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On Generalized Skew Semi-Derivations and Commutativity in Prime Rings

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Abstract This paper demonstrates the commutativity of a prime ring under both skew semi-derivations and generalized skew semi-derivations satisfying some specific identities.

1 Introduction

Throughout this paper, P denotes a prime ring with its centre $Z(P)$ unless explicitly stated otherwise. For any elements $p_1, p_2 \in P$, $[p_1, p_2]$ represents the commutator of p_1 and p_2 and defined as $p_1p_2 - p_2p_1$. Also, the anti-commutator is denoted by $p_1 \circ p_2$ and defined as $p_1p_2 + p_2p_1$. A ring P is a prime ring if $p_1Pp_2 = 0$ implies that $p_1 = 0$ or $p_2 = 0$. Further, a ring P is said to be 2-torsion free when $2p_1 = 0$ implies $p_1 = 0$ for all $p_1 \in P$. We use the fundamental commutator identities, namely, $[p_1, p_2z] = [p_1, p_2]z + p_2[p_1, z]$ and $[p_1p_2, z] = [p_1, z]p_2 + p_1[p_2, z]$ throughout this work. An additive mapping d from P to P is called a derivation if for all p_1 and p_2 in P, we have $d(p_1p_2) = d(p_1)p_2 + p_1d(p_2)$. On the other hand, a generalized derivation is an additive mapping $D : P \to P$ satisfies the condition: for all p_1 and p_2 in P and a derivation d, $D(p_1p_2) = D(p_1)p_2 + p_1d(p_2)$. In 1985, Leroy [11] introduced the concept of skew derivation as an additive map δ from P to P satisfies $\delta(p_1p_2) = \delta(p_1)p_2 + \phi(p_1)\delta(p_2)$ for all $p_1, p_2 \in P$ where ϕ represents an automorphism on P. In 1983, Bergan [6] introduced the concept of a semi-derivation in the context of a ring P. A semi-derivation is an additive map $h : P \to P$ satisfies $h(p_1p_2) = h(p_1)\mathfrak{g}(p_2) + p_1h(p_2) = h(p_1)p_2 + \mathfrak{g}(p_1)h(p_2)$, provided there exists a map $\mathfrak{g}$ such that $h(\mathfrak{g}(p_1)) = \mathfrak{g}(h(p_1))$ holds for all $p_1, p_2 \in P$. Further, an additive map $\mathcal{G} : P \to P$ is referred as a generalized semi-derivation if it satisfies $\mathcal{G}(p_1p_2) = \mathcal{G}(p_1)\mathfrak{g}(p_2) + p_1h(p_2) = \mathfrak{g}(p_1)\mathcal{G}(p_2) + h(p_1)p_2$, and $\mathcal{G}(\mathfrak{g}(p_1)) = \mathfrak{g}(\mathcal{G}(p_1))$ where $\mathfrak{g}$ is an arbitrary map, and h is a semi-derivation over P. If $\mathfrak{g}$ is the identity map, h corresponds to a derivation, whereas when $\mathfrak{g}$ is an automorphism on P, $\mathcal{G}$ becomes a generalized derivation. For more results, we refer [2, 3, 9].

In 2023, Aziz *et al.* [1] defined two new notions of derivation, which are as follows:

- A map $\mathfrak{f} : P \to P$ is called a skew semi-derivation if there exist a function $\mathfrak{g}$ and an automorphism ϕ on P such that for all $p_1, p_2 \in P$, the following conditions hold:

  (i) $\mathfrak{f}(p_1p_2) = \mathfrak{f}(p_1)\mathfrak{g}(p_2) + \phi(p_1)\mathfrak{f}(p_2) = \mathfrak{f}(p_1)\phi(p_2) + \mathfrak{g}(p_1)\mathfrak{f}(p_2)$;

  (ii) $\mathfrak{f}(\mathfrak{g}(p_1)) = \mathfrak{g}(\mathfrak{f}(p_1))$.

- Also, an additive map $\mathfrak{F} : P \to P$ is called a generalized skew semi-derivation if there exist a function $\mathfrak{g}$, an automorphism ϕ and a skew semi-derivation $\mathfrak{f}$ on P such that for all $p_1, p_2 \in P$, the following conditions hold:

  (i) $\mathfrak{F}(p_1p_2) = \mathfrak{F}(p_1)\mathfrak{g}(p_2) + \phi(p_1)\mathfrak{f}(p_2) = \mathfrak{g}(p_1)\mathfrak{F}(p_2) + \mathfrak{f}(p_1)\phi(p_2)$;

(ii) $\mathfrak{F}(\mathfrak{g}(p_1)) = \mathfrak{g}(\mathfrak{F}(p_1))$.

If ϕ acts as the identity map on P, then $\mathfrak{f}$ takes on the role of a semi-derivation, and when $\mathfrak{g}$ serves as the identity map on P, $\mathfrak{f}$ becomes a skew derivation. Similarly, if ϕ, the identity map over P, then $\mathfrak{F}$ transforms to a generalized semi-derivation, and if $\mathfrak{g}$ acts as the identity map on P, $\mathfrak{F}$ becomes a generalized skew derivation. An intriguing problem in the field of ring theory revolves around the investigation of conditions under which a ring becomes commutative. Over the past few decades, numerous authors have investigated the commutativity nature of prime rings by introducing certain additive mappings and have established several results that highlight the close relationship between the global structure of a ring P and the behaviour of additive mappings defined on P.

