $B \times_b F$ of two pseudo-Riemannian manifolds (B, g_B) and $(F, g_{F)})$ is defined by a positive-valued smooth function b on B, yielding the metric tensor $g = g_B \oplus b^2 g_F$. Here, (B, g_B) is referred to as the base manifold, while (F, g_F) represents the fiber, with b acting as the warping function. These warped product manifolds represent a natural extension of Riemannian product manifolds. On the other hand, bi-warped product submanifolds constitute natural extensions of both warped product submanifolds and Riemannian product manifolds, as introduced by Chen and Dillen [8].

The algebraic structure of a manifold is encoded in its homology groups, which serve as crucial topological characteristics. Homology theory has been applied in data analysis, especially in the field of topological data analysis [12]. The concept of integral currents plays a crucial role in offering topological insights by integrating the geometric structure of differentiable manifolds with homology groups employing integral coefficients. Federer and Fleming [10] showed the close connection between submanifold theory and homological theory, revealing that any non-trivial integral homological group $H_p(M, Z)$ is linked via stable currents. Subsequently, in 1970, Lawson and Simon [14] expanded this investigation to submanifolds of spheres, establishing the absence of an integral current under a pinching condition of the second fundamental form. Fu and Xu [11] proved the vanishing theorem for homology groups of compact submanifolds in hyperbolic space with negative constant curvature, and they also derived a topological sphere theorem. Following this, Lui and Zhang [15] demonstrated non-existence theorems for stable integral currents in certain classes of hypersurfaces or higher codimensional submanifolds in Euclidean spaces. In [16], it was proven by the authors that the homology groups were trivial and stable currents did not exist for a contact CR-warped product submanifold in an odd-dimensional unit sphere.

Furthermore, Alkhaldi et al. in [1] demonstrated the vanishing homology and absence of stable integral currents in compact oriented warped product pointwise semi-slant submanifolds of a complex space form, and they derived topological sphere theorems. More recently, Khan et al. [13] investigated the topological characteristics of compact bi-warped product submanifolds, revealing the non-existence of stable integral currents and the triviality of their homology groups

in compact oriented bi-warped product submanifolds in Euclidean space.
Inspired by the aforementioned studies, we show that if the Laplacian and gradient of the warping function of a compact bi-warped product submanifold in a Sasakian space form with constant sectional curvature $c = 1$ satisfy specific extrinsic restrictions, then these submanifolds have no stable integral currents, and their homology groups are trivial. We have also proven similar results for the Dirichlet energy functions $E(f_1)$ and $E(f_2)$ and the first non-zero eigenvalues λ_1 and λ_2.

2 Preliminaries

A $(2n+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be an almost contact metric manifold if it admit a $(1,1)$-tensor field ϕ, a characteristic vector field ξ, a 1-form η and a compatible metric denoted by g satisfying [3].

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all $X, Y \in \Gamma(T\tilde{M})$. Then (ϕ, ξ, η, g) is called an almost contact metric structure of $\tilde{M}$. An almost contact metric manifold is called a Sasakian manifold [3] if and only if

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

$$\tilde{\nabla}_X \xi = -\phi X, \tag{2.4}$$

for every $X, Y \in \Gamma(T\tilde{M})$, and $\tilde{\nabla}$ represents the Riemannian connection relative to g.

A Sasakian manifold $\tilde{M}$ is termed a Sasakian space form $\tilde{M}(c)$ if it possesses a constant ϕ-holomorphic sectional curvature c. The curvature tensor $\tilde{R}$ is expressed as [16]

$$\begin{aligned}\tilde{R}(X, Y, Z, W) =& \frac{c+3}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ &+ g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) \\ &+ g(\phi Y, Z)g(\phi X, W) + g(\phi Z, X)g(\phi Y, W) \\ &- 2g(\phi X, Y)g(\phi Z, W)\}\end{aligned} \tag{2.5}$$

for all $X, Y, Z, W \in \Gamma(T\tilde{M})$.

Suppose N is a submanifold isometrically immersed in a differentiable manifold $\tilde{M}$, with ∇ and $\nabla^{\perp}$ representing the induced Riemannian connections on the tangent bundle TN and the normal bundle $T^{\perp}N$, respectively.
The Gauss and Weingarten formulas are expressed as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.7}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, with h representing the second fundamental form, $\nabla^{\perp}$ denoting the normal connection, and A indicating the shape operator.

The decomposition for the vector fields $X \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$ is as follows:

$$\phi X = TX + FX \tag{2.8}$$

$$\phi V = tV + fV, \tag{2.9}$$

where TX and FX are the tangential and normal components of ϕX, while tV and fV are tangential and normal components of ϕV, respectively.

Suppose R represents the Riemannian curvature tensor of N. Then the Gauss equation for a submanifold N is expressed as:

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \tag{2.10}$$

for all $X, Y, Z, W \in \Gamma(TN)$.

