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On C-SMALL AND C-SUPPLEMENT SUBMODULES

M. Akbari Gelvardi and Y. Talebi

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Abstract Let R be an arbitrary ring, and Z be a submodule of an R-module W. Z is called C-small in W denoted by $Z \ll_c W$ if any proper submodule of Z is small in W. We characterize C-small submodules of an R-module W and classify C-small submodules in finite $\mathbb{Z}$-modules, and we obtained some related results. Due to definition of C-small, we define C-supplement submodule of a module. We also provide some results in C-supplement modules related to other concepts.

1 Introduction

In this paper, R is an associative ring with identity, and all modules are unitary right R-module. We use the notation $\subseteq$, $\leq$ and $<$ to show inclusion, $submodule$ and proper submodule, respectively, and $V \ll_{\oplus} W$ means that V is a direct summand (DS) of W.

A submodule G of W is small in W denoted by $G \ll W$ if $G + Z = W$ for any $Z \leq W$ implies $Z = W$. A module W is said hollow if $G < W$ implies $G \ll W$. More details about small submodules can be found in [2, 8, 9]. The concept of small submodules has been extended by some researchers, for this see [11, 3, 12]. A submodule Z of a module W is called C-small, denoted by $Z \ll_c W$ if $G < Z$ implies $G \ll W$. By the definition, every small submodule is C-small, but the converse is not true.

For two submodules V, G of a module W, V is called a supplement of G if V is minimal with respect to the property $W = V + G$, equivalently $W = V + G$ and $V \cap G \ll V$. V is weak supplement of G if $V + G = W$ and $V \cap G \ll W$. A module M is called supplemented (weakly supplemented) if every submodule of M has a supplement (weak supplement) in W and W is called amply supplemented if $X + Y = W$ for two submodules X and Y, then X has a supplement in W, contained in Y. To see more details about supplemented modules, see [10, 5, 4] . In this paper, we also extend the concept of supplemented modules by defining C-supplemented modules.

For a module W, let $S \leq T \leq W$. If $T/S \ll W/S$, then S is called cosmall submodule of T in W. The submodule T of W is called coclosed in W if T has no proper cosmall submodule.

In section 2, first we give the definition of C-small submodules and some properties and then the related results to other concepts. In section 3, we define the concept of C-supplement, amply C-supplement and weak C-supplement submodules. Any supplement submodule is C-supplement but the converse is not true, and we investigate the relation between C-supplemented, amply C-supplemented and weakly C-supplemented modules. In section 4, we obtain a classification of C-small modules in a finite $\mathbb{Z}$-module and show that every finite $\mathbb{Z}$-module is a direct sum of its C-small submodules.

2 C-small submodules and projective modules

In this section, first we define C-small submodules as a generalization of small submodules and show the related results with projective modules.

Definition 2.1. Let R be a ring and Z be submodule of an R-module W. Z is called C-small submodule in W, denoted by $Z \ll_c W$, if $G < Z$ implies $G \ll W$.

Under the above definition, if Z is a small submodule of W, then Z is also a C-small submodule of W. But the inverse is not always true. To see this, let W be an R-module and $G < V \ll W$ and $G + S = W$ for some $S \leq W$. Since $G < V$ we have $V + S = W$ that is $S = W$. So $G \ll W$ and therefore $V \ll_c W$. Now let $\mathbb{Z}_n$ be the ring of integers modulo n and $W = \mathbb{Z}_{12}$ be a $\mathbb{Z}$-module. We have $A = \{0, 3, 6, 9\} \leq W$. Since $\{0, 6\}$ and $\{0\}$ are small in W, so $A \ll_c W$. But let $B = \{0, 4, 8\} < W$. We see that $A + B = W$. Therefore A is not small in W.

Example 2.2. (a) Every proper submodule of hollow modules is C-small. For all integers n and prime number p, submodules of $\mathbb{Z}_{p^n}$ is C-small.
(b) Let $W = \mathbb{Z}_{18}$. Submodules $\{0\}$, $\{0, 9\}$, $\{0, 6, 12\}$ and $\{0, 2, 4, 6, 8, 10, 12, 14, 16\}$ are C-small submodules and {0,3,6,9,12,15} is not a C-small submodule for $\{0, 9\} < \{0, 3, 6, 9, 12, 15\}$ and $\{0, 9\}$ is not small in W.