Towards this direction of research work, recent literature has extensively explored commutativity within prime rings that allow certain restricted semi-derivations and generalized derivations. Moreover, many authors have made advancements in these findings, as exemplified in references such as [4, 5, 7, 8, 10, 12]. Our current study extends this investigative path by delving into the structural analysis of prime rings that admits a generalized skew semi-derivation along with a skew semi-derivation adhering to more precise identities.

2 Results

Here, we discuss several essential lemmas required for the main results.

Lemma 2.1 ([3], Lemma 1.2(iii)). *If $z \in Z(P) \setminus \{0\}$ and $a \in P$ satisfying $az \in Z(P)$, then $a \in Z(P)$.*

Lemma 2.2. *Let $z \in Z(P)$ and $\mathfrak{f}$ be a skew semi-derivation on P, associated with onto function $\mathfrak{g}$. Then $\mathfrak{f}(z) \in Z(P)$ and $\mathfrak{F}(z) \in Z(P)$.*

Proof. By definition of skew semi-derivation, we have

$$\mathfrak{f}(zp_1) = \mathfrak{f}(z)\mathfrak{g}(p_1) + \phi(z)\mathfrak{f}(p_1)$$

and

$$\mathfrak{f}(p_1 z) = \mathfrak{f}(p_1)\phi(z) + \mathfrak{g}(p_1)\mathfrak{f}(z).$$

Since $p_1 z = zp_1$ for all $p_1 \in P$, we have $\mathfrak{f}(p_1 z) = \mathfrak{f}(zp_1)$. Therefore, $\mathfrak{f}(z)\mathfrak{g}(p_1) = \mathfrak{g}(p_1)\mathfrak{f}(z)$, which gives us $[\mathfrak{g}(p_1), \mathfrak{f}(z)] = 0$ for all $p_1, z \in P$. As $\mathfrak{g}$ is onto, we have $\mathfrak{f}(z) \in Z(P)$.

Also, $\mathfrak{F}(p_1 z) = \mathfrak{g}(p_1)\mathfrak{F}(z) + \mathfrak{f}(p_1)\phi(z)$ and $\mathfrak{F}(zp_1) = \mathfrak{F}(z)\mathfrak{g}(p_1) + \phi(z)\mathfrak{f}(p_1)$, which implies $\mathfrak{g}(p_1)\mathfrak{F}(z) = \mathfrak{F}(z)\mathfrak{g}(p_1)$, since $\phi(z) \in Z(P)$. Therefore, we get

$$[\mathfrak{g}(p_1), \mathfrak{F}(z)] = 0 \text{ for all } p_1, z \in P.$$

Since $\mathfrak{g}$ is onto, we have $\mathfrak{F}(z) \in Z(P)$.
Similarly, $\mathcal{G}(z) \in Z(P)$ where $\mathcal{G}$ is a generalized semi-derivation on P. □

Lemma 2.3. *Let P be a prime ring and $a \in P$. Then $a\mathfrak{f}(p_1) = 0$ or $\mathfrak{f}(p_1)a = 0$ for all $p_1 \in P$ implies $a = 0$ or $\mathfrak{f} = 0$.*

Proof. First assume that $a\mathfrak{f}(p_1) = 0$.
Replacing p_1 by $p_1 p_2$, we get $a\mathfrak{f}(p_1 p_2) = 0$. This implies that

$$a\mathfrak{f}(p_1)\mathfrak{g}(p_2) + a\phi(p_1)\mathfrak{f}(p_2) = 0.$$

This gives $a\phi(p_1)\mathfrak{f}(p_2) = 0$ for all $p_1, p_2 \in P$. Therefore, $aP\mathfrak{f}(p_2) = 0$.
As P is prime, we have $a = 0$ or $\mathfrak{f} = 0$.
Similarly, $\mathfrak{f}(p_1)a = 0$ implies $a = 0$ or $\mathfrak{f} = 0$. □

By the same process, we can prove the following lemma.

Lemma 2.4. *If P is a prime ring and $\mathfrak{F}, \mathcal{G}$ are generalized skew semi-derivation and generalized semi-derivation on P, respectively, then*

(i) $a\mathfrak{F}(p_1) = 0$ *or* $\mathfrak{F}(p_1)a = 0$ *for all* $p_1 \in P$ *implies* $a = 0$ *or* $\mathfrak{F} = 0$*,*

(ii) $a\mathcal{G}(p_1) = 0$ *or* $\mathcal{G}(p_1)a = 0$ *for all* $p_1 \in P$ *implies* $a = 0$ *or* $\mathcal{G} = 0$*.*

Lemma 2.5. *If P is a prime ring which is 2-torsion free and $\mathfrak{f}$ is a skew semi-derivation on P associated with an onto function $\mathfrak{g}$ such that $\mathfrak{f}^2 = 0$, then $\mathfrak{f} = 0$.*