Consider $p \in N$ and $e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}$ is an orthonormal basis of the tangent space $\tilde{M}_{2m+1}$. When restricted to N, the vectors $e_1, ..., e_n$ are tangent to N at p, while $e_{n+1}, ..., e_{2m+1}$ are normal to N.

We denote by h^r_{ij}, where $i, j = 1, ..., n$ and $r = \{e_{n+1}, ..., e_{2m+1}\}$, the coefficients of the second fundamental form h with respect to the local frame field. Then, we have

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.11}$$

At every point $x \in N$, the wirtinger angle $\theta(X)$ between ϕX and $T_x N$ remains invariant regardless of the choice of the non-zero vector $X \in T_x N$. In such cases, the angle θ serves as a function on N, known as the slant function of the submanifold, and the submanifold is termed a pointwise slant submanifold. If the slant function $\theta(X)$ is constant across N, then N is recognized as a slant submanifold. Now, let us consider the following characterization for a submanifold N to be a slant submanifold [6, 7]:

$$T^2 = cos^2\theta(-I + \eta \circ \xi), \tag{2.12}$$

for $0 \leq \theta \leq \frac{\pi}{2}$ and T being an endomorphism defined in (2.8), the subsequent equation is derived from the aforementioned equation:

$$g(TX, TY) = cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{2.13}$$

$$g(FX, FY) = sin^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{2.14}$$

for all $X, Y \in \Gamma(TN)$. Invariant and anti-invariant submanifolds are categorized as slant submanifolds, with slant function $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

Definition 2.1. [8] If we have a Cartesian product $M = M_0 \times M_1 \times M_3 \times ... \times M_k$ of Riemannian manifolds $M_0, M_1, ..., M_k$, with canonical projection maps $\pi_i = M \longrightarrow M_k$ for $i = 0, 1, 2, ..., k$, and positive-valued functions $f_0, f_1, ..., f_k$ are defined such that $f_0, f_1, ..., f_k : M_0 \longrightarrow (0, \infty)$, then we define the Riemannian metric g as:

$$g(X,Y) = g(\pi_{1_*}X, \pi_{1_*}Y) + \sum_{i=1}^{k}(f_i \circ \pi) g(\pi_{i_*}X, \pi_{i_*}Y)$$

where $*$ denote the symbol for tangent map, and for any X, Y tangent to M, then M is termed a multiple warped product manifold. If we select two fibers of a multiple warped product $M_0 \times_{f_1} M_1 \times_{f_2} ... \times_{f_k} M_k$, such that $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2$, then M is defined as a bi-warped product submanifold, which satisfies the following:

$$\nabla_X Y = \sum_{i=1}^{2}(X ln f_i) Y_i,$$

for $X \in \Gamma(TM_0)$, $Y \in \Gamma(T(M_1 \times M_2))$ and Y_i is tangent to M_i, for i=1, 2, where, ∇ denotes the Levi-Civita connection on M.

The following property is directly derived from [9] as follows:

$$\nabla_Z X = \nabla_X Z = \frac{(X f_i)}{f_i} Z, \quad i = 1, 2 \tag{2.15}$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(T(N_\perp \times N_\theta))$.

The gradient $\nabla(lnf)$ of lnf, for all $X \in \Gamma(TN)$ is given by [4]

$$g(\nabla lnf, X) = X(lnf). \tag{2.16}$$

The following relation is provided in the reference cited in [9]

$$R(X,Z)Y = \frac{H^{f_1}(X,Y)}{f}Z, \tag{2.17}$$

$$R(X,U)Y = \frac{H^{f_2}(X,Y)}{f}Z, \tag{2.18}$$

for all $X, Y \in \Gamma(TN_T)$, $Z \in \Gamma(T(N_\perp))$, $U \in \Gamma(TN_\theta)$ and H^{f_1} and H^{f_2} are the Hessian tensor of warping functions f_1 and f_2.

Remark 2.2. A warped product manifold $N^n = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ is trivial if and only if f_1 and f_2 are constants along N_T^{p+1} and $N_\perp^t$, where $(p+1)$, t and s are the dimensions of the invariant submanifold, anti-invariant submanifold and pointwise slant submanifold, respectively.

For the Laplacian $\Delta(lnf_i)$ of the warping function f_i, i=1, 2, we have [5]

$$\begin{aligned}\Delta(lnf_i) &= -\,div(\frac{\nabla f_i}{f_i}) = -g(\nabla\frac{1}{f_i}, \nabla f_i) - \frac{1}{f_i}div(\nabla f_i)\\ &= \|\nabla lnf_i\|^2 + \frac{\Delta f_i}{f_i}.\end{aligned} \tag{2.19}$$

Further simplifying equation (2.19), we obtain

$$\frac{\Delta f_i}{f_i} = \Delta(lnf_i) - \|\nabla lnf_i\|^2. \tag{2.20}$$

3 Homology and stable currents in bi-warped product submanifolds

Now, let us investigate the homology and stable currents on a bi-warped product submanifold of the odd dimensional sphere $S^{2(\frac{p}{2}+q)+1}$ of constant holomorphic sectional curvature $c = 1$.

