Palestine Journal of Mathematics

Vol 14(1)(2025) , 1073–1079

Idempotent $*-$Invariant Principally Ordered Regular Semigroups

G. A. Pinto

Communicated by Ayman Badawi

Abstract. We start this paper by observing that a principally ordered regular semigroup S is an inverse semigroup if and only if the operation $x \to x^0$ is the identity, when restricted to the the set of idempotents of S, that is, $e = e^0$ for every $e \in E(S)$. We then study principally ordered regular semigroups S, for which the operation $x \to x^*$ satisfies $e = e^*$, for every idempotent $e \in S$, that we call idempotent $*-$invariant. We prove that under this condition the semigroup S is dually naturally ordered, inverse and $S = S^0$. We obtain the following results: (1) S is a semigroup with zero if and only if S has a greatest element which is idempotent; (2) S is a monoid if and only if S has a smallest idempotent; (3) If $E(S)$ is a finite set, then S has a greatest idempotent and hence $E(S)$ is a band with a zero element; (4) the $*$-subsemigroup generated by a pair of idempotents is described.

MSC2020 - Mathematical Sciences Classification System 20M17 and 06F05

Keywords ordered semigroups; regular; inverse; dually naturally ordered; principally ordered, zero, identity, greatest idempotent, smallest idempotent.

1 Introduction

We recall (see, for example [**1**]) that the *natural order*, $\leq_n$, on the idempotents of a regular semigroup S, is defined by

$$e \leq_n f \iff e = ef = fe$$

and that an ordered regular semigroup $(T, \leq)$ is said to be *dually naturally ordered* if the order reverses the natural order, in the sense that if $e \leq_n f$ then $f \leq e$.

An ordered regular semigroup S is said to be *principally ordered* if, for every $x \in S$, there exists $x^* = \max\{y \in S | xyx \leq x\}$.

The basic properties of the operation $x \to x^*$, in principally ordered regular semigroups, were established in [**3**] and [**4**] and are listed in [**1**, Theorem 13.26]. In particular, we recall for the reader's convenience that, in such a semigroup, the following properties hold, and will be used throughout in what follows:

(P_1) $\Big(\forall x \in S\Big)$ $x = xx^*x$;

(P_2) every $x \in S$ has a biggest inverse, namely $x^0 = x^*xx^*$;

(P_3) $\Big(\forall x \in S\Big)$ $x^0 \leq x^*$;

(P_4) $\Big(\forall x \in S\Big)$ $xx^0 = xx^*$ is the greatest idempotent in R_x;

(P_5) $\Big(\forall x \in S\Big)$ $x^0x = x^*x$ is the greatest idempotent in L_x;

(P_6) $\Big(\forall e \in E(S)\Big)$ $e \leq e^0 \leq e^*$;

In any ordered regular semigroup S, in which every $x \in S$ has a biggest inverse x^0, it was proven in [**8**], and is stated in [**1**, Theorem 13.22], that

(P_7) $\Big(\forall x \in S\Big)$ $(xx^0)^0 = x^{00}x^0$ and $(x^0x)^0 = x^0x^{00}$;

(P_8) $(x, y) \in \mathcal{R} \iff xx^0 = yy^0$; $\quad (x, y) \in \mathcal{L} \iff x^0x = y^0y$.

Properties (P_7) and (P_8) hold in a principally ordered regular semigroup since by (P_2), biggest inverses in such a semigroup exist.

In the next result, we present a necessary and condition for a principally ordered regular semigroups to be an inverse semigroup.