Proposition 2.3. *Let W be a module and $Z_1 \ll_c W$ and $Z_2 \ll_c W$. Then $Z_1 \cap Z_2 \ll_c W$.*

Proof. Let $G \leq Z_1 \cap Z_2$. Then $G \leq Z_1$. So if $G = Z_1$, then it is C-small and if $G < Z_1$, then $G \ll W$ and therefore $G \ll_c W$. □

Proposition 2.4. *Let W and V be two R-modules and $f : W \longrightarrow V$ be an R-homomorphism. If $Z \ll_c W$, then $f(Z) \ll_c f(W)$. In particular, if $Z \ll_c W \leq V$ then $Z \ll_c V$.*

Proof. Let $G < f(Z)$. Then there exists $S < Z$ such that $f(S) = G$. Now assume $G + H = f(S) + H = V$ for some $H \leq Z$. Then $S + f^{\leftarrow}(H) = W$. Since $Z \ll_c W$, so $S \ll W$, this implies $S \leq W = f^{\leftarrow}(H)$. So we have $f(S) \leq H$ and $f(S) = G$. Therefore $V = H$. □

Proposition 2.5. *Let $W = W_1 \oplus W_2$ and V be a submodule of W such that $(V + W_1)/W_1 \ll_c W/W_1$ and $W = V + W_2$. Then $(V + W_1)/V \ll_c W/V$.*

Proof. Let V be a $submodule$ of W such that $(V + W_1)/W_1 \ll_c W/W_1$ and $W = V + W_2$. We have $W/W_1 \cong W_2 \longrightarrow W_2/(W_2 \cap V) \cong (W_2 + V)/V = W/V$. So $(V + W_1)/W_1$ is mapped onto $(V + W_1)/V$. Now since $(V + W_1)/W_1 \ll_c W/W_1$, we have $(V + W_1)/V \ll_c W/V$. □

Proposition 2.6. *Let W be a module and $Z \leq V \leq W$. If $V \ll_c W$, then $Z \ll_c W$ and $V/Z \ll_c W/Z$.*

Proof. Let $V \ll_c W$. Since $Z \leq V$, it is small in W or it is equal to V. Both cases show that $Z \ll_c W$. Now we show that $V/Z \ll_c W/Z$. Let $G/Z < V/Z$ and $G/Z + S/Z = W/Z$ for some $S/Z \leq W/Z$. This implies that $G + S = W$. Since $G < V$ and $V \ll_c W$, then $S = W$ and $S/Z = W/Z$. So $G/Z \ll W/Z$ and therefore $V/Z \ll_c W/Z$. □

Proposition 2.7. *Let W be a module and $Z \leq V \leq W$. If $Z \ll V$ and $V/Z \ll_c W/Z$, then $V \ll_c W$.*

Proof. Let $G < V$ and $G + S = W$ for some $S \leq W$. Then $(G + S)/Z = W/Z$ and $(G + Z)/Z + (S + Z)/Z = W/Z$. Since $Z \ll V$, we have $G + Z \neq V$. So $(G + Z)/Z < V/Z$. Hence $(S + Z)/Z = W/Z$ (by $V/Z \ll_c W/Z$) and $S + Z = W$. Since $Z \ll V$, $Z \ll W$ and so $S = W$ and then $V \ll_c W$. □

Corollary 2.8. *Let W be a module and $H, Z \leq W$. If $H + Z \ll_c W$, then $Z \ll_c W$ and $H \ll_c W$.*

Corollary 2.9. *Let W be a module and $Z < V \leq W$. Then $V \ll_c W$ if and only if $Z \ll V$ and $V/Z \ll_c W/Z$.*

Definition 2.10. Let W and V be two R-modules and $f : W \longrightarrow V$ be an R-epimomorphism. f is called C-small in case $Ker f \ll_c W$.

Proposition 2.11. *Let W be an R-module and $Z \leq W$. The following statements are equivalent:*

(1) $Z \ll_c W$

(2) *The natural map $p_z : W \longrightarrow W/Z$ is C-small epimorphism.*

(3) *For every $G < Z$, every R-module V and every $h \in Hom_R(V, W)$, $(Imh) + G = W$ implies $Imh = W$*

Proof. (1) $\Rightarrow$ (2) It is clear. (2) $\Rightarrow$ (3) Since p_z is a C-small epimorphism and $Kerp_z = Z$, this implies $Z \ll_c W$. So for every $G < Z$ we have that $G \ll W$. Now if $(Imh) + G = W$, then $Imh = W$ for every R-module V and $h \in Hom_R(V, W)$. (3) $\Rightarrow$ (1) If we assume $V = W$ and $h : V \longrightarrow W$ be an inclusion map, then $G \ll W$ and so $Z \ll_c W$. □

Corollary 2.12. *Let $g : W \longrightarrow V$ be an R-epimorphism. If for some $0 \neq G \leq V$, $f^{\leftarrow}(G) \ll_c W$, then for all homomorphism h, if gh is epic, then h is epic.*

Lemma 2.13. *Let W be a module and $W = A + B = (A \cap B) + C$ for submodules $A, B, C \leq W$. Then $W = (B \cap C) + A = (A \cap C) + B$.*

Proof. See [6, Lemma 1.2]. □

Lemma 2.14. *Let W be a module and $A, B, C \leq W$ such that $A \cap B \leq A \cap C$. If $G < (A \cap C)/(A \cap B)$, then there exists C' such that $B \leq C' < C$ and $G = (A \cap C')/(A \cap B)$.*