Let us consider $N^n = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ as a bi-warped product submanifold of a Sasakian manifold $\tilde{M}^{2m+1}$, N_T^{p+1} represents a $(p+1)$-dimensional invariant submanifold tangent to ξ, $N_\perp^t$ denotes a t-dimensional anti-invariant submanifold, and N_θ^s stands for an s-dimensional pointwise slant submanifold and $n = p + t + s + 1$.
The tangent and normal spaces of N are given by the tangent bundles D^T, $D^\perp$ and D^θ of N_T, $N_\perp$ and N_θ, respectively. Thus, we have

$$TN = D^T \oplus D^\perp \oplus D^\theta, \quad T^\perp N = \phi D^\perp \oplus FD^\theta \oplus \mu,$$

where μ denote the invariant subbundle of $T^\perp N$.

To establish our main results, we utilize the information obtained by Lawson and Simons [14]. Here is a summary of their findings:

Lemma 3.1. *[14, 18] For the second fundamental form h and any positive integers p, q with $p + q = n$, if the following inequality*

$$\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}(2\|h(u_\alpha, u_\beta)\|^2 - g(h(u_\alpha, u_\alpha), h(u_\beta, u_\beta))) < pqc \tag{3.1}$$

is satisfied for an n-dimensional compact submanifold N^n in a space form $\tilde{M}(c)$ of constant curvature $c \geq 0$, then there exists no stable p-current in N^n and both $H_p(N^n, Z)$ and $H_q(N^n, Z)$ are zero. Here $H_\alpha(N^n, Z)$ represents the α-th homology group of N^n with integer coefficients, and $\{e_\alpha\}_{1\leq\alpha\leq n}$ forms an orthonormal basis of N^n.

The following lemma from [17] is utilized in the proof of our main results later.

Lemma 3.2. *[17] Let $N^n = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ be a non-trivial bi-warped product submanifold of a Sasakian manifold. Then* *(i)* $g(h(u_1, u_2), \phi u_3) = 0$,

(ii) $g(h(u_2, u_3), \phi u_3) = -\phi u_2(lnf_1)\|u_3\|^2$,

(iii) $g(h(u_2, u_3), Fu_4) = 0$,

for any $u_1, u_2 \in D^T$, $u_3 \in D^\perp$ *and* $u_4 \in D_\theta$.

Lemma 3.3. *[17] Let* $N^n = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ *be a non-trivial bi-warped product submanifold of a Sasakian manifold. Then*

(i) $g(h(u_2, u_4), Fu_4) = -\phi u_2(lnf_2)\|u_4\|^2$,

(ii) $g(h(u_2, u_4), FTu_4) = cos^2\theta u_2(lnf_2)\|u_4\|^2$,

(iii) $g(h(\phi u_2, u_4), Fu_4) = u_2(lnf_2)\|u_4\|^2$,

(iv) $g(h(\phi u_2, u_4), FTu_4) = cos^2\theta \phi u_2(lnf_2)\|u_4\|^2$,

for any $u_1, u_2 \in D^T$, $u_3 \in D^\perp$ *and* $u_4 \in D_\theta$.

Theorem 3.4. *Let* $N^{p+q+1} = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ *be a compact bi-warped product submanifold of* $S^{2(\frac{p}{2}+q)+1}(1)$. *If the following inequality holds*

$$
\begin{aligned}
t\Delta(lnf_1) + s\Delta(lnf_2) >& t(2-t)\|\nabla lnf_1\|^2 + s(2csc^2\theta - s)\|\nabla lnf_2\|^2 \\
&+ q - qg(\nabla lnf_1, \nabla lnf_2), \qquad (3.2)
\end{aligned}
$$

where $\nabla f_1, \nabla f_2$ *and* $\Delta f_1, \Delta f_2$ *denote the gradient and the laplacian of the warped product functions* f_1 *and* f_2*, respectively. Then the* $(p+1)$*-stable currents are absent in* N^{p+q+1}*. In addition,* $H_{p+1}(N^n, Z) = H_q(N^n, Z) = 0$*, where* $H_i(N^n, Z) = 0$ *is the* i*-th homology groups of* N^{p+q+1}*, and* $p+1, t, s$ *are the dimensions of* N_T^{p+1}*,* $N_\perp^t$ *and* N_θ^s*, respectively, with* $q = t + s$.