Lemma 1.1. *A principally ordered regular semigroup, S, is an inverse semigroup if and only if $e = e^0$, for every $e \in E(S)$.*

Proof. It is well known [**7**, Theorem 5.1.1] that a semigroup is inverse if and only if each element has a unique inverse, if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class have a unique idempotent. For an idempotent e in a principally ordered regular semigroup S, we have that e is an inverse of itself and, by (P_2), e^0 is also an inverse of e.
If, on one hand, S is an inverse semigroup we can immediately conclude that $e = e^0$.
If, on the other hand, $e = e^0$, for every $e \in E(S)$, let us assume that $f, g \in E(S)$ are such that $(f, g) \in \mathcal{L}$. Then, by (P_8), $f^0 f = g^0 g$ and therefore, by hypothesis, $f = g$ which means that every $\mathcal{L}$ has a unique idempotent. Similarly, every $\mathcal{R}$ has a unique idempotent. Therefore, S is an inverse semigroup. □

Corollary 1.2. *If S is a principally ordered inverse semigroup, then $S = S^0$.*

Proof. For any $x \in S$ we have, by (P_4) and (P_5), that xx^0 and x^0x are idempotents. Using Lemma 1 along with properties (P_2) and (P_7), we have that

$$x = xx^0xx^0x = (xx^0)^0x(x^0x)^0 = x^{00}x^0xx^0x^{00} = x^{00}$$

which immediately tells us that $S \subseteq S^0$. Since the reverse inclusion is obvious, we can conclude that $S = S^0$. □

Throughout this paper we consider S a principally ordered regular semigroup, with set of idempotents denoted by $E(S)$.

2 Idempotent $*-$Invariant

We say that a principally ordered regular semigroup S is *idempotent $*$-invariant* if $e = e^*$ for every $e \in E(S)$

Note that in a idempotent $*$-invariant principally ordered regular semigroup S, we have, using (P_6) that for every $e \in S$,

$$e \leq e^0 \leq e^* = e \implies e = e^0$$

from which the following results follow immediately from Lemma 1 and its Corollary.

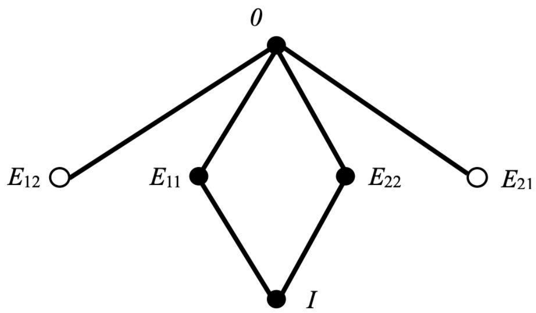
Theorem 2.1. *Let S be a idempotent $*$-invariant principally ordered regular semigroup. Then, S is an inverse semigroup.*

Theorem 2.2. *If S is a idempotent $*$-invariant principally ordered regular semigroup, then $S = S^0$.*

Example 2.3. It can be seen in [**2**, Example 1] that the set of 2×2 real matrices $S_6 = \{I, O, E_{11}, E_{12}, E_{21}, E_{22}\}$, where O is the zero matrix and

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with the usual product of matrices is an inverse ordered semigroup with a partial order defined by the following Hasse diagram



Routine calculations show that S_6 is principally ordered with
$$E_{11}^* = E_{11}, \quad E_{22}^* = E_{22}, \quad E_{12}^* = E_{21}, \quad E_{21}^* = E_{12}, \quad I^* = I, \quad O^* = O$$
and
$$E_{11}^0 = E_{11}, \quad E_{22}^0 = E_{22}, \quad E_{12}^0 = E_{21}, \quad E_{21}^0 = E_{12}, \quad I^0 = I, \quad O^0 = O$$
which tells us that $S_6 = S_6^* = S_6^0$.
Also, S_6 is dually naturally ordered and, since $E(S_6) = \{E_{11}, E_{22}, O, I\}$, it follows immediately that S_6 is idempotent $*$-invariant.

A natural question is: does the converse of Theorem 1 hold? The following example shows that it does not.