Proof. Assume that $\mathfrak{f}^2(p_1) = 0$ for all $p_1 \in P$.
Replace p_1 by p_1p_2, we obtain $\mathfrak{f}^2(p_1p_2) = 0$, i.e., $\mathfrak{f}(\mathfrak{f}(p_1p_2)) = 0$. By the definition of semi-derivation, we have

$$\mathfrak{f}(\mathfrak{f}(p_1)\mathfrak{g}(p_2) + \phi(p_1)\mathfrak{f}(p_2)) = 0.$$

This implies that

$$\mathfrak{f}(\mathfrak{f}(p_1)\mathfrak{g}(p_2)) + \mathfrak{f}(\phi(p_1)\mathfrak{f}(p_2)) = 0.$$

Again, by the definition of semi-derivation, we get

$$\mathfrak{f}(\mathfrak{f}(p_1))\phi(\mathfrak{g}(p_2)) + \mathfrak{g}(\mathfrak{f}(p_1))\mathfrak{f}(\mathfrak{g}(p_2)) + \mathfrak{f}(\phi(p_1))\mathfrak{g}(\mathfrak{f}(p_2)) + \phi(\phi(p_1))\mathfrak{f}(\mathfrak{f}(p_2)) = 0,$$

which gives us

$$\mathfrak{g}(\mathfrak{f}(p_1))\mathfrak{f}(\mathfrak{g}(p_2)) + \mathfrak{f}(\phi(p_1))\mathfrak{g}(\mathfrak{f}(p_2)) = 0.$$

By the property of skew semi-derivation, we have

$$(\mathfrak{f}(\mathfrak{g}(p_1)) + \mathfrak{f}(\phi(p_1)))\mathfrak{f}(\mathfrak{g}(p_2)) = 0 \text{ for all } p_1, p_2 \in P.$$

Since both $\mathfrak{g}$ and ϕ are onto, we can write $(\mathfrak{f}(P)+\mathfrak{f}(P))f(p_2') = 0$, where $\mathfrak{g}(p_2) = p_2'$. Therefore, $\mathfrak{f}(P)f(p_2') = 0$ for all $p_2' \in P$, which implies that $\mathfrak{f} = 0$. □

We can establish the subsequent lemma using a similar proof with some necessary modifications.

Lemma 2.6. *If P is prime ring which is 2-torsion free and $\mathfrak{F}, \mathcal{G}$ are generalized skew semi-derivation and generalized semi-derivation on P, respectively, then*

(i) $\mathfrak{F}^2 = 0$ *implies* $\mathfrak{F} = 0$*;*

(ii) $\mathcal{G}^2 = 0$ *implies* $\mathcal{G} = 0$*.*

Lemma 2.7. *Let P be a prime ring and $\mathfrak{f}$ is a skew semi-derivation on P associated with an onto function $\mathfrak{g}$ satisfying $\mathfrak{f}(P) \subset Z(P)$. Then $\mathfrak{f} = 0$ or P is commutative.*

Proof. For all $p_1, p_2 \in P$, we have $\mathfrak{f}(p_1p_2) \in Z(P)$.
By definition, we get $\mathfrak{f}(p_1)\mathfrak{g}(p_2) + \phi(p_1)\mathfrak{f}(p_2) \in Z(P)$. This implies that

$$[\mathfrak{f}(p_1)\mathfrak{g}(p_2) + \phi(p_1)\mathfrak{f}(p_2), \phi(p_1)] = 0.$$

This gives us

$$\mathfrak{f}(p_1)[\mathfrak{g}(p_2), \phi(p_1)] = 0,$$

i.e.,

$$\mathfrak{f}(p_1)[\mathfrak{g}(p_2), p_1'] = 0, \text{ where } \phi(p_1) = p_1'.$$

Since $\mathfrak{g}$ is onto and $\mathfrak{f}(p_1) \in Z(P)$, we obtain

$$\mathfrak{f}(p_1)P[p_2, p_1'] = 0 \text{ for all } p_2, p_1' \in R.$$

Primness of P forces that

$$\mathfrak{f} = 0 \ \text{ or } [p_2, p_1'] = 0 \text{ for all } p_2, p_1' \in R.$$

Hence, P is commutative.

Similarly, we can prove $\mathfrak{F}(P) \subset Z(P)$ and $\mathcal{G}(P) \subset Z(P)$ implies $\mathfrak{F} = 0$ and $\mathcal{G} = 0$, respectively, or P is commutative.

□

3 Skew semi-derivation

Theorem 3.1. *Let P be a prime ring and $\mathfrak{f}$ is a nonzero skew semi-derivation on P, then for any $p_1, p_2 \in P$, the following statements hold and are equivalent to each other:*

(i) $\mathfrak{f}(Z(P)) \neq \{0\}$ *and* $\mathfrak{f}[p_1, p_2] \in Z(P)$*;*

(ii) $[\mathfrak{f}(p_1), p_2] \in Z(P)$*;*

(iii) $\mathfrak{f}(p_1) \circ p_2 \in Z(P)$*;*

(iv) P *is commutative.*

Proof. (4) implies $(1), (2)$ *and* (3) are obvious.
$(1) \Rightarrow (4)$: By hypothesis, $\mathfrak{f}[p_1, p_2] \in Z(P)$. Let $0 \neq z \in Z(P)$. Now, replace p_2 by $p_2 z$, we have

$$\mathfrak{f}[p_1, p_2]\mathfrak{g}(z) + \phi[p_1, p_2]\mathfrak{f}(z) \in Z(P).$$