Proof. Suppose $dim(N_T^{p+1}) = p + 1 = 2\alpha + 1$, $dim(N_\perp^t) = t$, and $dim(N_\theta^s) = s = 2\beta$, where N_T, $N_\perp$ and N_θ^t are the integral manifolds of the distributions D_T, $D_\perp$ and D_θ^t, respectively. Let $\{u_0 = \xi, u_1, u_2, ..., u_\alpha, u_{\alpha+1} = \phi u_1, ..., u_{2\alpha} = \phi u_\alpha\}$, $\{u_{2\alpha+1} = \hat{u}_1, ..., u_{2\alpha+t} = \hat{u}_t\}$ and $\{u_{2\alpha+t+1} = u_1^*, ..., u_{2\alpha+t+\beta} = u_\beta^*, u_{2\alpha+t+\beta+1} = u_{\beta+1}^* = sec\theta Tu_1^*, ..., u_{2\alpha+t+2\beta} = u_{s=2\beta}^* = sec\theta Tu_\beta^*\}$ be the orthonormal frames of TN_T^{p+1}, $TN_\perp^t$ and TN_θ^s, respectively.
The orthonormal basis of the normal subbundle $\phi D^\perp$ and FD_θ are $\{u_{n+1} = \tilde{u}_1 = \phi\tilde{u}_1, ..., u_{n+t} = \tilde{u}_t = \phi\tilde{u}_t\}$ and $\{u_{t+1} = \bar{u}_1 = csc\theta Fu_1^*, ..., u_{t+\beta} = \bar{u}_\beta = csc\theta Fu_1^*, u_{t+\beta+1} = \bar{u}_{\beta+1} = csc\theta sec\theta\ FTu_1^*, ..., u_{t+2\beta} = \bar{u}_{2\beta} = csc\theta sec\theta FTu_\beta^*\}$ respectively. Thus, we have

$$
\begin{aligned}
&\sum_{i=0}^{p}\sum_{j=1}^{n}(2\|h(u_i, u_j)\|^2 - g(h(u_i, u_i), h(u_j, u_j))) \\
&\quad = \sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{t}(h_{ij}^r)^2 + \sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=t+1}^{s}(h_{ij}^r)^2 \\
&\quad + \sum_{i=0}^{p}\sum_{j=p+1}^{t}(\|h(u_i, u_j)\|^2 - g(h(u_i, u_i), h(u_j, u_j))) \\
&\quad + \sum_{i=0}^{p}\sum_{j=t+1}^{s}(\|h(u_i, u_j)\|^2 - g(h(u_i, u_i), h(u_j, u_j))). \qquad (3.3)
\end{aligned}
$$

Thus, utilizing the Gauss equation (2.10) for the unit sphere in odd dimensions, we obtain

$$\begin{aligned}
&\sum_{i=0}^{p}\sum_{j=1}^{n}(2\|h(u_i,u_j)\|^2-g(h(u_i,u_i),h(u_j,u_j)))\\
&\quad=\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{t}(h_{ij}^r)^2+\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=t+1}^{s}(h_{ij}^r)^2\\
&\quad+\sum_{i=0}^{p}\sum_{j=p+1}^{t}g(R(u_i,u_j)u_j,u_i)+\sum_{i=0}^{p}\sum_{j=t+1}^{s}g(R(u_i,u_j)u_j,u_i)\\
&\quad-\sum_{i=0}^{p}\sum_{j=1}^{n}g(\tilde{R}(u_i,u_j)u_j,u_i).
\end{aligned}\tag{3.4}$$

By employing formula (2.5) in (3.4) for an odd-dimensional sphere yields

$$\begin{aligned}
&\sum_{i=0}^{p}\sum_{j=1}^{n}(2\|h(u_i,u_j)\|^2-g(h(u_i,u_i),h(u_j,u_j)))\\
&\quad=\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{t}(h_{ij}^r)^2+\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=t+1}^{s}(h_{ij}^r)^2\\
&\quad+(p+1)q+\sum_{i=0}^{p}\sum_{j=p+1}^{t}g(R(u_i,u_j)u_j,u_i)\\
&\quad+\sum_{i=0}^{p}\sum_{j=t+1}^{s}g(R(u_i,u_j)u_j,u_i).
\end{aligned}\tag{3.5}$$

We now compute the last two terms on the right-hand side of equation (3.5) by employing equations (2.17) and (2.18) as follows:

$$\sum_{i=0}^{p}\sum_{j=p+1}^{t}g(R(u_i,u_j)u_j,u_i)=\frac{t}{f_1}\sum_{i=0}^{p}g(\nabla_{u_i}\nabla f_1,u_i),\tag{3.6}$$

and

$$\sum_{i=0}^{p}\sum_{j=t+1}^{s}g(R(u_i,u_j)u_j,u_i)=\frac{s}{f_2}\sum_{i=0}^{p}g(\nabla_{u_i}\nabla f_2,u_i).\tag{3.7}$$