Example 2.4. Let G be a discretely ordered group. Adjoin to G an element z and add the single relation $z < 1_G$. Extend the multiplication of G to $S = G \cup \{z\}$ by defining $z^2 = z$ and $xz = x = zx$, for all $x \in G$.
From [**1**, Exercise 13.2] we can say that S is an inverse semigroup. In [**6**, Example 2] it is stated that this semigroup is principally ordered. Routine calculations give us that $x^* = x^0 = x^{-1}$, for every $x \in G$. Also, $z^* = 1_G$ and $z^0 = z$, which means that S is not idempotent $*$-invariant. We obtain that $S = S^0$ but $S \neq S^*$.
Since $1_G \leq_n z$ and $z \leq 1_G$, we can conclude that S is dually naturally ordered.

In both Examples we have that the semigroup is dually naturally ordered. In fact, it is true in general that a idempotent $*$-invariant principally ordered regular semigroup, is dually naturally ordered.

Theorem 2.5. *If S is a idempotent $*$-invariant principally ordered regular semigroup, then, S is dually naturally ordered.*

Proof. Let $e, f \in E(S)$ be such that $e \leq_n f$, that is, $ef = e = fe$. Then,
$$efe = e \quad \Longrightarrow \quad f \leq e^* = e$$
which means that S is dually naturally ordered. □

In both Example 1 and Example 2 we can see that we have an identity element which is the smallest idempotent of the semigroup.

In Example 1 we have the presence of a zero element which is the greatest element of the semigroup, but in Example 2 none of them (neither a greatest element nor a zero element) exist.

This is not a coincidence. In the next two Theorems we relate the existence of a zero element with the existence of a greatest element, and the existence of an identity element with the existence of a smallest idempotent.

Theorem 2.6. *Let S be a idempotent $*$-invariant principally ordered regular semigroup. The following statements are equivalent*

(1) *S has a greatest element (in fact an idempotent).*
(2) *S is a semigroup with zero.*

Proof. (2) $\Longrightarrow$ (1): Let us assume that ξ is the zero element of S, which is, in particular, an idempotent. For any $x \in S$, we have that

$$\xi \cdot xx^* = \xi = xx^* \cdot \xi \quad \Longrightarrow \quad \xi \leq_n xx^*$$

and since, by Theorem 3, S is dually naturally ordered, we obtain that $xx^* \leq \xi$. Multiplying on the right by x gives

$$x = xx^*x \leq \xi x = \xi \quad \Longrightarrow \quad x \leq \xi.$$

Thus, ξ is the greatest element of S.

(1) $\Longrightarrow$ (2): Let us assume that ξ is the greatest element of S. We have that

$$\xi \cdot \xi \leq \xi = \xi\xi^*\xi = (\xi\xi^*)\xi \leq \xi \cdot \xi \quad \Longrightarrow \quad \xi \in E(S).$$

For any $e \in E(S)$, we have

$$\xi e = \xi eee \leq \xi e\xi e \leq \xi\xi\xi e = \xi e \quad \Longrightarrow \quad \xi e \in E(S).$$

Similarly, $e\xi \in E(S)$, $e\xi e \in E(S)$ and $\xi e\xi \in E(S)$.

Note that $\xi e\xi \in E(S)$ is such that $\xi e\xi \leq_n \xi$ and, using Theorem 3, we can say that

$$\xi \leq \xi e\xi \leq \xi\xi\xi = \xi$$

and therefore $\xi e\xi = \xi$.

Thus, $\xi \in V(e\xi e)$ and since $(e\xi e)^0$ is the greatest inverse of $e\xi e$, we have by (P_6) and the fact that S is idempotent $*$-invariant, that

$$\xi \leq (e\xi e)^0 \leq (e\xi e)^* = e\xi e$$

Then, since ξ is the greatest element of S,

$$\xi = e\xi e \quad \Longrightarrow \quad \xi\xi = \xi e\xi e \Longrightarrow \quad \xi = \xi e$$

and, similarly, $\xi = e\xi$.

Finally, for any $x \in S$, we have, since x^*x is an idempotent, that

$$\xi x \leq \xi\xi = \xi = \xi x^*x \leq \xi\xi x = \xi x \quad \Longrightarrow \quad \xi x = \xi$$

and, similarly, $x\xi = \xi$.