Therefore, $\phi[p_1, p_2]\mathfrak{f}(z) \in Z(P)$. Since $\mathfrak{f}(z) \in Z(P) \neq \{0\}$ and by Lemma 2.1, we have $\phi[p_1, p_2] \in Z(P)$. This implies that $[p_1, p_2] \in Z(P)$. For all $v \in P$, we can write $[[p_1, p_2], v] = 0$. Again, replacing p_2 by $p_1 p_2$, we get $[[p_1, p_1 p_2], v] = 0$ i.e., $[p_1[p_1, p_2], v] = 0$. This gives $[p_1, p_2][p_1, v] = 0$ and therefore, $[p_1, p_2]P[p_1, v] = 0$. Hence, $p_1 \in Z(P)$ for all $p_1 \in P$, i.e., P is commutative.

$(2) \Rightarrow (4)$. Given that $[\mathfrak{f}(p_1), p_2] \in Z(P)$ for all $p_1, p_2 \in P$.
Now, if $Z(P) = 0$, then $\mathfrak{f}(p_1) \in Z(P) \Rightarrow \mathfrak{f}(P) \subset Z(P)$.
Hence, P is commutative, by Lemma 2.7. Therefore, we assume $Z(P) \neq \{0\}$. So, for all $v \in P$, $[[\mathfrak{f}(p_1), p_2], v] = 0$. Replace p_2 by $\mathfrak{f}(p_1)p_2$, we get

$$[[\mathfrak{f}(p_1), \mathfrak{f}(p_1)p_2], v] = 0$$

which implies that

$$[\mathfrak{f}(p_1)[\mathfrak{f}(p_1), p_2], v] = 0.$$

This gives $[\mathfrak{f}(p_1), p_2][\mathfrak{f}(p_1), v] = 0$, i.e., $[\mathfrak{f}(p_1), p_2]P[\mathfrak{f}(p_1), v] = 0$, as $[\mathfrak{f}(p_1), p_2] \in Z(P)$. Since P is prime, in both cases, $\mathfrak{f}(p_1) \in Z(P)$, i.e., $\mathfrak{f}(P) \subset Z(P)$ and hence, by Lemma 2.7, P is commutative.

$(3) \Rightarrow (4)$: Let $\mathfrak{f}(p_1) \circ p_2 \in Z(P)$ for all $p_1, p_2 \in P$.
If $Z(P) = \{0\}$, then the above expression becomes $p_2\mathfrak{f}(p_1) = -\mathfrak{f}(p_1)p_2$. Replace p_2 by $p_2 z$ for any $z \in P$, we get $p_2 z\mathfrak{f}(p_1) = -\mathfrak{f}(p_1)p_2 z$. This gives $p_2 z\mathfrak{f}(p_1) = p_2\mathfrak{f}(p_1)z$, and hence,

$$p_2 P[z, \mathfrak{f}(p_1)] = 0 \text{ for all } p_1, p_2, z \in P.$$

Since P is prime, we have $\mathfrak{f}(p_1) \in Z(P)$ i.e., $\mathfrak{f} = 0$ for all $p_1 \in P$, which contradicts $\mathfrak{f}$ of being non-zero. Then, there exists $u \in Z(P)$ satisfying $u \neq 0$.
Now, replace p_2 by u in the given expression to get

$$(\mathfrak{f}(p_1) + \mathfrak{f}(p_1))u \in Z(P),$$

and by Lemma 2.1, $\mathfrak{f}(p_1) + \mathfrak{f}(p_1) \in Z(P)$, which implies $\mathfrak{f}(p_1 + p_1) \in Z(P)$. Replace $p_1 + p_1$ by p_1, we have

$$\mathfrak{f}(p_1) \in Z(P) \text{ for all } p_1 \in P.$$

i.e., $\mathfrak{f}(P) \subset Z(P)$ and therefore, by Lemma 2.7, P is commutative. □

Theorem 3.2. *Let $\mathfrak{f}$ be a nonzero skew semi-derivation associated with an onto map $\mathfrak{g}$ and an automorphism ϕ over a 2-torsion free prime ring P. If $\mathfrak{f}$ satisfies $\mathfrak{f}(p_1 \circ p_2) \in Z(P)$, then P is commutative.*

Proof. We have $\mathfrak{f}(p_1 \circ p_2) \in Z(P)$. If $Z(P) = \{0\}$, then $\mathfrak{f}(p_1 \circ p_2) = 0$. Replacing p_2 by $p_1 p_2$, we get $\mathfrak{f}(p_1 \circ p_1 p_2) = 0$, i.e., $\mathfrak{f}(p_1(p_1 \circ p_2)) = 0$. This gives

$$\mathfrak{f}(p_1)\phi(p_1 \circ p_2) + \mathfrak{g}(p_1)\mathfrak{f}(p_1 \circ p_2) = 0$$

which implies $\mathfrak{f}(p_1)\phi(p_1 \circ p_2) = 0$. Therefore, $\mathfrak{f}(p_1)P\phi(p_1 \circ p_2) = 0$. Since $\mathfrak{f}$ is nonzero and P is prime,

$$\phi(p_1 \circ p_2) = 0 \text{ for all } p_1, p_2 \in P.$$

This implies that $p_1 \circ p_2 = 0 \quad for\ all\ p_1, p_2 \in P$. Again, replace p_2 by $z \in Z(P)$, we get $2p_1 z = 0 \Rightarrow p_1 z = 0$. Since $z \neq 0$, $p_1 = 0$ for all $p_1 \in P$, i.e., P is a zero ring which is a contradiction.