Combining equations (3.5), (3.6) and (3.7), we derive

$$\begin{aligned}
\sum_{i=0}^{p}\sum_{j=1}^{n}(2\|h(u_i,u_j)\|^2-g(h(u_i,u_i),h(u_j,u_j)))&=(p+1)q+\frac{t}{f_1}\sum_{i=0}^{p}g(\nabla_{u_i}\nabla f_1,u_i)\\
&+\frac{s}{f_2}\sum_{i=0}^{p}g(\nabla_{u_i}\nabla f_2,u_i)+\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{t}(h_{ij}^r)^2\\
&+\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=t+1}^{s}(h_{ij}^r)^2.
\end{aligned}\tag{3.8}$$

We first compute the terms Δf_1 and Δf_2, which correspond to the Laplacian of the warping functions f_1 and f_2, respectively. This is done by using (2.15) as follows:

$$\begin{aligned}\Delta f_1 = -\sum_{k=1}^{n} g(\nabla_{u_k}\nabla f_1, u_k) &= -\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_1, u_i) - \sum_{j=1}^{t} g(\nabla_{\hat{u}_j}\nabla f_1, \hat{u}_j) \\ &\quad - \sum_{r=1}^{s} g(\nabla_{u_r^*}\nabla f_1, u_r^*) \\ &= -\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_1, u_i) - \frac{1}{f_1}\sum_{j=1}^{t} g(\hat{u}_j, \hat{u}_j)\|\nabla f_1\|^2 \\ &\quad - \frac{1}{f_2}\left(\sum_{r=1}^{s} g(u_r^*, u_r^*) - sec^2\theta \sum_{r=1}^{s} g(Tu_r^*, Tu_r^*)\right) g(\nabla f_1, \nabla f_2).\end{aligned} \tag{3.9}$$

Finally, the above equation simplifies to:

$$\Delta f_1 = -\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_1, u_i) - \frac{t}{f_1}\|\nabla f_1\|^2 - \frac{s}{f_2} g(\nabla f_1, \nabla f_2). \tag{3.10}$$

Likewise, we can calculate

$$\Delta f_2 = -\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_2, u_i) - \frac{t}{f_1} g(\nabla f_1, \nabla f_2) - \frac{s}{f_2}\|\nabla f_2\|^2. \tag{3.11}$$

Multiplying equation (3.10) by $\frac{1}{f_1}$, we obtain

$$\frac{\Delta f_1}{f_1} = -\frac{1}{f_1}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_1, u_i) - t\|\nabla ln f_1\|^2 - sg(\nabla ln f_1, \nabla ln f_2). \tag{3.12}$$

Applying equation (2.20) in (3.12), we derive

$$\Delta(ln f_1) - \|\nabla ln f_1\|^2 = -\frac{1}{f_1}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_1, u_i) - t\|\nabla ln f_1\|^2 - sg(\nabla ln f_1, \nabla ln f_2). \tag{3.13}$$

After rearraging the terms, we obtain

$$\frac{1}{f_1}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_1, u_i) = -\Delta(ln f_1) + (1-t)\|\nabla ln f_1\|^2 - sg(\nabla ln f_1, \nabla ln f_2). \tag{3.14}$$

Equation (3.11) can be expressed similarly as

$$\frac{1}{f_2}\sum_{i=0}^{p} g(\nabla_{u_i}\nabla f_2, u_i) = -\Delta(ln f_2) + (1-s)\|\nabla ln f_2\|^2 - tg(\nabla ln f_1, \nabla ln f_2). \tag{3.15}$$

On the other hand

$$\begin{aligned}\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n} (h_{ij}^r)^2 &= \sum_{r=1}^{t}\sum_{i=0}^{p}\sum_{j=1}^{t} g(h(u_i, \hat{u}_j), \tilde{u}_r)^2 + \sum_{r=1}^{2\beta}\sum_{i=0}^{p}\sum_{j=1}^{s} g(h(u_i, u_j^*), \bar{u}_r)^2 \\ &= \sum_{i=0}^{p}\sum_{j,r=1}^{\beta} \{g(h(u_i, u_j^*), csc\theta Fu_r^*)^2 + g(h(u_i, u_j^*), csc\theta sec\theta Fu_r^*)^2\} \\ &\quad + \sum_{i=0}^{\alpha}\sum_{j,r=1}^{\beta} \{g(h(\phi u_i, u_j^*), csc\theta Fu_r^*)^2 + g(h(\phi u_i, u_j^*), csc\theta sec\theta Fu_r^*)^2\} \\ &\quad + \sum_{i=0}^{p}\sum_{j,r=1}^{t} \{g(h(u_i, \hat{u}_j), \phi\hat{u}_r)^2.\end{aligned} \tag{3.16}$$