Therefore, ξ is the zero element of S, and S is a semigroup with zero. □

Next example shows that the equivalence in the previous Theorem does not hold, if we do not consider the hypothesis that the semigroup is idempotent $*$-invariant.

Example 2.7. Let $S = \{e, f, g\}$ be a band where g is a zero element and $ef = f = fe$, chain-ordered in the following way: $f < g < e$.

Routine calculations allows us to conclude that S is a principally ordered inverse semigroup where $f^* = g^* = e^* = e$, hence it is not idempotent $*$-invariant. We have that e is the greatest element of S, but it is not the zero element of S.

Therefore, we can conclude, from this example, that in the previous Theorem the hypothesis idempotent $*$-invariant, is essential.

Theorem 2.8. *Let S be a idempotent $*$-invariant principally ordered regular semigroup. The following statements are equivalent*

(1) *S has a smallest idempotent.*

(2) *S is a monoid.*

Proof. (2) $\Longrightarrow$ (1): Assuming that α is the identity element of S we have, for every $e \in E(S)$, using Theorem 3, that

$$\alpha \cdot e = e = e \cdot \alpha \quad \Longrightarrow \quad e \leq_n \alpha \quad \Longrightarrow \quad \alpha \leq e$$

and therefore α is the smallest idempotent of S.

(1) $\Longrightarrow$ (2): Let us now suppose that α is the smallest idempotent of S. For any $e \in E(S)$

$$\alpha e = \alpha\alpha\alpha e \leq \alpha e \cdot \alpha e \leq \alpha eee = \alpha e \quad \Longrightarrow \quad \alpha e \in E(S)$$

and, similarly, $e\alpha, e\alpha e, \alpha e\alpha \in E(S)$.

Since $e\alpha e \leq_n e$ we conclude, using Theorem 3, that

$$e \leq e\alpha e \leq eee = e \quad \Longrightarrow \quad e\alpha e = e.$$

Then, $e \in V(\alpha e\alpha)$ and therefore, using the fact that S is idempotent $*$-invariant, we obtain

$$e \leq (\alpha e\alpha)^0 \leq (\alpha e\alpha)^* = \alpha e\alpha \leq \alpha ee = \alpha e \leq ee = e.$$

Thus, $\alpha e = e$ and, similarly, $e\alpha = e$.

Now, for any $x \in S$,

$$\alpha x = \alpha(xx^*x) = (\alpha \cdot xx^*)x = xx^* \cdot x = x$$

and, similarly, $x\alpha = x$. Therefore, α is the identity element of S, that is S is a monoid. □

Example 1, shows that, in a idempotent $*$-invariant principally ordered regular monoid S, the identity element is not, in general, the smallest element of S.

The following example illustrate that, in the previous Theorem, the hypothesis that it is idempotent $*$-invariant, is crucial.

Example 2.9. Let $S = \{z, e, f\}$ be a band where z is a zero element, and $ef = e = fe$, with the following order relations $e < f < z$.
Straightforward calculations give us that S is a principally ordered semigroup with $e^* = f$, $f^* = f$ and $z^* = z$. Also, $e^0 = e$, $f^0 = f$ and $z^0 = z$, which means, by Lemma 1, that S is an inverse semigroup.
Now, S has a smallest idempotent, e, which is not an identity element. This does not contradicts Theorem 5 since S is not idempotent $*$-invariant (in fact, $e \neq e^*$).
Also, $e \leq_n f$ and $z \leq_n f$, but $e \leq f$ and $f \leq z$, from which we can conclude that S is neither naturally ordered nor dually naturally ordered.

We now explore what happens when the set of idempotents is finite.