Proof. Let $G < (A \cap C)/(A \cap B)$. Then $G = Z/(A \cap B)$ such that $A \cap B \leq Z$ and $Z < A \cap C$. Now we consider $Z + B = C'$. Then $A \cap (Z + B) = Z + (A \cap B) = Z$. This implies $G = Z/(A \cap B) = (A \cap (Z + B))/(A \cap B) = (A \cap C')/(A \cap B)$ such that $B \leq C' < C$. □

Proposition 2.15. *Let W be a module such that $W = A + B$ for $A, B \leq W$. If $B \leq C$ and $C/B \ll_c W/B$, then $(A \cap C)/(A \cap B) \ll_c W/(A \cap B)$.*

Proof. Let $Z < (A \cap C)/(A \cap B)$. Then by Lemma 2.14, there exists a $C' \leq W$ such that $B \leq C' < C$ and $Z = (A \cap C')/(A \cap B)$. Now let $W/(A \cap B) = (A \cap C')/(A \cap B) + X/(A \cap B)$ for some $A \cap B \leq X \leq W$. Then $W = (A \cap C') + X$ .By Lemma 2.13 , $W = C' + (A \cap X)$. Since $C/B \ll_c W/B$, $W = B + (A \cap X)$. Again by Lemma 2.13 , $W = X + (A \cap B)$. Hence $W = X$. Thus $(A \cap C)/(A \cap B) \ll_c W/(A \cap B)$. □

Proposition 2.16. *Let W be a module and $Z \leq V \leq W$ such that $V \leq_{\oplus} W$. If $Z \ll_c W$, then $Z \ll_c V$.*

Proof. Let $G < Z$ and $G + S = V$ for some $S \leq V$. Since V is a DS, there exists a $H \leq W$ and $V \oplus H = W$. So we have $G + S + H = W$. Since $Z \ll_c W$ and $G < Z$, $S + H = V + H = W$. Therefore $S = V$ and $Z \ll_c V$. □

Proposition 2.17. *Let W be an R-module and $Z \ll_c W$. If there exists $G < Z$ and W/G is indecomposable, then W is indecomposable.*

Proof. Let $W = A \oplus B$ for some $A, B \leq W$. So $(A + G)/G + (B + G)/G = W/G$. We show that $(A+G) \cap (B+G) = G$. Since $G \leq W$, we have $G = A' \oplus B'$ for some $A' \leq A$ and $B' \leq B$. Let $x \in (A + G) \cap (B + G)$. Thus $x = a + g_1 = b + g_2$ for some $a \in A, b \in B, g_1, g_2 \in G$. Now since $G = A' \oplus B'$, we have $g_1 = a_1 + b_1$ and $g_2 = a_2 + b_2$ where $a_1, a_2 \in A'$ and $b_1, b_2 \in B'$. Therefore $a + a_1 + b_1 = b + a_2 + b_2$ and $a + a_1 - a_2 = b + b_2 - b_1$. Since $A \cap B = 0$, we conclude $a = a_2 - a_1 \in A'$. Therefore $a \in G$ and so $x \in G$. This implies that $(A + G)/G \oplus (B + G)/G = W/G$. Since W/G is indecomposable, we may assume $A + G = W$. Since $G < Z$ and $Z \ll_c W$, $G \ll W$. Thus $A = W$ and $B = 0$. This implies W is indecomposable. □

Remark 2.18. Let W be a module. If $V \leq W$ is hollow module, then V is C-small submodule in W but the converse is not true. To see this, let $V \leq W$ a hollow module. If $G < V$, then $G \ll V$ and so $G \ll W$. Therefore $V \ll_c W$. For the converse, let $W = \mathbb{Z}_{36}$ and $A = \{0, 6, 12, 18, 24, 30\}$. Then $A \ll_c W$ but it is not hollow module for we have $B = \{0, 12, 24\} < A$ and $C = \{0, 18\} < A$ and $B + C = A$.

Theorem 2.19. *Let W be a module. Every C-small submodule of W which is a DS, is a hollow module.*

Proof. Let $V \ll_c W$ be a DS in W and $G < V$. Since $V \ll_c W$ we have that $G \ll W$ and since V is a DS in W, $G \ll V$. Therefore V is hollow. □

Proposition 2.20. *Let W be a module and $Z \leq W$. Then the following are equivalent:*

(1) *There is a decomposition $W = G \oplus G'$ with $G \leq Z$ and $G' \cap Z \ll_c W$;*

(2) *There is an idempotent $e \in End(W)$ such that $(W)e \leq Z$ and $(Z)(1-e) \ll_c (W)(1-e)$;*

