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SOME RESULTS ON MENGER HYPERALGEBRAS

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Abstract Menger hyperalgebras of rank n, where n is a fixed natural number, can be considered as a suitable generalization of arbitrary semihypergroups. In this article, we establish the so-called a *diagonal semihypergroup* which is induced by a Menger hyperalgebra of rank n. Moreover, we investigate some algebraic properties of the diagonal semihypergroups. In particular, we discovery that an element of the diagonal semihypergroups satisfying some special conditions can be acted as a right (left) scalar identity element. Furthermore, we also show that a scalar diagonal unit of the diagonal semigroups is unique.

1 Introduction

In 1946, Menger, K. [20] studied an algebraic property of composition of multiplace functions, which are also called the functions of many variables. Such the algebraic property of composition is called the *superassociative law*. The algebraic structure of multiplace functions satisfying the superassociative law is called a *Menger algebra of rank* n, where n is a natural number. Menger algebras can be considered as a generalization of semigroups in various kinds of n-ary semigroups. Menger algebras have been studied in different directions of theoretical and applied mathematics (c.f. [20, 21]).

The theories of Menger algebras and its applications are developed by Dudek, W.A. and Trokhimenko, V.S. who presented the concept of principal v-congruences on Menger algebras, which is a generalization of the principal left and right congruences on semigroups (see [10]). In 1997, Trokhimenko, V.S. [28] studied the special class of Menger algebras, the so-called v-regular Menger algebras of rank n. In 2012, they studied and presented the idea of subtraction Menger algebras (see [14]). Based on the concept, they also investigated some of the results related to subtraction Menger algebras in 2016 (see [13]). Up to 2018, they proved that the set of all idempotent n-ary operations defined on a given nonempty set forms a Menger algebra, and verified some of its interesting properties (see, [27]).

Nowadays, the concept of Menger algebras have been widely extended to study in various aspects. Based on this concept, Denecke, K. established Menger algebras and clone of terms (see [7]). In 2021, Denecke, K. and Hounnon, H. [8] presented the notion of Menger algebras of terms. Moreover, Wattanatripop, K. and Changphas, T. [29] studied and investigated the properties of Menger algebras of terms induced by order-decreasing transformations. According to important results on semigroups, every semigroup can be constructed as a ternary semigroup, while there are some ternary semigroups can not be reduced to semigroups (c.f. [22]). As a result, the authors established the so-called ternary Menger algebras of rank n, which can be regarded as a natural generalization of ternary semigroups (see [25]). Moreover, the authors used this structure to constructed v-regular ternary Menger algebras of rank n in 2021 (see [23]). For more information related to Menger algebras, see [11, 12, 16, 30].

The French mathematician Marty, F. [19] who have been credited as an initiator in the study of hyperstructure theory, when he defined hypergroups based on the notion of hyperoperation in

1934. Such algebraic hyperstructures can be regarded as a generalization of classical algebraic structures. Based on classical structures, the composition of two elements (or n elements where n is a fixed natural number) is an element. While in algebraic hyperstructures, the composition of two elements (or n elements) is a set. Since then, there are many mathematicians who have studied algebraic properties on this topic. Moreover, there are several books which were written on this topic. A book on hyperstructures [3] have pointed out on some applications of hyperstructures in graphs, hypergraphs cryptography, fuzzy set theory, rough set theory, codes, automata, geometry, and lattices. For another book, see [6].

The concept of semihypergroups is a suitable generalization of the concept of semigroups which is different from Menger algebras and n-ary semigroups. As a result, many authors investigated algebraic properties of semihypergroups in different aspects. In 2011, Heidari, D. and Davvaz, B. [15] introduced the concept of ordered semihypergroups, which can be considered as a generalization of ordered semigroups. Indeed, every ordered semigroup is an ordered semihypergroup. In 2014, some algebraic properties of hyperideals in ordered semihypergroups were investigated by Changphas, T. and Davvaz, B. (see [1]). Up to 2021, Daengsaen, J. and Leeratanavalee, S. [4] studied some algebraic properties related to the concept of regularities in ordered n-ary semihypergroups. Recently, Kumduang, T. and Leeratanavalee, S. [17] used the concept of algebraic hyperstructures to establish the so-called a Menger hyperalgebra and studied their representations. For more information related to semihypergroups, see [2, 5, 24, 26]