Theorem 2.10. *Let S be a idempotent $*$-invariant principally ordered regular semigroup, with a finite set of idempotents. Then S has a greatest idempotent.*

Proof. Let us assume that $E(S) = \{e_1, e_2, \cdots, e_n\}$ and consider the multiplication of its elements $z = e_1 e_2 \cdots e_n$. By Theorem 1, S is inverse and therefore the idempotents commute. Then,
$$z^2 = (e_1 e_2 \cdots e_n)(e_1 e_2 \cdots e_n) = {e_1}^2 {e_2}^2 \cdots {e_n}^2 = e_1 e_2 \cdots e_n = z$$
which means that z is an idempotent. Now, for every $i \in \{1, 2, \cdots, n\}$,
$$z \cdot e_i = e_1 e_2 \cdots e_n \cdot e_i = e_1 \cdots e_i e_i \cdots e_n = e_1 \cdots e_i \cdots e_n = z$$
and, similarly, $e_i \cdot z = z$. Then, $z \leq_n e_i$ which, by Theorem 3, tells us that $e_i \leq z$ and therefore z is the greatest idempotent of S. □

Corollary 2.11. *If S is a idempotent $*$-invariant principally ordered regular semigroup, with a finite set of idempotents, then $E(S)$ is a band with a zero element.*

Proof. Let us assume that $E(S) = \{e_1, e_2, ..., e_n\}$. By Theorem 1, S is an inverse semigroup and therefore its idempotents commute, which implies that $E(S)$ is a subsemigroup of S and, in fact, a band.
By Theorem 6, $z = e_1 e_2 \cdots e_n$ is the greatest idempotent of S and, in particular, is the greatest element of $E(S)$.
Finally, by Theorem 4, z is the zero element of S and since it belongs to $E(S)$, it has to be the zero element of $E(S)$. □

Note that in Example 1, $E(S) = \{I, E_{11}, E_{22}, O\}$ is a finite band with a zero element, O.

In the next Theorem we present a partial converse of the previous result.

Theorem 2.12. *Let S be a principally ordered regular semigroup, such that $E(S)$ is a commutative band with a zero element, z, which is the greatest element of S.*
If S has no chain of idempotents with length bigger than 2, then S is idempotent $*$-invariant.

Proof. Since $E(S)$ is a commutative band we can say that S is an inverse semigroup and therefore, by Lemma 1, $e = e^0 \leq e^* \leq z$. By hypothesis we must have that $e^* = z$ or $e^* = e$.
If $e^* = z$ then, using the fact that z is the zero element of S,
$$e = e^0 = e^* e e^* = zez = z$$
and therefore $e = z = e^*$.
Thus, in either case, we may conclude that $e = e^*$ which means that S is idempotent $*$-invariant. □

In [**5**, Theorem 2] Blyth and Pinto proved that if S is a principally ordered inverse semigroup such that $x \to x^*$ is *weakly isotone*, that is, for every $e, f \in E(S)$ such that $e \leq f$, we obtain $e^* \leq f^*$, then the $*$-subsemigroup generated by $\{e, f\}$ with $e < f$ and $e^* < f^*$ is a band with at most seven elements, in which every connection in the Hasse diagram also indicate the natural order or the dual natural order.

In this context of a idempotent $*$-invariant principally ordered regular semigroup, we can formulate the following result.

Theorem 2.13. *Let S be a idempotent $*$-invariant principally ordered regular semigroup. For $e, f \in E(S)$, we have that*

(1) *if $e < f$ then the $*$-subsemigroup generated by $\{e, f\}$ is a chain with exactly two elements e, f, where $ef = f$ is the zero (semigroup) element.*

(2) *if e and f are incomparable, then the $*$-subsemigroup generated by $\{e, f\}$ is a band with exactly three elements e, f, ef, that verify $e < ef$ and $f < ef$, where ef is the zero element.*

Proof. Let $e, f \in E(S)$. In any of the cases we have, by Theorem 3, that

$$ef \cdot e = ef = e \cdot ef \quad \Longrightarrow \quad ef \leq_n e \quad \Longrightarrow \quad e \leq ef$$

and, similarly, $f \leq ef$.