(3) *There is a DS G of W such that $G \leq Z$ and $Z/G \ll_c W/G$.*

Proof. (1) $\Longrightarrow$ (2) For $W = G \oplus G'$, there exists an idempotent $e \in End(W)$ such that $(W)e = G$ and $(W)(1-e) = G'$. Since $G \leq Z$, we conclude $(Z)(1-e) \leq Z \cap (W)(1-e) = G' \cap Z \ll_c W$. Since G' is a DS in W and $G' \cap Z < G'$, we have $G' \cap Z \ll_c G'$ and so $(Z)(1-e) \ll_c (W)(1-e)$.
(2) $\Longrightarrow$ (3) We can choose $G = (W)e$. So $G \leq Z$. Then $W = G \oplus (W)(1-e)$ and since $(Z)(1-e) \ll_c (W)(1-e)$, we have $Z/G \ll_c W/G$.
(3) $\Longrightarrow$ (1) There exists $G \oplus G' = M$. So $Z = G \oplus (G' \cap Z)$ by modularity. Also we have $G' \cap Z \cong Z/G \ll_c W/L \cong G'$. Thus $G' \cap Z \ll_c G'$ and so $G' \cap Z \ll_c W$.

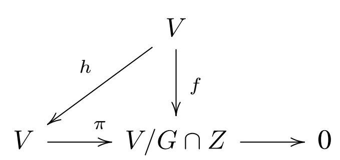
□

Proposition 2.21. *Let W be a module and $V \leq W$. If V is cyclic and has a unique maximal submodule, then $V \ll_c W$.*

Proof. Let V be cyclic and has a unique maximal submodule G. Then V is finitely generated. Now let $F < V$ and $F + S = V$ for some $S \leq V$. Since V is finitely generated, every proper submodule of V is contained in a maximal submodule and we have that $G + S = V$. By the same way S can not be proper in V. Therefore $S = V$ and $F \ll V$. So $F \ll W$. This implies $V \ll_c W$. □

Proposition 2.22. *Let W be a module and $V \leq W$. If $End(V)$ is local ring and V is self-projective, then $V \ll_c W$.*

Proof. Let $G < V$ and $G + Z = V$ for some $Z \leq V$. Now since $V = G + Z$, for every $v \in V$, we have $v = g + z$ for some $g \in G$ and $z \in Z$. Let $f : V \to V/G \cap Z$ be defined by $f(n) = Z + G \cap Z$. It is easy to see that f is well-defined and is a homomorphism. Since V is self-projective, there exists a homomorphism $h : V \to V$ such that the following diagram



commutes, where $\pi : V \to V/G\cap Z$ is a natural epimorphism. Since π is natural epimorphism and the diagram commutes, $h(v) + G \cap Z = z + G \cap Z$. This implies $h(V) \leq Z$. Let $x \in Z$, then $\pi(x - h(x)) = \pi(x) - \pi o h(x) = x + G \cap Z - f(x) + G \cap Z$. This implies $x - h(x) \in ker\pi$. So $Z = h(V) + G \cap Z$. Now since $V = G + Z$, then $V = G + h(V) + G \cap Z$. But $G \cap Z \leq G$. Therefore $V = G + h(V)$. Hence $h(V)$ is not a small submodule of V. Thus $h \notin RadEnd(V)$ that is h is a isomorphism. This implies $Z \leq V = h(V)$ and so $Z = V$. Therefore $G \ll V$ and so $G \ll W$. This shows that $V \ll_c W$.

□

Theorem 2.23. *Let W be a module. If every proper submodule of W is contained in a maximal submodule of W and if $G/RadW < W/RadW$ is a C-small submodule in $W/RadW$, then $G/RadW$ is a DS.*

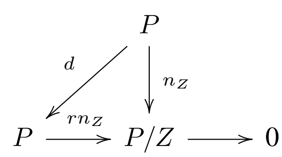
Proof. Let Z be a proper submodule of W that is contained in a maximal submodule say, H. Since we have $RadW = \cap\{X \leq W | X\ is\ maximal\ in\ W\}$, this implies $RadW \leq H$. So $Z + RadW \leq H \neq W$. By arbitrary choice of Z, $RadW \ll Z$. This shows that $W/RadW$ has no non-zero small submodule. Cause if $G/RadW$ is small, then $G \ll W$. To see this, let $G + Z = W$. Then $(G/RadW) + (Z + RadW)/RadW = W/RadW$, that implies $Z + RadW = W$ and $Z = W$ and it is a contraction to proper assumption of Z. So $G/RadW$ is a C-small submodule of $W/RadW$ which is not Small. Therefore there exists a $S/RadW < W/RadW$ and $G/RadW + S/RadW = W/RadW$. Since $G/RadW \cap S/RadW \leq G/RadW$, we have $G/RadW \cap S/RadW = 0$ and therefore $G/RadW \leq_{\oplus} W/RadW$. □

Proposition 2.24. *Let W be semisimple module. Then $Z \leq W$ is C-small if and only if Z is simple.*

Proof. Let $Z \leq W$ be a simple. So it is clear that $Z \ll_c W$. Conversely let $0 \neq Z \ll_c W$ and $G < Z$. Since W is semisimple, every submodule of W is a DS. Since $Z \ll_c W$, $G \ll W$ and so $G = 0$. □

