Palestine Journal of Mathematics

Vol 14(1)(2025) , 183–194

NULL CARTAN CURVE'S GENERALIZED INVOLUTE AND EVOLUTE CURVE COUPLE IN $\mathbb{E}_1^4$

M. Hanif, A. Ali, A.Ahmed and M.Ilyas, K.H. Shah

Communicated by Zafar Ahsan

MSC 2010 Classifications:Primary 53-XX, 53Axx . Secondary 53A04.

Keywords and phrases: Evolute, Null Cartan curve, Minkowski space.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.
Corresponding Author: Muhammad Hanif, Assistant Professor University of Narowal

Abstract *In this article, we obtained a special type of generalized involute curve in E_1^4 from null cartan curve. This study provides the conditions that are sufficient and required for a curve in E_1^4 to have both a generalized evolute and an involute curve.*

1 Introduction

In differential geometry, curves have various significant implications and attributes. These curves are vastly being studied by several researchers. Subsequently, researchers often introduce new curves based on previous studies. Among them are Involute [26] and evolute curves. Our knowledge of the local and global geometry of the general theory of curves in a Euclidean space, or more specifically in a Riemannian manifold, is now very extensive due to its long history of revelation. If two curves have been present and tangent line of second curve is perpendicular to the tangent line of the first curve, the second curve is referred to as the involute of the first curve. Moreover, a new approach to studying surface curves has been taken recently by Shaikh et al. [17]. They focus on rectifying, osculating, and normal curves on a surface by taking into account isometry and conformal maps between two surfaces and examining their invariance under such maps. In addition, curves on a smooth surface were also studied by Shaikh et al. [16]. Consequently, the position vector constantly remains in the surface's tangent plane, demonstrating the isometry of surfaces. However, a plenty of significant and compelling work has been done on normal curves, rectifying and osculating curves, rectifying curves under conformal transformation, and rectifying and osculating curves. [1],[18],[19],[20],[21],[22],[23],[24]. In a further step, orthonormal of the same space can be achieved by producing an evolute Frenet apparatus by an involute apparatus in four-dimensional Euclidean space, as discovered by Ozyilmaz and Yilmaz [14]. Frenet frame of involute curves depends on the curvatures of the provided curve, according to Bukcu and Karacan [4]. Sato [15] explored the singularities and geometric characteristics of pseudo-spherical evolutes of curves on a space-like surface in 3-dimensional Minkowski space. According to Izumiya. S.and Takahashi. M. [10], iteration of involutes generates a pair of curve sequences with respect to the Minkowski metric and its dual. Null curves have different properties than the other curves, such as time-like, space-like, or Euclidean curves. So, the partner curves of a null curve are also interesting and fascinating. For null curves, there are different conditions for the cases of space-like or time-like curves. According to Nolasco and Rui 12, the correspondence between plane curves and null curves exists in Minkowski 3-space, and the geometry of null curves according to the curvature of the corresponding plane curves was described. In 4-dimensional Minkowski space, Coken and Ciftci [5] differentiate pseudo-spherical null curves from Bertrand null curves. Sakaki 16 made two specific contributions: firstly, the evolute of a null curve in R_1^4; secondly, the involute of a spacelike curve in R_1^4, and the correspondence between them, which is very similar to the plane evolute and involute. We also refer to the papers [6], [7], [8],[25] for more results of finding involute and evolute. In the present *study, we define a novel class of specialized involute and evolute curves from null cartan curves in a 4-dimensional Minkowski space. We were successfully able to come up with the required and adequate conditions for the curve to have both evolute and involute properties.*

2 Preliminaries

The Minkowski spacetime E_1^4 is the Euclidean 4-space E^4 with a metric defined by

