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Shifted Chebyshev polynomials for Volterra-Fredholm integral equations of the first kind

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Abstract *The goal of this work is to solve the ill-posed problem composed by the sum of two compact operators, one from Volterra and the other from Fredholm. Noting that the traditional regularization of theses problems is based on the adjoint operators where in our cases the Volterra adjoint operator is not suitable. Using our regularized problem based on the perturbation of the equation and solving this latter by the third and the fourth Chebyshev polynomials, we obtain the good precision as well as the efficiency of this technique by means of examples.*

1 Introduction

One calls two problems inverse to each other if the formulation of one problem involves the other one. (J. Keller). If a plane lacks kerosene, it risks a crash, this is direct problem, whereas if a plane crashes, we cannot determine the causes, there are many ones, this is inverse problem, because to determine what causes a plane to crash we can take in-depth research and investigation to understand what happened. Among the family of inverse problems we find the ill-posed problem is a problem where it has no solutions or has many solutions or it has solutions but unstable say, for an arbitrarily small errors in the measurement data lead to a large errors in the solutions. Most difficulties in solving ill-posed problems are caused by the instability of solutions, so ill-posed problems correspond to all problems have unstable solutions.
Most studies of ill-posed problems are made for Volterra and Fredholm integrals of the first kind but never both. In [1, 5] for solving Fredholm integral equations of the first kind the authors utilize Chebyshev and the Legendre wavelets method constructed on the unit interval as basis in Galerkin method. A direct method for solving Volterra integral equation of the first kind by using block-pulse functions and their operational matrix of integration [2]. In [3, 6, 10, 11, 12] the authors suggest a method for solving Fredholm and Volterra integral equations of the first kind based on the wavelet bases. The Haar, continuous Legendre, CAS, Chebyshev wavelets. Comparison between Taylor and perturbed method for Fredholm and Volterra integral equation of the first kind are studied in [2, 7, 8]. Approxamation method to Volterra-Fredholm integral equations of the first kind [9].
Let $A = V + F$ be the sum of two linear compact operators Volterra-Fredholm integral equation of first kind defined from Hilbert space H into itself over R . We explicit this problem as

$$A\varphi = V\varphi + F\varphi = f, \tag{1.1}$$

where V is Volterra compact operator and F Fredholm compact operator defined for $\varphi \in H$ by

$$V\varphi = \int_a^x k_1(x,t)\varphi(t)dt \tag{1.2}$$

and

$$F\varphi = \int_a^b k_2(x,t)\varphi(t)dt \tag{1.3}$$

or sill the equation (1.1) becomes

$$\int_a^x k_1(x,t)\varphi(t)dt + \int_a^b k_2(x,t)\varphi(t)dt = f(x), \tag{1.4}$$

where the kernels k_1 and k_1 are continuous functions non-degenerates on $[a,b]\times[a,b]$, we will suppose that f is such that there exists a unique solution $\varphi \in H$ of the equation (1.1). Noting that the equation (1.1) admits a unique solution in direct sense or in the last square sense provided the right-hand side f is in $R(A)$ or in $R(A)+R(A)^{\perp}$, respectively. Due to the non closedness of range $R(A)$ the solution is not stable. This means that for any approximation data f_δ of f with the relation

$$\|f - f_\delta\| \leq \delta, \tag{1.5}$$

for some $\delta > 0$ and small, the solution ϕ_δ of the equation $A\varphi_\delta = f_\delta$ may be so far from the solution φ of the initial problem (1.1).

2 Regularization by differentiation of Volterra-Fredholm integral equation

Supposing the smoothness of functions k_1, k_2 and f. By the differentiation t0 x of two sides of the equation (1.1) we ontain a regularized equation say, Volterra-Fredholm integral equation of the second kind.

$$k_1(x,x)\varphi(x) + \int_a^x \frac{\partial k_1}{\partial x}(x,t)\varphi(t)dt + \int_a^b \frac{\partial k_2}{\partial x}(x,t)(x,t)\varphi(t)dt = f'(x), \tag{2.1}$$

If $k_1(x,x) \neq 0$ for all $x \in [a,b]$, we obtain the well-posed problem Volterra-Fredholm integral equation of the second kind by the mean of the division of the equation (2.1) by the factor $k_1(x,x)$.