Proposition 2.25. *In a projective R-module P with endomorphism ring $D = End(P)$, Let $e \in D$. If for every $G < Ime$, there exists $r \in J(D)$ such that $Imr = G$, then $Ime \ll_c P$.*

Proof. Let $G < Ime$. So there exists $r \in J(D)$ such that $Imr = G$. Let $G + Z = Imr + Z = P$ for some $Z \leq P$. Then we readily see if $n_Z : P \to P/Z$ is the natural epimorphism, $rn_Z : P \longrightarrow P/Z$ is epic. So we can choose $d \in D$ such that the diagram

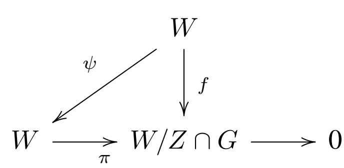


commutes. We have $(1 - dr)n_Z = 0$. But since $r \in J(D)$, $1 - dr$ is invertible and $n_Z = 0$. Therefore $Z = P$. So $G \ll P$ and then $Ime \ll_c P$.

□

Proposition 2.26. *Let W be a self-projective module and $V \leq W$. If for every $G < V$, $Hom(W, G) = 0$, then $V \ll_c W$.*

Proof. For proving that V is a C-small submodule in W, let $G < V$. By hypothesis, $Hom(W, G) = 0$. We show $G \ll W$. Let $G + Z = W$. Now we consider the following diagram



where $\pi : W \longrightarrow W/Z \cap G$ is the natural epimorphism and $f : W \longrightarrow W/Z \cap G$ is defines as follow. For every $w \in W$, $f(w) = g + z \cap G$ where $g \in G$ and $z \in Z$ such that $w = g + z$. Now we show $f = 0$. Let $f \neq 0$. Since W is self-projective, then there exists $\psi : W \longrightarrow W$ such that $\pi o \psi = f$. Now for $w \in W$, we have $(\pi o \psi)(w) = f(w)$, that is $\psi(w) + Z \cap G = g + Z \cap G$, where $w = g + z$ for some $g \in G$ and $z \in Z$. Now $\psi(w) - g \in Z \cap G \leq G$ which implies that $\psi(w) \in G$. Thus $\psi(W) \leq G$. But $Hom(W, G) = 0$. That is $\psi = 0$ which is a contradiction with $\pi o \psi = f$ cause $f \neq 0$. So $f = 0$. This implies $G \leq Z$ and hence $W = G + Z = Z$. Therefore $G \ll W$ and so $V \ll_c W$. □

3 C-supplemented modules

Let G and V be submodules of an R-module W. V is called a supplement of G in W if it is minimal with respect to the property $W = V + G$, equivalently $W = V + G$ and $V + G \ll V$.

Let W be a module and $V, V' \leq W$. V' is called C-supplement of V if $V + V' = W$ and $V \cap V' \ll_c V'$. V' is called weak C-supplement of V if $V + V' = W$ and $V \cap V' \ll_c W$. A submodule V of W is called a C-supplement (weak C-supplement) submodule, if there exists a submodule Z of W such that V is a C-supplement (weak C-supplement) of Z in W.

Module W is called C-supplemented (weakly C-supplemented) if every submodule of W has a C-supplement (weak C-supplement) in W and W is called amply C-supplemented if $W = A + B$ implies A has a C-supplement Z contained in B.

For two submodules $Z \leq V \leq W$, we say Z is a C-cosmall submodule of V in W if $V/Z \ll_c W/Z$. The submodule V of W is called C-coclosed in W if V has no proper C-cosmall submodule, equivalently $V/Z \ll_c W/Z$ implies $V = Z$ for any submodule $Z \leq V$.

Proposition 3.1. *Let W be a module and $Z \leq V \leq W$. If V is a C-coclosed submodule of W and $Z \ll_c W$, then $Z \ll_c V$.*

Proof. Let $G < Z$ and $G+S = V$ for some $S \leq V$. Suppose $V'/S < V/S$ for some $S \leq V' \leq V$ and $V'/S + Y/S = W/S$ for some $S \leq Y \leq W$. So $V' + Y = W$. This implies $G + S + Y = W$ and so $S + Y = W = Y$.Hence $V/S' \ll_c W/S$ and this implies $V = S$. □

Corollary 3.2. *Let W be a weakly C-supplemented module. Then every C-coclosed submodule of W has a C-supplemented.*

Corollary 3.3. *Let W be a module and every submodule of W is C-coclosed. Then W is C-supplemented if and only if W is weakly C-supplemented.*