$$g(y) = -dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2$$

where (y_1, y_2, y_3, y_4) represents the rectangular coordinate system of E_1^4. Vector $v \in E_1^4 \setminus \{0\}$ can be space-like if $g(v, v) > 0$, time-like if $g(v, v) < 0$, and light-like (null) if $g(v, v) = 0$. $||v|| = \sqrt{|g(v, v)|}$ is the norm of vector v. We call two vectors v and w orthogonal if $g(v, w) = 0$. An arbitrary curve $\gamma(s)$ in E^4 can be locally space-like, time-like, or light-like if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like, or light-like. A curve $\gamma(s)$ in E_1^4 can be space-like, time-like, or light-like if its velocity vectors $\gamma'(s)$ are commonly space-like, time-like, or null. Consider $I \subset (a, b)$ to be an open interval in R and $\gamma : I \subset (a, b) \rightarrow E_1^4$ is a regular curve in E_1^4 parameterized by the arc length parameter s, and $\{T, N_1, N_2, N_3\}$ denote the moving Frenet Frame along γ, which consists of the tangent vector T, the principal normal vector N_1, the first binormal vector N_2, and the second binormal vector N_3, respectively. so that $T \wedge N_1 \wedge N_2 \wedge N_3$, coincides with the standard orientation of E_1^4. Then
$g(T, T) = \epsilon_1, g(N_1, N_1) = \epsilon_2, g(N_2, N_2) = \epsilon_3, g(N_3 \cdot N_3) = \epsilon_4, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1, \epsilon_i \in \{1, -1\}, i \in \{1, 2, 3, 4\}$.
In particular, the following conditions hold:
$g(T, N_1) = g(T, N_2) = g(T, N_3) = g(N_1, N_2) = g(N_1, N_3) = g(N_2, N_3) = 0$.
From [20] the Frenet-Serret Formula for α in E_1^4 is given by

$$\begin{aligned} T' &= \epsilon_2 k_1 N \\ N' &= -\epsilon_1 k_1 T + \epsilon_3 k_2 B_1 \\ B_1' &= -\epsilon_2 k_2 N - \epsilon_1\epsilon_2\epsilon_3 k_3 B_2 \\ B_2' &= -\epsilon_3 k_3 B_1 \end{aligned} \tag{2.1}$$

. A null curve γ is parameterized by pseudo-arc s if $g(\gamma''(s), \gamma''(s)) = 1$ [21]. Further more non-null curve γ, we have this condition $g(\gamma'(s), \gamma'(s)) = \pm 1$. From [22] if γ is null Cartan curve, the Cartan Frenet frame is given by

$$\begin{aligned} T' &= \kappa_1 N_1 \\ N_1' &= \kappa_2 T - \kappa_1 N_2 \\ N_2' &= -\kappa_2 N_1 + \kappa_3 N_3 \\ N_3' &= -\kappa_3 T \end{aligned} \tag{2.2}$$

, where $g(T, T) = g(N_2, N_2) = 0$, and $g(N_1, N_1)$=$g(N_3, N_3) = 1$, also $g(T, N_1) = g(T, N_3) = g(N_1, N_2) = g(N_1, N_3) = g(N_2, N_3) = 0$, and $g(T, N_2) = 1$.

3 A null cartan curve's generalized involute curve in E_1^4

In this portion we present generalized involute curves of a light-like(null) cartan curve in E_1^4. First, we define two types of involute curves of a light-like cartan curve as follows:
Definition 3.1.((1,2)-type of generalized involute curve)*:*
A curve α is called (1,2)-type of generalized involute of β in 4-dimensional Minkowski space if $\alpha = \beta + \lambda T_\beta$, where T_β is orthogonal to (1,2)-plane spanned by $\{T_\alpha, N_\alpha\}$.
Definition 3.2.((1,3)-type of generalized involute curve)*:*
A curve α is called (1,3)-type of generalized involute of β in 4-dimensional Minkowski space if $\alpha = \beta + \lambda T_\beta$, where T_β is orthogonal to (1,3)-plane spanned by $\{T_\alpha, B_\alpha\}$.

The tangent T^* of evolute Γ^* of a null cartan curve is colinear with N_3, where N_3 is binormal of null cartan curve. We generalized this condition by assuming that T^* lies in the planes $\{N_3, T\}$, $\{N_3, N_1\}$, $\{N_3, N_2\}$. In special case, T^* which lies in the plane $\{N_3, T\}$ can be collinear with N_3 and then we get ordinary Evolute.

Let γ be a Cartan light-like curve in E_1^4 with Cartan frame $\{T, N_1, N_2, N_3\}$. Then Sakaki [16] gives the evolute of gamma,

$$\gamma^*(s^*) = \gamma(s) + \left(\frac{1}{\kappa_3}\right) N_3.$$

Differentiating, we get

$$\frac{ds^*}{ds}T^* = T(s) + \left(\frac{1}{\kappa_3}\right)' N_3 + \left(\frac{1}{\kappa_3}\right)(-\kappa_3 T) = \left(\frac{1}{\kappa_3}\right)' N_3.$$

Therefore

$$T^* = N_3.$$

This implies that γ^* is a space-like curve. We will generalize this condition assuming that space-like planes coincide, i.e

$$\text{span}\{T^*, N_2^*\} = \text{span}\{N_1, N_3\}.$$

This condition implies that time-like planes coincide, that is

$$\text{span}\{N_1^*, N_3^*\} = \text{span}\{T, N_2\}.$$

Since $T^* \in \text{span}\{N_1, N_3\}$, we have

$$T^* = \Phi(s)N_1(s) + \Psi(s)N_3,$$

where $\Phi^2 + \Psi^2 = 1$. Hence parametric equation of Generalized evolute of γ reads

$$\gamma^*(s^*) = \gamma(s) + \left(\frac{1}{\kappa_3}\right) T^* = \gamma(s) + \left(\frac{1}{\kappa_3}\right)(\Phi(s)N_1(s) + \Psi(s)N_3),$$

for $\Phi(s) = 0$ and $\Psi(s) = 1$, we get standered evolute.