Therefore, $Z(P) \neq \{0\}$. Fix $0 \neq z_0 \in Z(P)$, Now, replace p_2 by z_0 in the given expression, we get $\mathfrak{f}(p_1 \circ z_0) \in Z(P)$, and by definition, we have

$$\phi(p_1 \circ p_2)\mathfrak{f}(z_0) + \mathfrak{f}(p_1 \circ p_2)\mathfrak{g}(z_0) \in Z(P).$$

Therefore, $\phi(p_1 \circ p_2)\mathfrak{f}(z_0) \in Z(P)$. Hence, by Lemma 2.1,

$$\phi(p_1 \circ p_2) \in Z(P) \ or \ f(z_0) = 0$$

i.e.,

$$p_1 \circ p_2 \in Z(P) \ \ or \ f(z_0) = 0.$$

Now, if $p_1 \circ p_2 \in Z(P)$, then $p_1 \circ z_0 = z_0(p_1 + p_1) \in Z(P)$ and $p_1^2 \circ z_0 = z_0(p_1^2 + p_1^2) \in Z(P)$. From Lemma 2.1,

$$p_1 + p_1 \in Z(P) \ \ and \ \ p_1^2 + p_1^2 \in Z(P) \text{ for all } p_1 \in P.$$

So, for all $v \in P$, we obtain

$$(p_1 + p_1)p_1 v = (p_1^2 + p_1^2)v$$

which implies that $(p_1 + p_1)p_1 v = v(p_1^2 + p_1^2)$. This gives us

$$(p_1 + p_1)p_1 v = vp_1(p_1 + p_1).$$

This implies that

$$(p_1 + p_1)p_1 v = v(p_1 + p_1)p_1,$$

i.e.,

$$(p_1 + p_1)p_1 v = (p_1 + p_1)vp_1.$$

Hence, we have $(p_1 + p_1)[p_1, v] = 0$ and so, $(p_1 + p_1)P[p_1, v] = 0$ for all $p_1, v \in P$. Thus, by primness of P, the ring P is commutative.

Next, let $\mathfrak{f}(z_0) = 0$ for all $z_0 \in Z(P)$ i.e., $\mathfrak{f}(Z(P)) = 0$.
Then for any $z \in Z(P)$, we have $\mathfrak{f}(p_1 \circ z) \in Z(P)$, and therefore,

$$(\mathfrak{f}(p_1) + \mathfrak{f}(p_1))\phi(z) \in Z(P),$$

since $\mathfrak{f}(z) = 0$. By Lemma 2.1, we get

$$\mathfrak{f}(p_1) + \mathfrak{f}(p_1) \in Z(P),$$

this implies that $\mathfrak{f}(\mathfrak{f}(p_1) + \mathfrak{f}(p_1)) = 0$ and hence, $\mathfrak{f}^2(p_1) = 0$ for all $p_1 \in P$. By Lemma 2.5, we have $\mathfrak{f} = 0$, which contradicts our assumption. Hence, the result follows. □

Theorem 3.3. *Let $\mathfrak{f}$ be a nonzero skew semi-derivation associated with a surjective map $\mathfrak{g}$ and an automorphism ϕ over a prime ring P with non-zero center satisfying $[\mathfrak{f}(p_1), \mathfrak{f}(p_2)] \in Z(P)$ for all $p_1, p_2 \in P$. Then P is commutative.*

Proof. Assume $[\mathfrak{f}(p_1), \mathfrak{f}(p_2)] \in Z(P)$.
Now, replace p_1 by zp_2 where $0 \neq z \in Z(P)$, we get

$$[\mathfrak{f}(z)\mathfrak{g}(p_2) + \phi(z)\mathfrak{f}(p_2), \mathfrak{f}(p_2)] \in Z(P),$$

which implies

$$\mathfrak{f}(z)[\mathfrak{g}(p_2), \mathfrak{f}(p_2)] \in Z(P), \ \ as \ \phi(z) \in Z(P).$$

Therefore, $[\mathfrak{g}(p_2), \mathfrak{f}(p_2)] \in Z(P)$. Since $\mathfrak{g}$ is onto and by Theorem 3.1, P is commutative. □

Theorem 3.4. *Let $\mathfrak{f}$ be a nonzero skew semi-derivation associated with a surjective map $\mathfrak{g}$ and an automorphism ϕ over a prime ring P such that $\mathfrak{f}$ satisfies $\mathfrak{f}[p_1, p_2] = [p_1, p_2]$ for all $p_1, p_2 \in P$. Then P is indeed a commutative ring.*

Proof. Given that $\mathfrak{f}[p_1, p_2] = [p_1, p_2] \quad for\ all\ p_1, p_2 \in P$. Replacing p_2 by $p_2 p_1$, we obtain $\mathfrak{f}([p_1, p_2]p_1) = [p_1, , p_2]p_1$. This implies that

$$f[p_1, p_2]\mathfrak{g}(p_1) + \phi[p_1, p_2]\mathfrak{f}(p_1) = [p_1, p_2]p_1.$$

As $\mathfrak{g}$ is onto, we can write $\phi[p_1, p_2]\mathfrak{f}(p_1) = 0$ for all $p_1 \in P$. Again, by replacing p_2 by $p_2 p_1$, we get $\phi[p_1, p_2]\phi(p_1)\mathfrak{f}(p_1) = 0$, which means $\phi[p_1, p_2]P\mathfrak{f}(p_1) = 0$. Since P is prime and $\mathfrak{f}$ is nonzero, $\phi[p_1 p_2] = 0$ for all $p_1, p_2 \in P$ which implies that $[p_1, p_2] = 0$ for all $p_1, p_2 \in P$. Hence, P is commutative.