In this paper, we begin with recalling some significant results of semihypergroups, Menger algebras, and Menger hyperalgebras. In Section 3, we construct a new semihypergroup, which is called a *diagonal semihypergroup*, such that its binary hyperoperation is induced by an $(n+1)$-ary operation of a Menger hyperalgebra of rank n. Then, we investigate some interesting algebraic properties of the diagonal semihypergroup. In particular, we show that some elements on the diagonal semihypergroup satisfying some necessary conditions can be a right (left) scalar identity element. However, we also prove that a scalar diagonal unit of the diagonal semigroup is unique.

2 Preliminaries

In this section, we recall some important results on semihypergroups, Menger algebras of rank n, Menger hyperalgebras of rank n, and v-regular Menger algebras of rank n. Moreover, we present some of their special elements, and examples.

Let S be a nonempty set. The set S together with a mapping $\circ : S \times S \longrightarrow P^*(S)$, where $P^*(S)$ is the set of all nonempty subsets of S, is called a *hypergroupoid* and the mapping $\circ$ is called a *binary hyperoperation* on S. For any nonempty subsets X and Y of S,

$$\circ(X,Y) = \bigcup_{x\in X, y\in Y} x \circ y.$$

A hypergroupoid $(S,\circ)$ is said to be a *semihypergroup* if the hyperoperation $\circ$ on S satisfies the associative law, i.e., $\circ(\circ(x,y),z) = \circ(x,\circ(y,z))$ for all $x,y,z \in S$. It means that

$$\bigcup_{a\in\circ(x,y)} \circ(a,z) = \bigcup_{b\in\circ(y,z)} \circ(x,b).$$

According to the algebraic structures between semigroups and semihypergroups, we have a relationship between the two structures as follows: every semigroup can be considered as a special case of semihypergroups.

Now, let H be a nonempty set together with an $(n+1)$-ary operation $\bullet$ defined on H. where n is a fixed natural number. Then, the pair $(H,\bullet)$ is called a *Manger algebra of rank* n (also, an *abstract Menger algebra of rank* n or an *n-dimensional superpositive system*) if the structure satisfies the so-called a *superassociative law* given as follows:

$$\begin{aligned}&\bullet(\bullet(x,y_1,\ldots,y_n),z_1,\ldots,z_1)\\ &\qquad = \bullet(x,\bullet(y_1,z_1,\ldots,z_n),\ldots,\bullet(y_1,z_1,\ldots,z_n))\end{aligned} \tag{2.1}$$

for all $x,y_i,z_i \in H, i = 1,\ldots,n$. We can see immediately that the algebraic structure of Menger algebras of rank n can be regarded as a generalization of semigroups. In fact, in case $n = 1$, the superassociative law is reduced to the usual associative law and the Menger algebras of rank n is reduced to semigroups.

Definition 2.1. [9]. Let $(H, \bullet)$ be a Menger algebra of rank n. A nonempty subset I of H is said to be:

(i) an s-ideal, if for every $a, x_i \in H, i = 1, \ldots, n$

$$a \in I \Longrightarrow \bullet(a, x_1, \ldots, x_n) \in I;$$

(ii) a v-ideal, if for every $a_i, x \in H, i = 1, \ldots, n$

$$a_1, \ldots, a_n \in I \Longrightarrow \bullet(x, a_1, \ldots, a_n) \in I.$$

Now, some special elements of Menger algebras of rank n can be provided as follows:

Definition 2.2. [9]. Let $(H, \bullet)$ be a Menger algebra of rank n. An element $e \in H$ is said to be:

(i) a *left diagonal unit* if $x = \bullet(e, x^n)$ for all $x \in H$;

(ii) a *right diagonal unit* if $\bullet(x, e^n)$ for all $x \in H$;

(iii) a *diagonal unit* if $x = \bullet(e, x^n) = \bullet(x, e^n)$ for all $x \in H$,

where x^n is a sequence $x, x, \ldots, x$ such that x appears n terms.