Considering Lemma (3.2) and Lemma (3.3), we have

$$\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 = (csc^2\theta + cot^2\theta)\sum_{i=0}^{p}\sum_{j,r=1}^{\beta}(\phi u_i ln f_2)^2 g(u_j^*, u_j^*)^2$$
$$+ (csc^2\theta + cot^2\theta)\sum_{i=0}^{p}\sum_{j,r=1}^{\beta}(u_i ln f_2)^2 g(u_j^*, u_j^*)^2$$
$$+ \sum_{i=0}^{p}\sum_{j=1}^{t}(\phi u_i ln f_1)^2 g(\hat{u}_j, \hat{u}_j)^2. \tag{3.17}$$

After some computations, we find

$$\sum_{r=n+1}^{p+2t+2s+1}\sum_{i=0}^{p}\sum_{j=p+1}^{n}(h_{ij}^r)^2 = s(csc^2\theta + cot^2\theta)\|\nabla ln f_2\|^2 + t\|\nabla ln f_1\|^2. \tag{3.18}$$

By substituting equations (3.14), (3.15) and (3.18) into (3.8), we obtain

$$\sum_{i=0}^{p}\sum_{j=1}^{n}(2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))) - pq = -t\Delta(ln f_1) - s\Delta(ln f_2) + t(2-t)\|\nabla ln f_1\|^2$$
$$+ s(2csc^2\theta - s)\|\nabla ln f_2\|^2 + q$$
$$- qg(\nabla ln f_1, \nabla ln f_2). \tag{3.19}$$

We can derive the following inequality, assuming condition (3.2) holds:

$$\sum_{i=0}^{p}\sum_{j=1}^{n}(2\|h(u_i,u_j)\|^2 - g(h(u_i,u_i),h(u_j,u_j))) < pq. \tag{3.20}$$

By applying Lemma (3.1) on the odd-dimensional sphere with constant sectional curvature, specifically when $c = 1$, we arrive at the theorem's ultimate conclusion. □

Remark 3.5. If $N_\theta^s = 0$, the bi-warped product submanifold $N^n = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ transforms into the CR-warped product submanifold $N^n = N_T^{p+1} \times_{f_1} N_\perp^t$. However, if $N_\perp^t = 0$, then the bi-warped product submanifold $N^n = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ becomes the pointwise semi-slant warped product submanifold $N^n = N_T^{p+1} \times_{f_2} N_\theta^s$.

From Theorem (3.4) and Remark (3.5), we derive the following corollaries:

Corollary 3.6. *Consider $N^{p+t+1} = N_T^{p+1} \times_{f_1} N_\perp^t$ as a compact CR-warped product submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$. If the following condition holds:*

$$t\Delta(ln f_1) > t(2-t)\|\nabla ln f_1\|^2 + t, \tag{3.21}$$

then there does not exist integral $(p+1)$-stable currents in N^{p+t+1}. Futhermore, $H_{p+1}(N^{p+t+1}, Z) = H_t(N^{p+t+1}, Z) = 0$, where $H_i(N^{p+t+1}, Z) = 0$ represents the i-th homology groups of N^{p+t+1}, and $p+1, t$ denote the dimensions of N_T^{p+1} and $N_\perp^t$ respectively.

Corollary 3.7. *Consider $N^{p+s+1} = N_T^{p+1} \times_{f_2} N_\theta^s$ as a compact pointwise semi-slant warped product submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$. If the following condition holds,*

$$s\Delta(ln f_2) > s(2csc^2\theta - s)\|\nabla ln f_2\|^2 + s, \tag{3.22}$$

then there does not exist integral $(p+1)$-stable currents in N^{p+s+1}. Futhermore, $H_{p+1}(N^{p+s+1}, Z) = H_s(N^{p+s+1}, Z) = 0$, where $H_i(N^{p+s+1}, Z) = 0$ represents the i-th homology groups of N^{p+t+1}, and $p+1, s$ denote the dimensions of N_T^{p+1} and N_θ^s respectively.

In [5], Calin and Chang investigated geometric mechanics on Riemannian manifolds, where they introduced a positive differentiable function f(denoted as $f \in F(M^n)$) on a compact Riemannian manifold M^n. The Dirichlet energy of a function f and a Lagrangian are defined as follows in [6]:

$$E(f) = \frac{1}{2}\int_{M^n} \|\nabla f\|^2 dV, \qquad 0 < E(f) < \infty. \tag{3.23}$$

Utilizing the Dirichlet energy formula (3.23) for a compact manifold without boundary, in conjunction with Theorem (3.4), we derive the following result.