$$\varphi(x) + \int_a^x k_3(x,t)\varphi(t)dt + \int_a^b k_4(x,t)\varphi(t)dt = g(x), \tag{2.2}$$

where the functions

$$k_3(x,t) = \frac{1}{k_1(x,x)}\frac{\partial k_1}{\partial x}(x,t), \tag{2.3}$$

$$k_4(x,t) = \frac{1}{k_1(x,x)}\frac{\partial k_2}{\partial x}(x,t) \text{ and } g(x) = \frac{f'(x)}{k_1(x,x).} \tag{2.4}$$

If $k(x,x) = 0$ for $x \in [a,b]$, we derive several times until obtaining $\frac{\partial^q k_1}{\partial x^q}(x,t) \neq 0$ for all $x \in [a,b]$, with k_1, k_2 and f in $C^{q+1}([a,b]\times[a,b])$ and $C^{q+1}([a,b])$ respectively, where q represents the smallest integer for which the derivative k_1 of order q does not null.

$$\frac{\partial^q k_1}{\partial x^q}(x,t)\varphi(x) + \int_a^x \frac{\partial^{q+1} k_1}{\partial x^{q+1}}(x,t)\varphi(t)dt + \int_a^b \frac{x^{q+1}(x,t)}{\partial x^{q+1}}(x,t)\varphi(t)dt = f^{(q+1)}(x). \tag{2.5}$$

3 Regularization by perturbation of Volterra-Fredholm integral equation

Theorem 3.1. *If for any positive constant $\alpha > 0$ and small, such that*

$$\|(I + \frac{1}{\alpha}V)^{-1}\|\|F\| < \alpha. \tag{3.1}$$

Then one can repleace the equation (1.1) by its auxiliary one

$$\alpha\varphi_\alpha + V\varphi_\alpha + F\varphi_\alpha = f_\delta, \tag{3.2}$$

where we add the term $\alpha\varphi_\alpha$ to the operator $A\varphi = V\varphi + F\varphi$ for α positive and small, the equation (3.2) admits a unique stable solution φ_α, *Indeed, The Volterra integral operator of the second kind $(I + \frac{1}{\alpha}V)$ is invertible and the expression (3.1) leads us to the relation $\|(\alpha I + V)^{-1}\|\|F\| < 1$ which gives the existence and uniqueness of the solution of the equation (3.2) as a sum of two operators the first invertible and the second compact.*

Lemma 3.2. *[8] The problem (3.2) is well posed with the norm $\|(aI+A)^{-1}\| = O(\frac{1}{\sqrt{\alpha}})$ provided that the operator A verified the relation (3.1) and positive definite*

Proposition 3.3. *The relation (3.1) for operator A leads to the existence and uniqueness of the solution of the auxiliary problem (3.2) Besides, the solution φ_α converges to the exact solution φ of the initial problem (1.1), provided that $\frac{\delta}{\sqrt{\alpha}} \to 0$ as α goes to zero, say*

$$\lim_{\alpha\to 0} \|\varphi - \varphi_\alpha\| = 0 \tag{3.3}$$

Indeed,

$$\begin{aligned}
\|\varphi - \varphi_\alpha\| &= \|\varphi - (\alpha I + A)^{-1} f_\delta\| \\
&\leq \|(\alpha I + A)^{-1}\|\|\alpha\varphi + f - f_\delta\| \\
&\leq \alpha\|(\alpha I + A)^{-1}\|\|\varphi\| + \|(\alpha I + A)^{-1}\|\|f - f_\delta\|.
\end{aligned}$$

Therefore

$$\|\varphi - \varphi_\alpha\| = O(\sqrt{\alpha}) + \frac{\delta}{\sqrt{\alpha}}.$$

i. ***The third-kind polynomial*** V_n

The Chebyshev polynomial $V_n(x)$ of the third kind is a polynomial in x of degree n; defined by the relation

$$V_n(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos\frac{1}{2}\theta} \quad \texttt{when } x = \cos\theta \tag{3.4}$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\cos(n + \frac{1}{2})\theta + \cos(n - 2 + \frac{1}{2})\theta = 2\cos\theta\cos(n - 1 + \frac{1}{2})\theta,$$

which becomes

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, ...$$

With

$$V_0(x) = 1, \;\; V_1(x) = 2x - 1$$

Noting that the functions $\{V_n(x), \;\; n = 0, 1, 2, ...\}$ form an orthogonal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \sqrt{\frac{1+x}{1-x}}$ and so the polynomial system $S_n(x)$ given by