Definition 2.3. [28]. Let $(H, \bullet)$ be a Menger algebra of rank n. An element $(a_1, \ldots, a_n)$ is called:

(i) an idempotent element if

$$a_i = \bullet(a_i, a_1, \ldots, a_n) \text{ for all } i = 1, \ldots, n,$$

(ii) a v-regular element if there exists $x \in H$ such that

$$a_i = \bullet(\bullet(a_i, x^n), a_1, \ldots, a_n) \quad \text{for all } i = 1, \ldots, n.$$

The following examples are presented to illustrate the algebraic structure of Menger algebras of rank n and their some special elements.

Example 2.4. [9].

(i) On the set $\mathbb{R}^+$ of all positive real numbers, one can define an $(n+1)$-ary operation

$$\bullet(x, y_1, \ldots, y_n) = x \times \sqrt[n]{y_1 \times \ldots \times y_n}$$

for all $x, y_i \in H, i = 1, \ldots, n$, where $\times$ is the usual multiplication on $\mathbb{R}^+$. Then, $(H, \bullet)$ forms a Menger algebra of rank n

(ii) Define an $(n+1)$-ary operation $\bullet$ on a nonempty set H by

$$\bullet(x, y_1, \ldots, y_n) = x$$

for all $x, y_i \in H, i = 1, \ldots, n$. So, $(H, \bullet)$ forms a Menger algebra of rank n which all elements are idempotent elements and right diagonal units.

Now, we complete the section by giving the definition of Menger hyperalgebras of rank n. Let H be a nonempty set and $\cdot : H^{n+1} \longrightarrow P^*(H)$ be an $(n+1)$-ary hyperoperation on H. If $X, Y_i, i = 1, \ldots, n$ are nonempty subsets of H, then we define

$$\cdot(X, Y_1, \ldots, Y_n) = \bigcup\{\cdot(x, y_1, \ldots, y_n) \mid x \in X, y_i \in Y_i, i = 1, \ldots, n\}.$$

Definition 2.5. [17]. A nonempty set H together with an $(n+1)$-ary hyperoperation $\cdot : H^{n+1} \longrightarrow P^*(H)$ is said be an $(n+1)$-ary hypergroupoid and denoted by a pair $(H, \cdot)$.

Definition 2.6. [17]. An $(n+1)$-ary hypergroupoid $(H, \cdot)$ is called a *Menger hyperalgebra of rank* n if the $(n+1)$-ary hyperoperation $\cdot$ satisfies the superassociative law defined as in (2.1).

Remark 2.7. Any Menger hyperalgebras of rank n can be considered as a generalization of semihypergroups by setting $n = 1$. Then, Menger hyperalgebras of rank 1 are immediately reduced to be semihypergroups.

Similar to the relationship between the algebraic structures of semigroups and semihypergroups, the following theorem was given.

Theorem 2.8. [17]. *Every Menger algebra of rank n induces a Menger hyperalgebra of rank n.*

Example 2.9. [17]. Some examples of Menger hyperalgebras of rank n were already presented.

(i) Let H be a closed interval $[0, 1]$. Define an $(n+1)$-ary hyperoperation $\cdot$ on H by

$$\cdot(x, y_1, \ldots, y_n) = \left[0, \tfrac{x \times y_1 \times \ldots \times y_n}{n}\right]$$

for all $x, y_i \in H, i = 1, \ldots, n$, where $\times$ is the usual multiplication. Hence, $(H, \cdot)$ forms a Menger hyperalgebra of rank n.

(ii) Consider the set $\mathbb{N}$ of all natural numbers under an $(n+1)$-ary hyperoperation $\cdot$ on $\mathbb{N}$ given by:

$$\cdot(x, y_1, \ldots, y_n) = \{z \in \mathbb{N} \mid z \geq \max\{x, y_1, \ldots, y_n\}\}$$

for all $x, y_i \in \mathbb{N}, i = 1, \ldots, n$. Thus, $(\mathbb{N}, \cdot)$ forms a Menger hyperalgebra of rank n.

3 Diagonal Semihypergroups

In this section, we introduce a new binary hyperoperation that is induced by an $(n+1)$-ary operation defined on a base set of a Menger hyperalgebra of rank n and then we obtain the so-called a diagonal semihypergroup. Moreover, we investigate some of its algebraic properties.