(1) Note that, by Theorem 1, S is an inverse semigroup and, for idempotents e, f, such that $e \leq f$, we have that $e^* = e \leq f = f^*$, which means that $x \to x^*$ is, trivially, weakly isotone. Now, if $e < f$, then the hypothesis of [**5**, Theorem 2] hold and we can conclude that the $*$-subsemigroup, T, generated by $\{e, f\}$ is a band with the seven elements: $e, ef, e^*f, f, f^*, e^*, e^*f^*$. Since S is idempotent $*$-invariant they are reduce to three, since $e = e^*$, $ef = e^*f = e^*f^*$ and $f = f^*$. Additionally, we have that

$$e = ee \leq ef \leq ff = f$$

and therefore, since $f \leq ef$, we have that $ef = f$, which implies that $T = \{e, f\}$ is a two-element chain, with $e < f$. Also,

$$e \cdot f = ef = f \qquad \text{and} \qquad f \cdot e = fe = ef = f$$

which means that $ef = f$ is the zero element of T.

(2) Let us now assume that $e \not\leq f$ and $f \not\leq e$. If $e = ef$, then $f \leq ef = e$, which is a contradiction. Similarly, we obtain a contradiction assuming that $f = ef$. Therefore, we have that $e < ef$ and $f < ef$. We can immediately conclude that e, f, ef are all distinct and that they form a band with ef as its zero element. □

In Example 1, if we take idempotents I, E_{11}, then the $*$-subsemigroup T_1 generated by $\{I, E_{11}\}$ is equal to $T_1 = \{I, E_{11}\}$, where E_{11} is the zero element of T_1, since $I \cdot E_{11} = E_{11} = E_{11} \cdot I$.

Taking now, still in Example 1, the incomparable idempotents E_{11} and E_{22}, then the $*$-subsemigroup T_2 generated by $\{E_{11}, E_{22}\}$ is equal to $T_2 = \{E_{11}, E_{22}, O\}$, where $E_{11} < O$, $E_{22} < O$ and O is the zero element of T_2.

Example 2.14. Consider the set $\mathbb{Z}$ of integer numbers as a join semilattice under the definition

$$m \vee n = \max\{m, n\}$$

It is easy to verify that we obtain a principally ordered inverse semigroup, with $m^* = m$, and therefore idempotent $*$-invariant .

Let us now take S_6 as in Example 1 and consider the cartesian ordered set $S_6 \times \mathbb{Z}$, with the multiplication

$$(A, m)(B, n) = (AB, m \vee n).$$

Then, we can see that we obtain a idempotent $*$-invariant principally ordered regular (in fact, inverse) semigroup, with an infinite set of idempotents.

References

[1] T. S. Blyth; *Lattices and Ordered Algebraic Structures*. Springer, 2005.

[2] T. S. Blyth; Emília Giraldes and M. Paula O. Marques-Smith; *Dually Perfect Orthodox Dubreil-Jacotin Semigroups*. Communications in Algebra, **21**, 1993, 1773-1783.

[3] T. S. Blyth and G. A. Pinto; *Principally ordered regular semigroups*. Glasgow Math. J., 1990, **32**, 389-416.

[4] T. S. Blyth and G. A. Pinto; *Idempotents in principally ordered regular semigroups*. Communications in Algebra, **19**, 1991, 1549-1563.

[5] T. S. Blyth and G. A. Pinto; *On Idempotent-generated Subsemigroups of Principally Ordered Regular Semigroups*. Semigroup Forum **68** (2004) 47–58.

[6] T. S. Blyth and M. H. Almeida Santos; *Ordered Regular Semigroups with Biggest Associates*. Discussiones Mathematicae, General Algebra and Applications **39** (2019) 5–21.

[7] John M. Howie; *Fundamentals of Semigroup Theory*. Oxford University Press, 1995.

[8] Tatsuhiko Saito; *Naturally ordered regular semigroups with maximum inverses*. Proc. Edinburgh Math. Soc., **32**, 1989, 33-39.

Author information

G. A. Pinto, G. A. Pinto: General Education Department, New Uzbekistan University, Tashkent, Uzbekistan.
E-mail: g.pinto@newuu.uz; gcapinto@gmail.com

Received: 2024-04-12.
Accepted: 2024-10-23