Theorem 3.8. *Suppose $N^{p+q+1} = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ is a compact oriented bi-warped product submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$ without boundary. The following condition holds:*

$$2t(t-2)E(f_1) + 2s(s - 2csc^2\theta)E(f_2) > q\int_{N^n} (1 - g(\nabla lnf_1, \nabla lnf_2))dV. \tag{3.24}$$

where, $E(f_1)$ and $E(f_2)$ represent the Dirichlet energies of the warping functions f_1 and f_2 with respect to the volume element dV. Consequently, there are no stable $(p+1)$-currents in N^{p+q+1} and $H_{p+1}(N^{p+q+1}, Z) = H_q(N^{p+q+1}, Z) = 0$, with $q = t + s$.

Proof. By integrating along the volume element dV in equation (3.2), we obtain

$$\int_{N^n} t\Delta(lnf_1)dV + \int_{N^n} s\Delta(lnf_2)dV > \int_{N^n} t(2-t)\|\nabla lnf_1\|^2 dV + \int_{N^n} s(2csc^2\theta - s)\|\nabla lnf_2\|^2 dV$$
$$+ q\int_{N^n} (1 - g(\nabla lnf_1, \nabla lnf_2))\, dV. \tag{3.25}$$

On the other hand, for a compact oriented Riemannian manifold without boundary, we have $\int_{N^n} \Delta f = 0$ in [19]. Using this fact in the inequality (3.25), we get

$$\int_{N^n} t(t-2)\|\nabla lnf_1\|^2 dV + \int_{N^n} s(s - 2csc^2\theta)\|\nabla lnf_2\|^2 dV >$$
$$q\int_{N^n} (1 - g(\nabla lnf_1, \nabla lnf_2))dV. \tag{3.26}$$

By applying the Dirichlet energy formula (3.23) in (3.26), we find

$$2t(t-2)E(f_1) + 2s(s - 2csc^2\theta)E(f_2) >$$
$$q\int_{N^n} (1 - g(\nabla lnf_1, \nabla lnf_2))dV. \tag{3.27}$$

This concludes the proof of the theorem. □

The following corollaries are derived from the aforementioned theorem.

Corollary 3.9. *Suppose $N^{p+t+1} = N_T^{p+1} \times_{f_1} N_\perp^t$ is a compact oriented CR-warped product submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$ without boundary. If the following condition holds:*

$$E(f_1) > \int_{N^n} \frac{1}{2(t-2)} dV, \tag{3.28}$$

then there are no stable $(p+1)$-currents in N^{p+t+1} and $H_{p+1}(N^{p+t+1}, Z) = H_t(N^{p+t+1}, Z) = 0$, where, $E(f_1)$ represent the Dirichlet energy of the warping function f_1 with respect to the volume element dV.

Corollary 3.10. *Suppose $N^{p+s+1} = N_T^{p+1} \times_{f_2} N_\theta^s$ is a compact oriented pointwise semi-slant warped product submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$ without boundary. If the following condition holds:*

$$E(f_2) > \int_{N^n} \frac{1}{2(s - 2csc^2\theta)} dV, \tag{3.29}$$

then there are no stable $(p+1)$-currents in N^{p+s+1} and $H_{p+1}(N^{p+s+1}, Z) = H_s(N^{p+s+1}, Z) = 0$, where, $E(f_2)$ represent the Dirichlet energy of the warping function f_2 with respect to the volume element dV.

The Laplace eigenvalue equation is defined such that a real number λ is called an eigenvalue if there exists a non-vanishing function f_1, that satisfies the following equation:

$$\Delta f_1 = \lambda f_1, \quad on \quad N^n,$$

with appropriate boundary conditions. Considering a Riemannian manifold N^n without a boundary, the first nonzero eigenvalue of Δ, denoted by λ_1, is defined in [2].
Building upon the characterization established in [2] and utilizing both the first non-zero eigenvalue of the Laplace operator and the maximum principle for the first non-zero eigenvalue λ_1, we derive the following:

Theorem 3.11. *Let $N^{p+q+1} = N_T^{p+1} \times_{f_1} N_\perp^t \times_{f_2} N_\theta^s$ be a compact oriented bi-warped product submanifold of $S^{2(\frac{p}{2}+q)+1}(1)$, where f_1 and f_2 denote non-constant eigenfunctions associated with the first non-zero eigenvalues λ_1 and λ_2, respectively. If the following inequality holds:*

$$\int_{N^n} \left\{t(t-2)\lambda_1 f_1^2 + s(s-2csc^2\theta)\lambda_2 f_2^2 - q(1-g(\nabla ln f_1, \nabla ln f_2))\right\} dV > 0, \tag{3.30}$$

then there are no stable $(p+1)$-currents in N^{p+q+1} and $H_{p+1}(N^{p+q+1}, Z) = H_q(N^{p+q+1}, Z) =$ 0, with $q = t + s$.