$$\left\{ S_0(x) = \sqrt{\frac{1}{\pi}}V_0(x),\; S_1(x) = \sqrt{\frac{1}{\pi}}V_1(x),\; S_2(x) = \sqrt{\frac{1}{\pi}}V_2(x), ...S_n(x) = \sqrt{\frac{1}{\pi}}V_n(x)... \right\},$$

form an orthonormal system on the interval $[-1, 1]$ with respect to the weight $w(x) = \sqrt{\frac{1+x}{1-x}}$. In other words

$$\langle S_k(x), S_l(x)\rangle = \int_{-1}^{1} S_k(x)S_l(x)\sqrt{\frac{1+x}{1-x}}dx = \begin{cases} 0 & \texttt{if} \quad k \neq l \\ 1 & \texttt{if} \quad k = l \end{cases}$$

ii. ***Shifted third Chebyshev Polynomial*** V_n^s

For the construction of the shifted Chebyshev polynomials we use the change of variable $x = \frac{2}{b-a}t - \frac{b+a}{b-a}$. So, the shifted Chebyshev polyno- mials $V_n^s(t), \;\; t \in [a, b], \;\; a, b \in R$ is given as

$V_n^s(t) = V_n(\frac{2}{b-a}t - \frac{b+a}{b-a})$ *and so* $V_0^s(x) = 1$, $V_1^s(x) = 2(\frac{2}{b-a}x - \frac{b+a}{b-a})$. *Therefore, we get the fundamental relation for the shifted polynomial*

$$V_n^s(x) = 2(\frac{2}{b-a}t - \frac{b+a}{b-a})V_{n-1}^s(x) - V_{n-2}^s(x)$$

Noting that the functions $\{V_n^s,\ n = 0, 1, ..\}$ *form an orthogonal system on the interval [a,b] with respect to the weight function*

$$w^s(x) = \sqrt{\frac{1 + (\frac{2}{b-a}x - \frac{b+a}{b-a})}{1 - (\frac{2}{b-a}x - \frac{b+a}{b-a})}}$$

and so the polynomial system $S_n^s(x)$ *given by*

$$\{S_n^s(x) = \sqrt{\frac{2}{b-a}}S_n(\frac{2}{b-a}x - \frac{b+a}{b-a})\}$$

form an orthonormal system on the interval [a,b] with respect to the weight function (10)

iii. ***The fourth-kind polynomial*** W_n

The Chebyshev polynomial $W_n(x)$ *of the fourth kind is a polynomial in* x *of degree* n; *defined by the relation*

$$W_n(x) = \frac{\sin(n + \frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \quad \texttt{when } x = \cos\theta \tag{3.5}$$

The three term recurrence formula satisfied by Chebyshev polynomials is the translation of the elementary trigonometric identity

$$\sin(n + \frac{1}{2})\theta + \sin(n - 2 + \frac{1}{2})\theta = 2\cos\theta\sin(n - 1 + \frac{1}{2})\theta,$$

which becomes

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), \quad n = 2, 3, ...$$

With

$$W_0(x) = 1,\ \ W_1(x) = 2x + 1.$$

Noting that the functions $\{V_n(x),\ \ n = 0, 1, 2, ...\}$ *form an orthogonal system on the interval* $[-1, 1]$ *with respect to the weight* $w(x) = \sqrt{\frac{1+x}{1-x}}$ *and so the polynomial system* $S_n(x)$ *given by*

$$\left\{S_0(x) = \sqrt{\frac{1}{\pi}}W_0(x),\ S_1(x) = \sqrt{\frac{1}{\pi}}W_1(x),\ S_2(x) = \sqrt{\frac{1}{\pi}}W_2(x), ...S_n(x) = \sqrt{\frac{1}{\pi}}W_n(x)...\right\},$$

form an orthonormal system on the interval $[-1, 1]$ *with respect to the weight* $w(x) = \sqrt{\frac{1-x}{1+x}}$. *In other words*

$$\langle S_k(x), S_l(x)\rangle = \int_{-1}^{1} S_k(x)S_l(x)\sqrt{\frac{1-x}{1+x}}dx = \begin{cases} 0 & \texttt{if} \quad k \neq l \\ 1 & \texttt{if} \quad k = l \end{cases}$$