Proposition 3.1. *Let $(H, \cdot)$ be a Menger hyperalgebra of rank n. Define a binary hyperoperation $\circ$ on H by*

$$\circ(a, b) = \cdot(a, b^n) \quad \text{for all } a, b \in H. \tag{3.1}$$

Then $(H, \circ)$ forms a semihypergroup.

Proof. By the superassociative law on the Menger hyperalgebra $(H, \cdot)$ of rank n, we obtain that

$$\begin{aligned} \circ(\circ(a, b), c) &= \cdot(\cdot(a, b^n), c^n) \\ &= \cdot(a, (\cdot(b, c^n))^n) \\ &= \circ(a, \circ(b, c)) \end{aligned}$$

holds for all $a, b, c \in H$. Thus, the binary hyperoperation $\circ$ on H is associative, which implies that $(H, \circ)$ forms a semihypergroup. □

According to Proposition 3.1, for each Menger algebra $(H, \cdot)$ of rank n, we call the semihypergroup $(H, \circ)$, which its binary hyperoperation $\circ$ is defined as in (3.1), a *diagonal semihypergroup*. By using the definition of a binary hyperoperation $\circ$ which is defined as in (3.1), we get the following important remark.

Remark 3.2. Let $(H, \cdot)$ be a Menger hyperalgebra of rank n. Then a diagonal semihypergroup $(H, \circ)$ is unique.

Definition 3.3. A diagonal semihypergroup $(H, \circ)$ of $(H, \cdot)$ is commutative, if the following condition holds:

$$\circ(x, y) = \circ(y, x) \quad \text{for all } x, y \in H.$$

Next, we introduce some special elements on Menger hyperalgebras of rank n, and show a relationship between a Menger hyperalgebra of rank n and a diagonal semihypergroup.

Definition 3.4. Let $(H, \cdot)$ be a Menger hyperalgebra of rank n. An element $e \in H$ is called:

(i) a left (scalar) diagonal unit if $x \in \cdot(e, x^n)$ $(\{x\} = \cdot(e, x^n))$ for all $x \in H$;

(ii) a right (scalar) diagonal unit if $x \in \cdot(x, e^n)$ $(\{x\} = \cdot(x, e^n))$ for all $x \in H$;

(iii) a (scalar) diagonal unit if $x \in \cdot(e, x^n) \cap \cdot(x, e^n)$ $(\{x\} = \cdot(e, x^n) = \cdot(x, e^n))$ for all $x \in H$.

Example 3.5. (i) Let m be a fixed natural number. Consider the set $H = \{1, 2, 3, \ldots, m\}$ with an $(n+1)$-ary hyperoperation $\cdot$ on H defined by

$$\cdot(x, y_1, \ldots, y_n) = \{z \in H \mid z \leq \min\{x, y_1, \ldots, y_n\}\}$$

for all $x, y_i \in H, i = 1, \ldots, n$. Then $(H, \cdot)$ forms a Menger hyperalgebra of rank n such that the element m is a diagonal unit.

(ii) [17]. An $(n+1)$-ary groupoid $(H, \cdot)$ together with the following $(n+1)$-ary hyperoperation $\cdot$ on H given by

$$\cdot(x, y_1, \ldots, y_n) = \{x\}$$

for all $x, y_i \in H, i = 1, \ldots, n$, forms a Menger hyperalgebra of rank n such that each element is right scalar unit, while $(H, \cdot)$ has no left (scalar) unit.

(iii) [17]. Let $\mathbb{R}$ be the set of all real numbers. Define an $(n+1)$-ary hyperoperation $\cdot$ on $\mathbb{R}$ by

$$\cdot(x, y_1, \ldots, y_n) = \left\{x + \tfrac{y_1 + \cdots + y_n}{n}\right\}$$

for all $x, y_i \in \mathbb{R}, i = 1, \ldots, n$. Then $(\mathbb{R}, \cdot)$ is a Menger hyperalgebra of rank n. Moreover, an element $0 \in \mathbb{R}$ is a scalar diagonal unit.

Remark 3.6. Non-isomorphic Menger hyperalgebras of rank n may have the same diagonal semihypergroup. It is illustrated in Example 3.7.