3.1 Theorem

Let $\gamma : I \to E_1^4$ be a Cartan light-like curve with arc-length parameter s such that $\kappa_1 = 1$ and $\kappa_2, \kappa_3 \neq 0$. Then the curve γ is an $(1,3)$-evolute curve, and its evolute mate curve is a space-like or time-like curve with curvatures not equal to zero iff $\exists$ Φ, Ψ scaler functions of arc-length parameter s and real constants $\Lambda \neq \pm$, Ω satisfying

$$(\Phi'\kappa_3 - \Phi\kappa_3') = \Lambda(\Psi'\kappa_3 - \Psi\kappa_3'), \tag{3.1}$$

$$\Omega\Lambda\kappa_1 = \Lambda\kappa_2 - \kappa_3, \tag{3.2}$$

$$-2\Lambda^3\kappa_1\kappa_2 + \Lambda^2\kappa_1\kappa_3 \neq 0. \tag{3.3}$$

Proof. *Let $\gamma : I \to E_1^4$ be a regular curve with an arc-length parameter s, so that$\kappa_1, \kappa_2, \kappa_3 \neq 0$. Let $\gamma^* : I \to E_1^4$ be the $(1,3)$-evolute of γ. Denote$\{T^*, N_1^*, N_2^*, N_3^*\}$to be the Frenet frame along γ^* and κ_1^*, κ_2^*, and κ_3^* are the curvatures of γ^*. Then*

$$\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}, \qquad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}.$$

Further, we may express the curve $\gamma*$ as follows:

$$\gamma^*(s^*) = \gamma^*(g(s)) = \gamma(s) + \frac{1}{\kappa_3}(\Phi(s)N_1(s) + \Psi(s)N_3), \tag{3.4}$$

$\forall s^* \in I^*$, $s \in I$ *where $\Phi(s)$ and $\Psi(s)$ are C^∞ functions on I.*

Differentiating (3.4) by using equation (2.2), we will get

$$T^*g' = \left(1+\frac{1}{\kappa_3}(\Phi\kappa_2-\Psi\kappa_3)\right)T(s)+\left(\frac{\Phi'\kappa_3-\Phi\kappa_3'}{\kappa_3\kappa_3'}\right)N_1-\left(\frac{\Phi\kappa_1}{\kappa_3}\right)N_2+\left(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{\kappa_3\kappa_3'}\right)N_3. \tag{3.5}$$

Taking inner product of (3.5) by T and N_2 respectively, we get

$$g^{'}T^* = \left(\frac{\Phi'\kappa_3-\Phi\kappa_3'}{\kappa_3\kappa_3'}\right)N_1+\left(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{\kappa_3\kappa_3'}\right)N_3. \tag{3.6}$$

So equation (3.6) gets the form

$$T^* = \left(\frac{\Phi'\kappa_3-\Phi\kappa_3'}{g'\kappa_3\kappa_3'}\right)N_1+\left(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{g'\kappa_3\kappa_3'}\right)N_3. \tag{3.7}$$

Multiplying (3.7) by itself, we get

$$\epsilon_1^*(g^{'})^2 = \left(\frac{\Phi'\kappa_3-\Phi\kappa_3'}{\kappa_3\kappa_3'}\right)^2+\left(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{\kappa_3\kappa_3'}\right)^2. \tag{3.8}$$

If we denote

$$\alpha = \left(\frac{\Phi'\kappa_3-\Phi\kappa_3'}{g'\kappa_3\kappa_3'}\right), \beta = \left(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{g'\kappa_3\kappa_3'}\right). \tag{3.9}$$

Using equation (3.9) in (3.7), we get

$$T^* = \alpha N_1+\beta N_3. \tag{3.10}$$

Taking derivative of equation (3.10) using (2.2), we get

$$\epsilon_2^* g'\kappa_1^* N_1^* = (\alpha\kappa_2-\beta\kappa_3)T+\alpha' N_1-\alpha\kappa_1 N_2+\beta' N_3. \tag{3.11}$$