□

Corollary 3.5. *Let $\mathfrak{f}$ be a nonzero skew semi-derivation associated with a surjective map $\mathfrak{g}$ and an automorphism ϕ over a prime ring P such that $\mathfrak{f}$ satisfies the identity $\mathfrak{f}[p_1, p_2] = -[p_1, p_2]$ for all $p_1, p_2 \in P$. Then P is a commutative ring.*

4 Generalized skew semi-derivation

Theorem 4.1. *Let $\mathfrak{F}$ be a generalized skew semi-derivation defined over a prime ring P associated with both a nonzero skew semi-derivation $\mathfrak{f}$ and an onto function $\mathfrak{g}$. Then for all $p_1, p_2 \in P$, the subsequent statements hold true and equivalent:*

(i) $\mathfrak{f}(Z(P)) \neq \{0\}$ *and* $\mathfrak{F}[p_1, p_2] \in Z(P)$*;*

(ii) $[\mathfrak{F}(p_1), p_2] \in Z(P)$*;*

(iii) $\mathfrak{F}(p_1) \circ p_2 \in Z(P)$*;*

(iv) *P is commutative.*

Proof. Clearly, (4) implies (1), (2) and (3).
$(1) \Rightarrow (4)$: From condition (1), $\mathfrak{F}[p_1, p_2] \in Z(P)$. Let $z \in Z(P)$. Replace p_2 by $p_2 z$ in the above, we get

$$\mathfrak{F}[p_1, p_2]\mathfrak{g}(z) + \phi[p_1, p_2]\mathfrak{f}(z) \in Z(P).$$

Therefore, $\phi[p_1, p_2]\mathfrak{f}(z) \in Z(P)$. Since $\mathfrak{f}(z) \in Z(P) \neq \{0\}$ and by Lemma 2.1, we have $\phi[p_1, p_2] \in Z(P)$, *i.e.*, $[p_1, p_2] \in Z(P)$. Hence, for all $v \in P$, we obtain $[[p_1, p_2], v] = 0$. Replacing p_2 by $p_1 p_2$, we have

$$[[p_1, p_1 p_2], v] = 0.$$

This implies that $[p_1[p_1, p_2], v] = 0$, and therefore, we can write

$$[p_1, p_2]P[p_1, v] = 0, \text{ since } [p_1, p_2] \in Z(P).$$

Hence, $p_1 \in Z(P)$ for all $p_1 \in P$, i.e., P is commutative.

$(2) \Rightarrow (4)$: Given that $[\mathfrak{F}(p_1), p_2] \in Z(P)$ for all $p_1, p_2 \in P$. Now, if $Z(P) = 0$. Then $\mathfrak{F}(p_1) \in Z(P)$ for all $p_1 \in P$, which implies that

$$\mathfrak{F}(P) \subset Z(P)$$

and hence, P is commutative by Lemma 2.7.

Therefore, we assume $Z(P) \neq \{0\}$. Then $[[\mathfrak{F}(p_1), p_2], v] = 0$ for all $v \in P$. Replacing p_2 by $\mathfrak{F}(p_1)p_2$, we get $[[\mathfrak{F}(p_1), \mathfrak{F}(p_1)p_2], v] = 0$, which gives us

$$[\mathfrak{F}(p_1), p_2][\mathfrak{F}(p_1), v] = 0.$$

Therefore, we can write it as

$$[\mathfrak{F}(p_1), p_2]P[\mathfrak{F}(p_1), v] = 0, \text{ since } [\mathfrak{F}(p_1), p_2] \in Z(P).$$

Also, P is prime, in both cases, $\mathfrak{F}(p_1) \in Z(P)$ for all $p_1 \in P$, i.e., $\mathfrak{F}(P) \subset Z(P)$ and hence by Lemma 2.7, P is commutative.