Proof. Applying the minimum principle to the first eigenvalues λ_1 and λ_2, the results as derived in [2] can be obtained. Here, we make the assumption that f_1 and f_2 are non-constant warping functions. Then

$$\lambda_1 \int_{N^n} f_1^2 dV \leq \|\nabla f_1\|^2 dV \tag{3.31}$$

and

$$\lambda_2 \int_{N^n} f_2^2 dV \leq \|\nabla f_2\|^2 dV. \tag{3.32}$$

The equality in (3.31) and (3.32) valid if and only if $\Delta f_1 = \lambda_1 f_1$ and $\Delta f_2 = \lambda_2 f_2$. Integrating equation (3.26) and using Green's lemma, we obtain:

$$\int_{N^n} t(t-2)\|\nabla ln f_1\|^2 dV + \int_{N^n} s(s-2csc^2\theta)\|\nabla ln f_2\|^2 dV > \\ q\int_{N^n} (1-g(\nabla ln f_1, \nabla ln f_2))dV. \tag{3.33}$$

Utilizing equations (3.31) and (3.32) in (3.33), we derive

$$\int_{N^n} \left\{t(t-2)\lambda_1 f_1^2 + s(s-2csc^2\theta)\lambda_2 f_2^2 - q(1-g(\nabla ln f_1, \nabla ln f_2))\right\} dV > 0. \tag{3.34}$$

This completes the proof. □

4 Conclusion remarks

It has been shown in this paper that if the Laplacian and gradient of the warping function of a compact bi-warped product submanifold of an odd dimensional sphere satisfy specific extrinsic restrictions, then these submanifolds have no stable integral currents, and have trivial homology groups. Additionally, we have established similar results for the Dirichlet energy functions $E(f_1)$ and $E(f_2)$, as well as for the first non-zero eigenvalues λ_1 and λ_2.

References

[1] A.H. Alkhaldi, P. Laurian-Ioan, I. Ahmad, and A. Ali, *Vanishing homology of warped product submanifolds in complex space forms and applications*, Mathematics, **10(20)**, (2022), 3884.

[2] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre D'une Variétés Riemannienne*; Springer: Berlin, Germany, 1971.

[3] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture notes in Math., Springer, Berlin, 509 (1976).

[4] R.L. Bishop and B. O' Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145**, 1–49, (1965).

[5] O. Calin and D.C. Chang, *Geometric Mechanics on Riemannian Manifolds: Applications to Partial Differential Equations*; Springer Science and Business Media: New York, USA, 2006.

[6] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J., **42(1)**, 125–138, (2000).

[7] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata, **78(2)**, 183–199, (1999).

[8] B.Y. Chen and F. Dillen, *Optimal inequalities for multiply warped product submanifolds*, Int. Electron. J. Geom., **1(1)**, 1–11, (2008).

[9] F. Dobarro and B. Unal, *Curvature of multiply warped products*, J. Geom. Phys., **55(1)**, 75–106, (2005).

[10] H. Federer and W. Fleming, *Normal and integral currents*, Ann. of Math., **72(2)**, 458–520, (1960).

[11] H. Fu and H. Xu, *Vanishing and topological sphere theorems for submanifolds in a hyperbolic space*, Int. J. Math., **19**, 811–822, (2008).

[12] T. Gowdridge, N. Dervilis and K. Worden, *On topological data analysis for structural dynamics: An introduction to persistent homology*, ASME Open J. Engineering., **1**, 011038, (2022).

[13] M.A. Khan, C. Ozel, M.D. Siddiqi, S.K. Chaubeyi and S.S. Ali, *Analysis of stable currents and homology of biwarped product submanifolds in the Euclidean space*, Filomat, **37(21)**, 7105–7113, (2023).

[14] H.B. Lawson and J. Simons, *On stable currents and their application to global problems in real and complex geometry*, Ann. of Math., **98(2)**, 427–450, (1973).

[15] J. Liu and Q. Zhang, *Non-existence of stable currents in submanifolds of the Euclidean spaces*, J. Geom., **96(1-2)**, 125–133, (2009).

[16] B. Sahin and F. Sahin, *Homology of contact CR-warped product submanifolds of an odd-dimensional unit sphere*, Bull. Korean Math. Soc., **52(1)**, 215–222, (2015).

[17] A. AL-Jedani, S. Uddin, A. Alghanemi and I. Mihai, *Bi-warped products and applications in locally product Riemannian manifolds*, J. Geom. Phys., **144**, 358–369, (2019).

[18] Y.L. Xin, *An application of integral currents to the vanishing theorems*. Sci. Sinica Ser. A, **27(3)**, 233–241, (1984).

[19] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Progr. Math., Birkhauser, Boston, MA, **30** (1983).

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