Remark 3.4. (1) It is easy to see that if A is a C-coclosed submodule of W and $B \leq A$, then A/B is C-coclosed in W/B. For this $(A/B)/(C/B) \ll_c (W/B)/(C/B)$ where $B \leq C \leq A$, then $A/C \ll_c W/C$ and so $A = C$.
(2) Let W be a module and $G \leq W$ a supplement submodule, then for every $H \leq G$, G/H is a supplement $submodule$ in W/H. To see this, since G is supplement in W, there exists $Z \leq W$ and $W = G + Z$ and $G \cap Z \ll_c G$. Therefore $G/H + (Z + H)/H = W/H$. Now $G/H \cap (Z+H)/H = G \cap (Z+H)/H = H + (G \cap Z)/H$. Now Let $\pi : L \to G/H$ be a natural epimorphism. Since $G \cap Z \ll_c L$ so $\pi(Z \cap G) = H + (Z \cap G) \ll_c G/H$.
(3) For a module W and $Z \leq G \leq W$, if Z is C-coclosed in W, it is clear that Z is C-coclosed in G and when G is a C-supplement $submodule$ in W, the inverse is also true that is if Z is C-coclosed in G, then Z is C-coclosed in W. For proving this let $Z/H \ll_c W/H$ for some $H \leq Z \leq W$. By (2) G/H is a C-supplement in W/H. So we have $Z/H \ll_c G/H$. Hence $Z = H$.

Proposition 3.5. *Every DS of an amply C-supplemented module is amply C-supplemented.*

Proof. Let W be a module and $W = Z \oplus Z'$. Now suppose $Z = C + D$, then $W = C + (D \oplus Z')$. Since W is amply C-supplemented, $W = E + (D \oplus Z')$ and $E \cap (D \oplus Z') \ll_c E$ for some $E \leq C$. Thus $Z = Z \cap W = E + D$ and $E \cap D = E \cap (D \oplus Z') \ll_c E$. So Z is amply C-supplemented. □

Theorem 3.6. *Let W be a module and $T \leq W$. Then the following are equivalent:*

(1) *There is decomposition $W = Q + Q'$ with $Q \leq T$ and $T \cap Q' \ll_c Q'$;*

(2) *There is an idempotent $e \in End(W)$ such that $(W)e \leq T$ and $(T)(1-e) \ll_c (W)(1-e)$;*

(3) *There is $Q \leq_\oplus W$ such that $Q \leq T$ and $T/Q \ll_c W/Q$;*

(4) *T has a C-supplement D in W such that $T \cap D \leq_\oplus T$.*

Proof. (1) $\Longrightarrow$ (2) For $W = Q + Q'$, there exists an idempotent $e \in End(W)$ such that $(W)e = Q$ and $(W)(1-e) = Q'$. Now since $Q \leq T$, we have $(T)(1-e) \leq T \cap (W)(1-e) \ll_c (W)(1-e)$.
(2) $\Longrightarrow$ (3) We can take $Q = (W)e$. Then $W = Q \oplus (W)(1-e)$ and since by $modularity$ $T = Q \oplus ((W)(1-e) \cap T)$, we have $T/Q \ll_c W/Q$.
(3) $\Longrightarrow$ (4) Let $Q \leq_\oplus W$ and $Q \leq T$. So $W = Q \oplus Q'$ and $T = Q \oplus (Q' \cap T)$. Take $D = Q'$. Then $T \cap D \leq_\oplus T$ and $T \cap D \cong T/Q \ll_c W/Q \cong D$. Thus $T \cap D \ll_c D$.
(4) $\Longrightarrow$ (1) Let D be a C-supplement of T in W and also $T = Q \oplus (T \cap D)$ for some $Q \leq T$. Then $W = T + D = Q + (T \cap D) + D = Q + D$ and $Q \cap D = (Q \cap T) \cap D = Q \cap (T \cap D) = 0$. So $Q \leq_\oplus W$ and also Since D is a C-supplement of T, $T \cap D \ll_c D$.

□

Proposition 3.7. *Let W be a module such that every submodule of W is C-supplemented. Then W is amply C-supplemented.*

Proof. Let $A, B \leq W$ and $W = A + B$. Then since $A \cap B$ is a submodule of A and A is C-supplemented, there exists $X \leq A$ and $(A \cap B) + X = A$ and $(A \cap B) \cap X = B \cap X \ll_c X$. Therefore $W = A + B = (A \cap B) + X + B = X + B$. So W is amply C-supplemented. □

Corollary 3.8. *Let R be a ring. Then every R-module is amply C-supplemented if and only if every R-module is C-supplemented.*

Proposition 3.9. *Let W be a π-projective module. Then W is a C-supplemented module, if and only if W is amply C-supplemented module.*

Proof. Let $W = Z + G$ for $G, Z \leq W$. Then there is $e \in End(W)$ such that $(W)e \leq Z$ and $(W)(1-e) \leq G$. Now suppose that $V \leq W$ be a C-supplement of Z in W. We have $W = (W)e + (W)(1-e) = (W)e + (Z+V)(1-e) \leq Z + (V)(1-e)$. So $W = Z + (V)(1-e)$ where $(V)(1-e) \leq G$. Now since $Z \cap V \ll_c V$ we have $Z \cap (V)(1-e) = (Z \cap V)(1-e) \ll_c (V)(1-e)$. Therefore W is amply C-supplemented. Converse is clear. □