(3) $\Rightarrow$ (4) : Let $\mathfrak{F}(p_1) \circ p_2 \in Z(P)$ for all $p_1, p_2 \in P$.
If $Z(P) = \{0\}$, then the above expression becomes $p_2\mathfrak{F}(p_1) = -\mathfrak{F}(p_1)p_2$. Replacing p_2 by p_2z for any $z \in P$, we get $p_2z\mathfrak{F}(p_1) = -\mathfrak{F}(p_1)p_2z$, which implies that $p_2z\mathfrak{F}(p_1) = p_2\mathfrak{F}(p_1)z$. After simplifying, we obtain

$$p_2P[z, \mathfrak{F}(p_1)] = \text{ for all } p_1, p_2, z \in P.$$

Since P is prime, $\mathfrak{F}(p_1) \in Z(P)$ for all $p_1 \in P$, i.e., $\mathfrak{F} = 0$ for all $p_1 \in P$, a contradiction to our assumption. Hence, there exists $u \in Z(P)$ satisfying $u \neq 0$.
Now, replacing p_2 by u in the given expression, we get $(\mathfrak{F}(p_1)+\mathfrak{F}(p_1))u \in Z(P)$, and by Lemma 2.1, $\mathfrak{F}(p_1) + \mathfrak{F}(p_1) \in Z(P)$. This implies that $\mathfrak{F}(p_1 + p_1) \in Z(P)$. Replacing $p_1 + p_1$ by p_1, we have

$$\mathfrak{F}(p_1) \in Z(P) \text{ for all } p_1 \in P.$$

i.e., $\mathfrak{F}(P) \subset Z(P)$ and therefore, by Lemma 2.7, P is commutative. □

Corollary 4.2. *Let $\mathcal{G}$ be a generalized semi-derivation defined over a prime ring P associated with a nonzero semi-derivation $\mathfrak{h}$. Then the following statements hold and are equivalent to each other for all $p_1, p_2 \in P$:*

(i) $\mathfrak{h}(Z(P)) \neq \{0\}$ *and* $\mathcal{G}[p_1, p_2] \in Z(P)$*;*

(ii) $[\mathcal{G}(p_1), p_2] \in Z(P)$*;*

(iii) $\mathcal{G}(p_1) \circ p_2 \in Z(P)$*;*

(iv) *P is commutative.*

Theorem 4.3. *Let P be a prime ring which is 2-torsion free and $\mathfrak{F}$ be a generalized skew semi-derivation associated with a nonzero skew semi-derivation $\mathfrak{f}$, a surjective map $\mathfrak{g}$ and an automorphism ϕ over P. If $\mathfrak{F}$ satisfies $\mathfrak{F}(p_1 \circ p_2) \in Z(P)$, then P must be a commutative ring.*

Proof. By hypothesis $\mathfrak{F}(p_1 \circ p_2) \in Z(P)$. If $Z(P) = \{0\}$, then $\mathfrak{F}(p_1 \circ p_2) = 0$. Replacing p_2 by p_1p_2, we have $\mathfrak{f}(p_1 \circ p_1p_2) = 0$. This implies that $\mathfrak{F}(p_1(p_1 \circ p_2)) = 0$ i.e.,

$$\mathfrak{f}(p_1)\phi(p_1 \circ p_2) + \mathfrak{g}(p_1)\mathfrak{F}(p_1 \circ p_2) = 0$$

which gives us $\mathfrak{f}(p_1)\phi(p_1 \circ p_2) = 0$, and so, $\mathfrak{f}(p_1)P\phi(p_1 \circ p_2) = 0$. Since $\mathfrak{f}$ is nonzero and P is prime, we get

$$\phi(p_1 \circ p_2) = 0 \text{ for all } p_1, p_2 \in P$$

i.e., $p_1 \circ p_2 = 0$ for all $p_1, p_2 \in P$. Now, replace p_2 by $z \in Z(P)$ to get

$$2p_1z = 0 \Rightarrow p_1z = 0.$$

Since $z \neq 0$, $p_1 = 0$ for all $p_1 \in P$. Hence, P is a zero ring, which contradicts our assumption. Therefore, we assume $Z(P) \neq \{0\}$. Fix $0 \neq z_0 \in Z(P)$, Again, replace p_2 by z_0 in the given expression, we get $\mathfrak{F}(p_1 \circ z_0) \in Z(P)$, by expanding it, we have $\mathfrak{F}(p_1 \circ p_2)g(z_0) + \phi(p_1 \circ p_2)\mathfrak{f}(z_0) \in Z(P)$. Therefore, $\phi(p_1 \circ p_2)\mathfrak{f}(z_0) \in Z(P)$, and by Lemma 2.1,

$$\phi(p_1 \circ p_2) \in Z(P) \;\; or \; f(z_0) = 0,$$

i.e.,

$$p_1 \circ p_2 \in Z(P) \;\; or \;\; f(z_0) = 0.$$

Now, if $p_1 \circ p_2 \in Z(P)$, then $p_1 \circ z_0 = z_0(p_1 + p_1) \in Z(P)$, and $p_1^2 \circ z_0 = z_0(p_1^2 + p_1^2) \in Z(P)$. By Lemma 2.1, $p_1 + p_1 \in Z(P)$ and $p_1^2 + p_1^2 \in Z(P)$ for all $p_1 \in P$. Therefore, for all $v \in P$, we get

$$(p_1 + p_1)p_1v = (p_1^2 + p_1^2)v.$$

After calculating, we have $(p_1 + p_1)P[p_1, v] = 0$ for all $p_1, v \in P$. By using primness of P, we get $p_1 \in Z(P)$ for all $p_1 \in P$. Hence, by Lemma 2.7, P is commutative.
Next, if

$$\mathfrak{f}(z_0) = 0 \text{ for all } z_0 \in Z(P),$$

i.e., $\mathfrak{f}(Z(P)) = 0$. Then, for any $z \in Z(P)$, we have $\mathfrak{f}(p_1 \circ z) \in Z(P)$, and therefore,

$$(\mathfrak{f}(p_1) + \mathfrak{f}(p_1))\phi(z) \in Z(P), \ \ as\ \mathfrak{f}(z) = 0,$$

which implies that

$$\mathfrak{f}(p_1) + \mathfrak{f}(p_1) \in Z(P), \text{ by Lemma 2.1}.$$

This implies that

$$\mathfrak{f}(\mathfrak{f}(p_1) + \mathfrak{f}(p_1)) = 0.$$

Therefore, $\mathfrak{f}^2(p_1) = 0$ for all $p_1 \in P$. Also, by Lemma 2.4, $\mathfrak{f} = 0$, a contradiction to our assumption. Hence, the result follows. □