iv. ***Shifted fourth Chebyshev Polynomial*** W_n^s

For the construction of the shifted Chebyshev polynomials we use the change of variable $x = \frac{2}{b-a}t - \frac{b+a}{b-a}$. *So, the shifted Chebyshev polyno- mials* $U_n^s(t),\ t \in [a, b],\ a, b \in R$ *is given as* $U_n^s(t) = U_n(\frac{2}{b-a}t - \frac{b+a}{b-a})$ *and so* $U_0^s(x) = 1$, $U_1^s(x) = 2(\frac{2}{b-a}x - \frac{b+a}{b-a})$. *Therefore, we get the fundamental relation for the shifted polynomial*

$$U_n^s(x) = 2(\frac{2}{b-a}t - \frac{b+a}{b-a})U_{n-1}^s(x) - U_{n-2}^s(x)$$

Noting that the functions $\{U_n^s,\ n = 0, 1, ..\}$ *form an orthogonal system on the interval [a,b] with respect to the weight function*

$$w^s(x) = \sqrt{\frac{1 - (\frac{2}{b-a}x - \frac{b+a}{b-a})}{1 + (\frac{2}{b-a}x - \frac{b+a}{b-a})}}$$

and so the polynomial system $S_n^s(x)$ given by

$$\{S_n^s(x) = \sqrt{\frac{2}{b-a}} S_n(\frac{2}{b-a}x - \frac{b+a}{b-a})\}$$

form an orthonormal system on the interval [a,b] with respect to the weight function (10)

Discretization of integral equation

Applying a collocation method to the equation (3.2) in order to discredit and convert this equation to a system of linear equations on the interval $[a, b]$. Approximate the unknown function $\varphi(x)$ by a finite sum of the form

$$\varphi_\alpha(x) \simeq \sum_{k=0}^{N} \alpha_k S_k^s(x), \tag{3.6}$$

where $S_n^s(x)$ denotes the nth shifted Chebyshev polynomial of the third or the fourth kind. After substitution of the expansion (3.6) into the equation (3.2) this latter becomes an approximate equation as

$$\alpha \sum_{k=0}^{N} \alpha_k S_k^s(x) - V\left(\sum_{k=0}^{N} \alpha_k S_k^s(t)\right) - F\left(\sum_{k=0}^{N} \alpha_k S_k^s(t)\right) = f_\delta(x). \tag{3.7}$$

Choosing the Fourier's coefficients α_k such that (3.7) is satisfied on the interval $[a, b]$. For this technical we take the equidistant collocation points as follows

$$t_j = a + jh, \quad h = \frac{b-a}{N}, \quad j = 0, 1, ...N, \tag{3.8}$$

and define the residual as

$$R_N(x) = \alpha \sum_{k=0}^{N} \alpha_k S_k^s(x) - V\left(\sum_{k=0}^{N} \alpha_k S_k^s(t)\right) - F\left(\sum_{k=0}^{N} \alpha_k S_k^s(t)\right) - f_\delta(x). \tag{3.9}$$

Then, by imposing conditions at collocation points

$$R_N(x_j) = 0, \quad j = 0, 1, ...N, \tag{3.10}$$

the integral equation (3.7) is converted to a system of linear equations.

4 Illustrating examples

Example 1. *Consider the linear Volterra-Fredholm integral equation*

$$\int_0^x \cos(x-t)\varphi(t)dt + \int_0^1 \sin(x-t)\varphi(t)dt = f(x),$$

where the function $f(x) = \frac{1}{2}e^x - \cos x + \frac{1}{2}e\left(\cos(x-1) + \sin(x-1)\right)$ chosen so that the solution $\varphi(x)$ is given by

$$\varphi(x) = e^x$$

Applying the shifted third Chebyshev polynomial $V_8^s(x)$ to approximate the solution $\varphi_\alpha(x)$, that is to say solution of the algebraic system of linear equations for $\alpha = 10^{-10}$

Points of x	*Exact sol*	*Approx sol*	*Error N=8*
0.0000e+00	*1.0000e+00*	*1.0000e+00*	*4.4283e-10*
2.5000e-01	*1.2840e+00*	*1.2840e+00*	*5.5645e-11*
3.7500e-01	*1.4549e+00*	*1.4549e+00*	*1.9957e-10*
5.0000e-01	*1.6487e+00*	*1.6487e+00*	*4.0386e-10*
7.5000e-01	*2.1170e+00*	*2.1170e+00*	*1.3097e-10*
8.7500e-01	*2.3988e+00*	*2.3988e+00*	*2.7559e-10*
1.0000e+00	*2.7182e+00*	*2.7182e+00*	*7.6849e-09*