Example 3.7. Let $(H, *)$ be a nontrivial commutative semihypergroup. Define two $(n+1)$-ary hyperoperations, where $n \geq 2$, on H by

$$\cdot_1(x, y_1, \ldots, y_n) = *(x, y_1) \quad \text{and}$$
$$\cdot_2(x, y_1 \ldots, y_n) = *(x, y_2) \quad \text{for all } x, y_i \in H, i = 1, \ldots, n.$$

It is easy to verify that $(H, \cdot_1)$ and $(H, \cdot_2)$ form two Menger hyperalgebras of rank n that are non-isomorphic. Moreover, these Menger hyperalgebras of rank n have the same diagonal semihypergroup $(H, \circ)$, where the binary hyperoperation $\circ$ is defined as in (3.1).

Proposition 3.8. *If a Menger hyperalgebra $(H, \cdot)$ of rank n contains a right (left) scalar diagonal unit e, then each element $f \in H$ satisfying $\{e\} = \cdot(e, f^n)$, is a right (left) scalar diagonal unit.*

Proof. Let $(H, \cdot)$ be a Menger hyperalgebra of rank n containing a left scalar diagonal unit e. Indeed, for every $x \in H$, we have

$$\begin{aligned}\{x\} &= \cdot(x, e^n)\\ &= \cdot(x, (\cdot(e, f^n))^n)\\ &= \cdot(\cdot(x, e^n), f^n)\\ &= \cdot(x, f^n).\end{aligned}$$

Consequently, $\{x\} = \cdot(x, f^n)$ for all $x \in H$, which yields that the element f is also a right scalar diagonal unit of a Menger hyperalgebra $(H, \cdot)$ of rank n. Similar to the above argument, we can show the rest. □

According to Proposition 3.8, we can see immediately that a right (left) scalar diagonal unit of a Menger hyperalgebra of rank n need not be unique. However, a scalar diagonal unit of a Menger hyperalgebra of rank n must be unique. It can be shown as the following proposition.

Proposition 3.9. *If a Menger hyperalgebra $(H,\cdot)$ of rank n contains a scalar diagonal unit, then the scalar diagonal unit is unique.*

Proof. Assume that a Menger hyperalgebra $(H,\cdot)$ of rank n contains two distinct scalar diagonal units, say e and f. Hence,

$$\{x\} = \cdot(e, x^n) = \cdot(x, e^n) \text{ and } \{x\} = \cdot(f, x^n) = \cdot(x, f^n) \text{ for all } x \in H.$$

It immediately yields that $\{e\} = \cdot(e, f^n) = \{f\}$ and hence $e = f$. That is a contradiction with the assumption. It means that the scalar diagonal unit of a Menger hyperalgebra $(H,\cdot)$ of rank n is unique. □

Proposition 3.10. *Let $(H,\cdot)$ be a Menger hyperalgebra of rank n such that H contains a left scalar diagonal unit e and for every $x \in H$ there exists an element $y \in H$ satisfying $\{e\} = \cdot(y, x^n)$. Then the diagonal semihypergroup $(H,\circ)$ of $(H,\cdot)$ is left cancellative, i.e.,*

$$\circ(x,a) = \circ(x,b) \Longrightarrow a = b$$

for all $a, b, x \in H$.

Proof. Assume that $\circ(x,a) = \circ(x,b)$ holds for all $a, b, x \in H$. By the assumption, we get $\cdot(x,a,a,\ldots,a) = \cdot(x,b,b,\ldots,b)$. Now, we obtain

$$\begin{aligned}
\{a\} &= \cdot(e, a^n) \\
&= \cdot(\cdot(y, x^n), a^n) \\
&= \cdot(y, \cdot(x, a^n), \cdot(x, a^n), \ldots, \cdot(x, a^n)) \\
&= \cdot(y, \cdot(x, b^n), \cdot(x, b^n), \ldots, \cdot(x, b^n)) \\
&= \cdot(\cdot(y, x^n), b^n) \\
&= \cdot(e, b^n) \\
&= \{b\}.
\end{aligned}$$

Then $\{a\} = \{b\}$, and hence $a = b$. Therefore, a diagonal semihypergroup $(H,\circ)$ of $(H,\cdot)$ is left cancellative. □

Proposition 3.11. *Let $(H,\cdot)$ be a Menger hyperalgebra of rank n. For each diagonal semihypergroup $(H,\circ)$ of $(H,\cdot)$, the following equality holds:*

$$\cdot(\circ(x,y), z_1, \ldots, z_n) = \circ(x, \cdot(y, z_1, \ldots, z_n))$$

for all $x, y, z_i \in H, i = 1, \ldots, n$.