Corollary 4.4. *Let P be a 2-torsion free prime ring and $\mathcal{G}$ be a generalized semi-derivation associated with a nonzero semi-derivation $\mathfrak{f}$, a surjective map $\mathfrak{g}$ over P. If $\mathcal{G}$ satisfies $\mathcal{G}(p_1 \circ p_2) \in Z(P)$, then P is indeed commutative.*

Theorem 4.5. *Let P be a 2-torsion free prime ring with nonzero centre and $\mathfrak{F}$ be a nonzero generalized skew semi-derivation associated with a skew semi-derivation $\mathfrak{f}$, a surjective map $\mathfrak{g}$ and an automorphism ϕ over P such that $\mathfrak{F}$ satisfies $[\mathfrak{F}(p_1), \mathfrak{f}(p_2)] \in Z(P)$ for all $p_1, p_2 \in P$. Then P is a commutative ring.*

Proof. Assume $[\mathfrak{F}(p_1), \mathfrak{f}(p_2)] \in Z(P)$. Now, replacing p_1 by zp_2 where $0 \neq z \in Z(P)$, we get $[\mathfrak{F}(z)\mathfrak{g}(p_2)+\phi(z)\mathfrak{f}(p_2), \mathfrak{f}(p_2)] \in Z(P)$. This gives us $\mathfrak{F}(z)[\mathfrak{g}(p_2), \mathfrak{f}(p_2)] \in Z(P), \ as\ \phi(z), \mathfrak{F}(z) \in Z(P)$. Hence, $[\mathfrak{g}(p_2), \mathfrak{f}(p_2)] \in Z(P)$. Since $\mathfrak{g}$ is onto, using Theorem 3.1, we get our required result. □

Corollary 4.6. *Let P be a 2-torsion free prime ring with nonzero centre and $\mathcal{G}$ be a nonzero generalized semi-derivation associated with both a surjective map $\mathfrak{g}$ and an automorphism ϕ over P satisfying $[\mathcal{G}(p_1), \mathcal{G}(p_2)] \in Z(P)$ for all $p_1, p_2 \in P$. Then P is a commutative integral domain.*

Theorem 4.7. *Let $\mathfrak{F}$ be a generalized skew semi-derivation associated with a nonzero skew semi-derivation $\mathfrak{f}$, a surjective map $\mathfrak{g}$ and an automorphism ϕ over a prime ring P satisfying $\mathfrak{F}[p_1, p_2] = [p_1, p_2]$ for all $p_1, p_2 \in P$. Then P is a commutative integral domain.*

Proof. Given that $\mathfrak{F}[p_1, p_2] = [p_1, p_2]$ for all $p_1, p_2 \in P$. Replacing p_2 by p_2p_1, we obtain $\mathfrak{F}([p_1, p_2]p_1) = [p_1, p_2]p_1$. This implies that $\mathfrak{F}[p_1, p_2]\mathfrak{g}(p_1) + \phi[p_1, p_2]\mathfrak{f}(p_1) = [p_1, p_2]x$. Since $\mathfrak{g}$ is onto, we can write

$$\phi[p_1, p_2]\mathfrak{f}(p_1) = 0 \text{ for all } p_1 \in P.$$

Again, by replacing p_2 by p_2p_1, we get $\phi[p_1, p_2]\phi(p_1)\mathfrak{f}(p_1) = 0$, i.e., $\phi[p_1, p_2]P\mathfrak{f}(p_1) = 0$. Since P is prime and $\mathfrak{f}$ is nonzero, $\phi[p_1, p_2] = 0$ for all $p_1, p_2 \in P$ i.e., $[p_1, p_2] = 0$ for all $p_1, p_2 \in P$. Hence, P is commutative.

□

Corollary 4.8. *Let $\mathfrak{F}$ be a generalized skew semi-derivation associated with a nonzero skew semi-derivation $\mathfrak{f}$, an onto map $\mathfrak{g}$ and an automorphism ϕ over a prime ring P satisfying $\mathfrak{F}[p_1, p_2] = -[p_1, p_2]$ for all $p_1, p_2 \in P$. Then P is a commutative ring.*

Corollary 4.9. *If $\mathcal{G}$ is a generalized semi-derivation associated with a nonzero semi-derivation $\mathfrak{f}$, a surjective map $\mathfrak{g}$ and an automorphism ϕ over a prime ring P such that $\mathcal{G}$ satisfies $\mathcal{G}[p_1, p_2] = \pm[p_1, p_2]$ for all $p_1, p_2 \in P$, then P is indeed a commutative ring.*

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Declarations

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