Proposition 3.10. *Let W be a weakly C-supplemented module. Then*

(1) *Every C-coclosed submodule of W is weakly C-supplemented.*

(2) *Every factor module of W is weakly C-supplemented.*

Proof. (1) Let Z be a C-coclosed submodule of W and $V \leq Z$. Since W is weakly C-supplemented, there exists a $G \leq W$ and $Z + G = W$ and $Z \cap G \ll_c W$. Thus $Z = V + (Z \cap G)$. Also $V \cap (Z \cap G) = V \cap G \ll_c Z$.
(2) Let V be a submodule of W and $G/V \leq W/V$. Since W is weakly C-supplemented, there exists $Z \leq W$ such that $W = Z + G$ and $G \cap Z \ll_c W$. So we have $W/V = G/V + (Z+V)/V$. Now Let $\phi : W \to W/V$ be natural epimorphism. Then $G/V \cap (Z+V)/V = (V + G \cap Z)/V = \phi(G \cap Z) \ll_c W/V$. So W/V is weakly C-supplemented. □

4 finite $\mathbb{Z}$-module decomposition

In Examples 2.2 we showed some C-small submodule especially in finite $\mathbb{Z}$-module $\mathbb{Z}_n$. Next we classify all C-small submodules in $\mathbb{Z}_n$.

Theorem 4.1. *Let $\mathbb{Z}_n$ be the ring of integers modulo n and $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_t^{\alpha_t}$ where all of p_i are distinct prime and $\alpha_i \geq 0$. Then $k\mathbb{Z}_n \ll_c \mathbb{Z}_n$ if and only if $k = qp_1^{\beta_1} p_2^{\beta_2} ... p_t^{\beta_t}$, where $gcd(q, n) = 1$ and for any $1 \leq i \leq t, 1 \leq \beta_i \leq \alpha_i$, or $k = qp_1^{\alpha_1} p_2^{\alpha_2} ... p_{i-1}^{\alpha_{i-1}} p_{i+1}^{\alpha_{i+1}} ... p_t^{\alpha_t}$ where $gcd(q, n) = 1$.*

Proof. Let $W = \mathbb{Z}_n$. It is clear to see that every submodule G of W can be written as $G = kW$ where k is an integer and if $k = qp_1^{\beta_1} p_2^{\beta_2} ... p_t^{\beta_t}$, where $gcd(q, n) = 1$ and for any $1 \leq i \leq t, 1 \leq \beta_i \leq \alpha_i$, then $kW = k'M$ such that $k' = p_1^{\beta_1} p_2^{\beta_2} ... p_t^{\beta_t}$ where for any $1 \leq i \leq t, 1 \leq \beta_i \leq \alpha_i$. Now let G be a submodule of W, satisfying the first condition in necessity and $G + S = W$ for some $S \leq W$. Therefore there exists an integer f, such that $S = fW$ and $f = p_1^{\beta_1} p_2^{\beta_2} ... p_t^{\beta_t}$ where $1 \leq i \leq t, 0 \leq \beta_i \leq \alpha_i$. Since $G + S = W$, then for some $g \in G$ and $s \in S$ we have $g + s = 1$. By Theory of Numbers we can conclude $gcd(f, k') = 1$. This implies $f = 1$. Therefore $S = W$. So $G \ll W$ and $G \ll_c M$. Now let G satisfies the second condition, Then for every $D < L$, it easy to see that D satisfies the first condition. So $D \ll W$. Therefore $G \ll_c W$. Conversely let $G = kW$ with $k = qp_1^{\beta_1} p_2^{\beta_2} ... p_t^{\beta_t}$, where $gcd(q, n) = 1$ and for any $1 \leq i \leq t, 0 \leq \beta_i \leq \alpha_i$ and there is a $1 \leq j \leq t$ such that $\beta_j = 0$ and there is $1 \leq u \leq t$ such that $0 \leq \beta_u < \alpha_u$ and $u \neq j$. Then $S = sW$ with $s = p_1^{\alpha_1} p_2^{\alpha_2} ... p_j^{0} ... p_t^{\alpha_t}$ is proper a submodule of G. Now let $D = p_j^{\alpha_j} M$. Since $gcd(s, p_j^{\alpha_j}) = 1$, by theory of numbers we have $S + D = W$. So G is not C-small in W. □

Example 4.2. Let $W = \mathbb{Z}_{900}$. Since $900 = 2^2 \times 3^2 \times 5^2$, $A = (3^2 \times 5^2)W$, $B = (2^2 \times 5^2)W$, $C = (2^2 \times 3^2)W$, $D = (2^2 \times 3^2 \times 5)W$, $E = (2^2 \times 3 \times 5^2)W$, $F = (2 \times 3^2 \times 5^2)W$, $G = (2 \times 3 \times 5^2)W$, $H = (2 \times 3^2 \times 5)W$, $I = (2^2 \times 3 \times 5)W$, and $J = (2 \times 3 \times 5)W$, are C-small submodules, while A, B, C are not small in W.