Proof. Indeed, for each $x, y, z_i \in H, i = 1, \ldots, n$, we have

$$\begin{aligned}
\cdot(\circ(x,y), z_1, \ldots, z_n) &= \cdot(\cdot(x, y^n), z_1, \ldots, z_n) \\
&= \cdot(x, \cdot(y, z_1, \ldots, z_n), \cdot(y, z_1, \ldots, z_n), \ldots, \cdot(y, z_1, \ldots, z_n)) \\
&= \cdot(x, (\cdot(y, z_1, \ldots, z_n))^n) \\
&= \circ(x, \cdot(y, z_1, \ldots, z_n)).
\end{aligned}$$

This completes the proof. □

Theorem 3.12. *Let $(H,*)$ be a semihypergroup and f be an n-ary hyperoperation on H such that it satisfies the following conditions:*

(i) $f(x^n) = \{x\}$ *for all $x \in H$;*

(ii) $*(f(x_1, \ldots, x_n), y) = f(*(x_1, y), \ldots, *(x_n, y))$ *for all $x_i, y \in H, i = 1, \ldots, n$.*

Then $(H,*)$ forms a diagonal semihypergroup of some Menger hyperalgebras of rank n.

Proof. Firstly, we assume that $(H,*)$ is a semihypergroup and the n-ary hyperoperation f on H satisfies all of the above conditions. Now, we define an $(n+1)$-ary hyperoperatation $\cdot$ on H by

$$\cdot(x,y_1,\ldots,y_n) = *(x,f(y_1,\ldots,y_n)) \quad \text{for all } x,y_i\in H, i=1,\ldots,n.$$

Secondly, we show that the $(n+1)$-ary hyperoperatation $\cdot$ on H satisfies the superassociative law. Indeed, for each $x,y_i,z_i\in H, i=1,\ldots,n$, we get

$$\begin{aligned}\cdot(\cdot(x,y_1,\ldots,y_n),z_1,\ldots,z_n) &= *(*(x,f(y_1,\ldots,y_n)),f(z_1,\ldots,z_n))\\ &= *(x,*(f(y_1,\ldots,y_n),f(z_1,\ldots,z_n))\\ &= *(a,f(*(y_1,f(z_1,\ldots,z_n)),\ldots,*(y_n,f(z_1,\ldots,z_n))))\\ &= *(a,f(\cdot(y_1,z_1,\ldots,z_n),\ldots,\cdot(y_n,z_1,\ldots,z_n)))\\ &= \cdot(x,\cdot(y_1,z_1,\ldots,z_n),\ldots,\cdot(y_n,z_1,\ldots,z_n)).\end{aligned}$$

It means that the hyperoperation $\cdot$ on H is superassociative, and hence $(H,\cdot)$ forms a Menger hyperalgebra of rank n.

Finally, suppose that $(H,\circ)$ is a diagonal semihypergroup of the Menger hyperalgebra $(H,\cdot)$ of rank n. So, for each $x,y\in H$, we have

$$\circ(x,y) = \cdot(x,y^n) = *(x,f(y^n)) = *(x,y).$$

It implies that $(H,\circ)$ and $(H,*)$ are the same. Consequently, $(H,*)$ forms a diagonal semihypergroup of some Menger hyperalgebras of rank n. □

Corollary 3.13. *Let $(H,*)$ be a semihypergroup with a left scalar identity element. The algebraic structure $(H,*)$ is a diagonal semihypergroup of some Menger hyperalgebras of rank n that has a left scalar diagonal unit if and only if there exists an n-ary hyperoperation on H satisfying all the conditions of Theorem 3.12.*