Table 1. *The exact and approximate solutions of example 1 in some arbitrary points, using the shifted third Chebyshev polynomial $V_8^s(x)$*

Example 2. *Consider the linear Volterra-Fredholm integral equation*

$$\int_0^x \left(3x+t^2\right)\varphi(t)dt + \int_0^1 \left(t^3 \sin x\right)\varphi(t)dt = f\left(x\right),$$

where the function $f\left(x\right) = \left(3x+1\right)\ln\left(x+1\right) + \frac{1}{2}x^2 - x - \frac{1}{6}\left(\sin x\right)\left(6\ln 2 - 5\right)$ *chosen so that the solution* $\varphi(x)$ *is given by*

$$\varphi(x) = \frac{1}{x+1}$$

Applying the shifted fourth Chebyshev polynomial $W_8^s(x)$ *to approximate the solution* $\varphi_\alpha(x)$, *that is to say solution of the algebraic system of linear equations for* $\alpha = 10^{-10}$

Points of x	*Exact sol*	*Approx sol*	*Error N=8*
0.0000e+00	*1.0000e+00*	*1.0000e+00*	*3.8242e-06*
2.5000e-01	*8.0000e-01*	*8.0000e-01*	*5.5706e-08*
3.7500e-01	*7.2727e-01*	*7.2727e-01*	*4.2328e-08*
5.0000e-01	*6.6666e-01*	*6.6666e-01*	*3.6107e-08*
7.5000e-01	*5.7142e-01*	*5.7142e-01*	*2.8193e-08*
8.7500e-01	*5.3333e-01*	*5.3333e-01*	*3.2583e-08*
1.0000e+00	*5.0000e-01*	*5.0000e-01*	*2.4834e-07*

Table 2. *The exact and approximate solutions of example 2 in some arbitrary points, using the shifted fourth Chebyshev polynomial* $S_8^s(x)$

Example 3. *Consider the linear Volterra-Fredholm integral equation*

$$\int_0^x -3^{(x-t)}\varphi(t)dt + \int_0^1 \left(1\right)\varphi(t)dt = f\left(x\right),$$

where the function $f(x) = -3^x\left(x+e^{-x}-1\right) - \frac{1}{\left(\ln 3\right)\left(\ln 3 - 1\right)}\left(3e^{-1}\ln 3 - 3\ln 3 + 2\right)$ *chosen so that the solution* $\varphi(x)$ *is given by*

$$\varphi(x) = 3^x\left(1-e^{-x}\right)$$

Applying the shifted third Chebyshev polynomial $V_8^s(x)$ *to approximate the solution* $\varphi_\alpha(x)$, *that is to say solution of the algebraic system of linear equations for* $\alpha = 10^{-10}$

Points of x	*Exact sol*	*Approx sol*	*Error N=8*
0.0000e+00	*0.0000e+00*	*1.6607e-07*	*1.6607e-07*
2.5000e-01	*1.5163e-01*	*1.5163e-01*	*2.7685e-09*
3.7500e-01	*1.7713e-01*	*1.7713e-01*	*2.0420e-09*
5.0000e-01	*1.8393e-01*	*1.8393e-01*	*4.8908e-09*
7.5000e-01	*1.6734e-01*	*1.6734e-01*	*5.1049e-09*
8.7500e-01	*1.5205e-01*	*1.5205e-01*	*5.6365e-09*
1.0000e+00	*1.3533e-01*	*1.3533e-01*	*3.9496e-08*

Table 3. *The exact and approximate solutions of example 3 in some arbitrary points, using the shifted third Chebyshev polynomial* $S_8^s(x)$

Example 4. *Consider the linear Volterra-Fredholm integral equation*

$$\int_0^x \left(\exp(x+t)\right)\varphi(t)dt + \int_0^1 \left(x-t\right)\varphi(t)dt = f\left(x\right),$$

where the function $f(x) = \frac{1}{4}\exp(-2)(5-3x) - \frac{1}{4}(5+3x) + \exp(x)$ *chosen so that the solution* $\varphi(x)$ *is given by*

$$\varphi(x) = x\exp(-2x)$$

Applying the shifted fourth Chebyshev polynomial $W_8^s(x)$ *to approximate the solution* $\varphi_\alpha(x)$, *that is to say solution of the algebraic system of linear equations for* $\alpha = 10^{-10}$