Let p be a prime number and W be a finite $\mathbb{Z}$-module. W is called p-module if for every $x \in W$, there exists a positive integer n such that $x^{p^n} = p^n x = 0$.

Lemma 4.3. *For every prime number p, finite $\mathbb{Z}$-module W has a submdule H and H is p-module and maximal with this property.*

Proof. If $W = 0$ or $p \nmid |W|$, then 0 is p-$submodule$ and it is maximal as a p-module. Let $p \mid |W|$. Therefore there is $H \leq W$ and $|H| = p$. Since W is finite, there is $S \leq W$ and S contains H such that S is maximal as a p-module. □

Proposition 4.4. *Let W be finite $\mathbb{Z}$-module and $G \leq W$. If G is cyclic p-module, then $G \ll_c W$.*

Proof. Let $|G| = P^n$. Since G is cyclic, then $G =< s >$ for some $s \in G$. Now let $D < G$ and $D + T = G$ for some $T \leq G$. Since G is P-module, D and T are also p-modules. Now $s = d + t$ for $d \in D, t \in T$. Since $o(s) = p^n$, we have $o(d + t) = p^n$. Since $D < G$, $|D| < p^n$ and let $|D| = p^{n_1}, n_1 < n$ and let $|T| = p^{n_2}$. If $n_2 \leq n_1$, then $o(d + t) = p^{n_1}$ and it is contradiction. If $n_1 < n_2$, then $o(d + t) = p^{n_2} = o(s) = p^n$. Therefore $n_2 = n$ and $T = G$. So $D \ll G$. Thus $D \ll W$ and $G \ll_c W$. □

Example 4.5. Let $W = \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. For the prime 3, $A = \mathbb{Z}_9 \times \mathbb{Z}_0 \times \mathbb{Z}_0$ is a 3-module and cyclic for A is generated by $(1, 0, 0)$ and obviously A is maximal as a 3-module. Therefore A is C-small. Now let $B = \mathbb{Z}_0 \times \mathbb{Z}_4 \times \mathbb{Z}_0$, $C = \mathbb{Z}_0 \times \mathbb{Z}_0 \times \mathbb{Z}_2$ and $D = \mathbb{Z}_0 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. For every $x \in B$, $y \in C$ and $s \in D$, $2^2x = 2^2s = 2y = 0$. So they are 2-modules. But D is not cyclic. So B and C are C-small.

Lemma 4.6. *Let p be a prime number and W be a finite $\mathbb{Z}$-module. If W is p-module, then W is direct sum of cyclic p-modules.*

Proof. Let $|W| = p^n$. If $n = 1$, then W is cyclic module and the result is obvious. Now let $a \in W$ such that for every $b \in W$, $o(b) \leq o(a)$. Then there is a submodule G of W such that $W =< a > \oplus G$. Clearly G is p-module and $|G| < |W|$. So by induction G is direct sum of Cyclic p-modules. Therefore W is direct sum of cyclic p-modules. □

Theorem 4.7. *Let W be finite $\mathbb{Z}$-module. Then $W = S_1 \oplus S_2 \oplus ... \oplus S_n$ such that $S_i \ll_c W$ for all $1 \leq i \leq n$.*

Proof. Let W be finite $\mathbb{Z}$-module. If $W = 0$ then result is obvious. Let $|W| = p_1^{\alpha_1} p_2^{\alpha_2} ... p_t^{\alpha_t}$ where p_i's be distinct prime numbers and α_i's be positives integer. By Lemma 4.3 for every $i = 1, 2, ..., t$, W has a p-module D and D is maximal with this property. Now let for every $i = 1, 2, ..., t$, $|D_i| = p_i^{\alpha_i}$. It is easy to see that $D_i \cap (D_1 + ... + D_{i-1} + D_{i+1} + ... + D_t) = 0$ for $1 \leq i \leq t$. Therefore $|D_1 + ... + D_t| = p_1^{\alpha_1} p_2^{\alpha_2} ... p_t^{\alpha_t} = |W|$. This implies $W = D_1 \oplus ... \oplus D_t$ which every D_i is a p-module. Now by Lemma 4.6 W is direct sum of cyclic submodules and now by proposition 4.4, $W = S_1 \oplus S_2 \oplus ... \oplus S_n$ where $S_i \ll_c W$ for any $1 \leq i \leq n$. □

Remark 4.8. In [1], authors have discussed chain conditions on small submodules, and in [7], authors have discussed chain conditions on non-small submodules. It will be interesting to discuss chain conditions on the class of small submodules and non-small submodules related to c-small submodules.

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Author information

M. Akbari Gelvardi, Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran.
E-mail: mohammad.akbarigl@gmail.com

Y. Talebi, Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran.
E-mail: talebi@umz.ac.ir

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