Proof. ($\Longrightarrow$) Assume that $(H,*)$ is a diagonal semihypergroup of Menger hyperalgebra $(H,\cdot)$ of rank n which has a left scalar diagonal unit e. Now, we define an n-ary hyperoperation f on H by

$$f(x_1,\ldots,x_n) = \cdot(e,x_1,\ldots,x_n) \quad \text{for all } x_i\in H.$$

For each $x,x_i,y\in H, i=1,\ldots,n$, we have

$$\begin{aligned}f(x^n) &= \cdot(e,x^n) = \{x\} \quad \text{and}\\ *(f(x_1,\ldots,x_n),y) &= *(\cdot(e,x_1,\ldots,x_n),y)\\ &= \cdot(\cdot(e,x_1,\ldots,x_n),y^n)\\ &= \cdot(e,\cdot(x_1,y^n),\ldots,\cdot(x_n,y^n))\\ &= \cdot(e,*(x_1,y),\ldots,*(x_n,y))\\ &= f(*(x_1,y),\ldots,*(x_n,y)),\end{aligned}$$

which yields that, the n-ary hyperoperation f defined on H satisfies all the conditions of Theorem 3.12.

($\Longleftarrow$) Assume that on a semihypergroup $(H,*)$ with a left scalar identity element there is an n-ary hyperoperation f defined on H satisfies all the conditions of Theorem 3.12.

By Theorem 3.12, the semihypergroup $(H,*)$ forms a diagonal semihypergroup of a Menger hyperalgebra $(H,\cdot)$ of rank n under an $(n+1)$-ary hyperoperation $\cdot$, which is given by:

$$\cdot(x,y_1,\ldots,y_n) = *(x,f(y_1,\ldots,y_n)) \tag{3.2}$$

for all $x,y_i\in H, i=1\ldots,n$.

Suppose that an element $e\in H$ is a left scalar identity element of $(H,*)$ Indeed, for every $x\in H$, we obtain

$$\cdot(e, x^n) = *(e, f(x^n)) = *(e, x) = \{x\}.$$

Consequently, the element e is also a left scalar diagonal unit of the Menger hyperalgebra $(H, \cdot)$ of rank n. □

Now, we give an example that satisfies Theorem 3.12 and Corollary 3.13.

Example 3.14. Let H be the closed interval $[0, 1]$. Then $(H, *)$ forms a semihypergroup under the binary hyperoperation $*$ which is defined on H by

$$*(x, y) = \{x\} \quad \text{for all } x, y \in H.$$

It is evident that each element of H is a right scalar identity element. Now, we define an n-ary hyperoperation f on H by

$$f(x_1, \ldots, x_n) = \{y \in H \mid y = \min\{x_1, \ldots, x_n\}\}$$

for all $x_i \in H, i = 1, \ldots, n$. Indeed, for every $x, y_i \in H, i = 1, \ldots, n$, we have

$$f(x^n) = \{y \in H \mid y = \min\{x, \ldots, x\}\} = \{x\} \quad \text{and}$$
$$*(f(x_1, \ldots, x_n), y) = f(x_1, \ldots, x_n) = f(*(x_1, y), \ldots, *(x_n, y)).$$

It follows that the n-ary hyperoperation f defined on H satisfies all the conditions of Theorem 3.12. By Corollary 3.13, the semihypergroup $(H, *)$ forms a diagonal semihypergroup of a Menger algebra $(H, \cdot)$ of rank n with each element is a right scalar diagonal unit, where the $(n + 1)$-ary hyperoperation is given as (3.2).

4 Conclusion remarks

In this paper, we constructed the so-called *diagonal semihypergroups* by using the algebraic hyperstructure of Menger hyperalgebras of rank n. Hence, some connections between their algebraic hyperstructures were arisen. In particular, we showed that some elements on diagonal semihypergroups which satisfied some necessary conditions in Proposition 3.8 can be acted as right (left) scalar identity elements. However, the scalar diagonal unit of diagonal semigroups is unique. Furthermore, we characterized that the semihypergroups with several specific conditions can be formed as diagonal semihypergroups of some Menger hyperalgebras of rank n.

References

[1] T. Changphas and B. Davvaz, *Properties of hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math., **33**, 425–432, (2014).

[2] S. Chaopraknoi and N. Tripop, *Regularity of semihypergroups of infinite matrices*, Thai J. Math., **4**, 7–14, (2006).

[3] P. Corsini and V . Leoreanu, Applications of hyperstructures theory, *Kluwer Academic Publishers, Dordrecht* (2003).

[4] J. Daengsaen and S. Leeratanavalee, *Regularities in ordered n-ary semihypergroups*, Mathematics, (16) **9**, Article ID: 1857, (2021).