Points of x	*Exact sol*	*Approx sol*	*Error N=8*
0.0000e+00	*1.0000e+00*	*1.0000e+00*	*3.8242e-06*
2.5000e-01	*8.0000e-01*	*8.0000e-01*	*5.5706e-08*
3.7500e-01	*7.2727e-01*	*7.2727e-01*	*4.2328e-08*
5.0000e-01	*6.6666e-01*	*6.6666e-01*	*3.6107e-08*
7.5000e-01	*5.7142e-01*	*5.7142e-01*	*2.8193e-08*
8.7500e-01	*5.3333e-01*	*5.3333e-01*	*3.2583e-08*
1.0000e+00	*5.0000e-01*	*5.0000e-01*	*2.4834e-07*

Table 4. *The exact and approximate solutions of example 2 in some arbitrary points, using the shifted fourth Chebyshev polynomial* $S_8^s(x)$

Example 5. *Consider the linear Volterra-Fredholm integral equation*

$$\int_{-1}^{x}(xt)\varphi(t)dt + \int_{-1}^{1}\cosh(x+t)\varphi(t)dt = f(x),$$

where the function $f(x) = \frac{1}{4}\cosh x(4+e^2-e^{-2}) + x^2\sinh x - x(\cosh x - e^{-1})$ *chosen so that the solution* $\varphi(x)$ *is given by*

$$\varphi(x) = \cosh x$$

Applying the third Chebyshev polynomial $V_8(x)$ *to approximate the solution* $\varphi_\alpha(x)$, *that is to say solution of the algebraic system of linear equations for* $\alpha = 10^{-10}$

Points of x	*Exact sol*	*Approx sol*	*Error N=8*
-1.0000e+00	*1.5430e+00*	*1.5430e+00*	*1.9541e-09*
-7.5000e-01	*1.2946e+00*	*1.2946e+00*	*2.0075e-10*
-5.0000e-01	*1.1276e+00*	*1.1276e+00*	*1.3849e-11*
0.0000e+00	*1.0000e+00*	*1.0000e+00*	*6.8679e-10*
5.0000e-01	*1.1276e+00*	*1.1276e+00*	*8.6020e-12*
7.5000e-01	*1.2946e+00*	*1.2946e+00*	*1.8078e-10*
1.0000e+00	*1.5430e+00*	*1.5430e+00*	*2.2303e-09*

Table 5. *The exact and approximate solutions of example 4 in some arbitrary points, using the third Chebyshev polynomial* $V_8(x)$

Example 6. *Consider the linear Volterra-Fredholm integral equation*

$$\int_{-1}^{x}(x+t)\varphi(t)dt + \int_{-1}^{1}e^{(x+t)}\varphi(t)dt = f(x),$$

where the function $f(x) = 2e^x - e^{-x} - 2xe^{-x} + xe$ *chosen so that the solution* $\varphi(x)$ *is given by*

$$\varphi(x) = e^{-x}$$

Applying the fourth Chebyshev polynomial $W_8(x)$ *to approximate the solution* $\varphi_\alpha(x)$, *that is to* *say solution of the algebraic system of linear equations for* $\alpha = 10^{-10}$

Points of x	*Exact sol*	*Approx sol*	*Error N=8*
-1.0000e+00	2.7182e+00	2.7182e+00	2.5395e-08
-7.5000e-01	2.1170e+00	2.1170e+00	8.8042e-09
-5.0000e-01	1.6487e+00	1.6487e+00	2.4954e-09
0.0000e+00	1.0000e+00	1.0000e+00	1.8502e-09
5.0000e-01	6.0653e-01	6.0653e-01	5.8265e-09
7.5000e-01	4.7236e-01	4.7236e-01	4.1970e-09
1.0000e+00	3.6787e-01	3.6787e-01	3.7479e-08

Table 6. *The exact and approximate solutions of example 5 in some arbitrary points, using the fourth Chebyshev polynomial* $W_8(x)$

5 Conclusion remarks

We can see that the sum of two compact operators is compact operator and so its range R(V+F)is not closed, consequently the inverse operator $(V + F)^{-1}$ *is never a continuous operator from its range to the whole space. The goal of this work is to replace the equation* (1.1) *ill posed Volterra-Fredholm integral equations of the first kind by a perturbed equation using Chebyshev polynomials of the third and the fourth kind to convert this perturbed equation to the system of linear equations. Finally, some numerical examples indicate the accuracy and the efficiency of this method.*

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