[5] J. Daengsaen, S. Leeratanavalee, and B. Davvaz, *On minimal and maximal hyperideals in n-ary semihypergroups*, Mathematics, **8**, Article ID: 1656, (2020).

[6] B. Davvaz and V. Leoreanu-Fotea, *Hyperring theory and applications*, Dordrecht: Kluwer Academic Publishers, 2003.

[7] K. Denecke, *Menger algebras and clone of terms*, East-West J. Math., **5**, 179–193, (2003).

[8] K. Denecke, and H. Hounnon, *Partial Menger algebras of terms*, Asian-Eur. J. Math., **14**, Article ID: 2150092, (2021).

[9] W.A. Dudek, and V.S. Trokhimenko, Algebras of multiplace functions, *Walter De Gruyter, Berlin* (2012).

[10] W.A. Dudek and V.S. Trokhimenko, *Congruences on Menger algebras*, Commun. Algebra, **42**, 3407–3426, (2014).

[11] W.A. Dudek and V.S. Trokhimenko, *Menger algebras of k-commutative n-place functions*, Georgian Math. J., **28**, 355–361, (2021).

[12] W.A. Dudek and V.S. Trokhimenko, *On σ-commutativity in Menger algebras of n-place functions*, Commun. Algebra, **93**, 375–386, (2017).

[13] W.A. Dudek and V.S. Trokhimenko, *On Some Subtraction Menger algebras of Multiplace Functions, Semigroup Forum*, **93**, 375–386, (2016).

[14] W.A. Dudek and V.S. Trokhimenko, *Subtraction Menger algebras*, Semigroup Forum, **85**, 111–128, (2012).

[15] D. Heidari and B. Davvaz, *On ordered hyperstructures*, UPB Sci. Bull. A: Appl. Math. Phys., **73**, 85–96, (2011).

[16] T. Kumduang and S. Leeratanavalee, *Left translations and isomorphism theorems for Menger algebras of rank n*, Kyungpook Math. J., **61**, 223–237, (2021).

[17] T. Kumduang and S. Leeratanavalee, *Menger hyperalgebras and their representations*, Commun. Algebra, **49**, 1513–1533, (2021).

[18] T. Kumduang and S. Leeratanavalee, *Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems*, Symmetry, **13**, Article ID: 558, (2021)

[19] F. Marty, *Sur une generalization de la notion de groupe*, In 8th congress Math. Scandinaves, 45–49, (1934).

[20] K. Menger, *Superassociative systems and logical functors*, Math. Ann., **157**, 278–295, (1946).

[21] K. Menger, *The algebra of functions: Past, Present, future*, Rendiconti Di Matematica, **20**, 409–430, (1961).

[22] A. Nongmanee and S. Leeratanavalee, *Quaternary rectangular bands and representations of ternary semigroups*, Thai J. Math., **20**, 729–745, (2022).

[23] A. Nongmanee and S. Leeratanavalee, *v-Regular ternary Menger algebras and left translation of ternary Menger algebras*, Mathematics, **9**, Article ID: 2691, (2021).

[24] A. Nongmanee and S. Leeratanavalee, *Representations, translations and reductions for ternary semihypergroups*, Axioms, **11**, Article ID: 266, (2022).

[25] A. Nongmanee and S. Leeratanavalee, *Ternary Menger algebras: a generalization of ternary semigroups*, Mathematics, **9**, Article ID: 553, (2021).

[26] M. Novak, *On EL-semihypergroups*, Eur. J. Comb., **44**, 274–286, (2015).

[27] V.S. Trokhimenko, *Menger algebras of idempotent n-ary operations*, Stud. Sci. Math. Hung., **55**, 260–269, (2018).

[28] V.S. Trokhimenko, *v-Regular Menger algebras*, Algebra Univers., **38**, 150–164, (1997).

[29] K. Wattanatripop and T. Changphas, *The Menger algebras of terms induced by order-decreasing transformation*, Commun. Algebra, **49**, 3114–3123, (2021).

[30] K. Wattanatripop, T. Kumduang, T. Changphas, T. and S. Leeratanavalee, *Power Menger algebras of terms induced by order-decreasing transformations and superpositions*, Int. J. Math. Comput. Sci., **16**, 1697–1707, (2021).

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