Since $\{N_1^*, N_3^*\}\perp\{N_1, N_3\}$, so we get

$$\alpha' = 0, \beta' = 0. \tag{3.12}$$

using equation (3.12) in equation (3.11), we will get

$$\epsilon_2^* g'\kappa_1^* N_1^* = (\alpha\kappa_2-\beta\kappa_3)T-\alpha\kappa_1 N_2. \tag{3.13}$$

Multiplying equation (3.13) by itself, we get

$$\epsilon_2^*(g'\kappa_1^*)^2 = -2\alpha\kappa_1(\alpha\kappa_2-\beta\kappa_3). \tag{3.14}$$

From equation (3.9), we get the result (3.1)

$$(\Phi'\kappa_3-\Phi\kappa_3') = \Lambda(\Psi'\kappa_3-\Psi\kappa_3'), \tag{3.15}$$

where $\Lambda = \left(\frac{\alpha}{\beta}\right)$, $\beta \neq 0$.
Using (3.9) in (3.14), we get

$$\epsilon_2^*(g')^2(\kappa_1^*)^2 = -2\left(\frac{\Phi'\kappa_3-\Phi\kappa_3'}{g'\kappa_3\kappa_3'}\right)^2\kappa_1\left(\frac{\Lambda\kappa_2-\kappa_3}{\Lambda}\right) \tag{3.16}$$

Using (3.15) in (3.8), we acquire

$$\epsilon_1^* g'^2 = \left(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{\kappa_3\kappa_3'}\right)^2(\Lambda^2+1). \tag{3.17}$$

Substituting equation(3.17) in (3.16), we get

$$(g')^2(\kappa_1^*)^2 = -2\frac{1}{(\Lambda^2+1)}[\Lambda\kappa_1(\Lambda\kappa_2-\kappa_3)]. \tag{3.18}$$

Denote

$$\Delta_1 = \frac{\alpha\kappa_2 - \beta\kappa_3}{g'\kappa_1^*} = \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{g'^2\kappa_1^*\kappa_3\kappa_3'}\right)[\Lambda\kappa_2 - \kappa_3], \tag{3.19}$$

$$\Delta_2 = \frac{\alpha\kappa_1}{g'\kappa_1^*} = \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{g'^2\kappa_1^*\kappa_3\kappa_3'}\right)(\Lambda\kappa_1). \tag{3.20}$$

Dividing (3.19)by (3.20), we accruire the result (3.2)

$$\Omega\Lambda\kappa_1 = \Lambda\kappa_2 - \kappa_3, \tag{3.21}$$

where $\Omega = -\left(\frac{\Delta_1}{\Delta_2}\right)$.
Using (3.19) and (3.20) in equation (3.13), we get

$$N_1^* = \Delta_1 T + \Delta_2 N_2. \tag{3.22}$$

Differentiating equation (3.22) by using equation (2.2), we obtain

$$-\epsilon_1^* g'\kappa_1^* T^* + \epsilon_3^* g'\kappa_2^* N_2^* = \Delta_1' T + (\Delta_1\kappa_1 - \Delta_2\kappa_2)N_1 + \Delta_2' N_2 + \Delta_2\kappa_3 N_3. \tag{3.23}$$

Since $\{T^*, N_2^*\}\perp\{T, N_2\}$*, so we obtain*

$$\Delta_1' = 0, \Delta_2' = 0. \tag{3.24}$$

Using (3.24) in (3.23), we obtain

$$-\epsilon_1^* g'\kappa_1^* T^* + \epsilon_3^* g'\kappa_2^* N_2^* = (\Delta_1\kappa_1 - \Delta_2\kappa_2)N_1 + \Delta_2\kappa_3 N_3. \tag{3.25}$$

Using the (3.6), (3.19) and (3.20) in (3.25), we obtain

$$\epsilon_3^* g'\kappa_2^* N_2^* = P(s)N_1 + Q(s)N_3, \tag{3.26}$$

where

$$P(s) = \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{g'^2\kappa_1^*\kappa_3\kappa_3'}\right)[-2\Lambda^3\kappa_1\kappa_2 + \Lambda^2\kappa_1\kappa_3], \tag{3.27}$$

$$Q(s) = \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{g'^2\kappa_1^*\kappa_3\kappa_3' h}\right)[-2\Lambda^3\kappa_1\kappa_2 + \Lambda^2\kappa_1\kappa_3]. \tag{3.28}$$

Since

$$\epsilon_3^* g'\kappa_2^* N_2^* \neq 0.$$

So, we get the result (3.3)

$$-2\Lambda^3\kappa_1\kappa_2 + \Lambda^2\kappa_1\kappa_3 \neq 0$$

Conversely, let $\gamma : I \subset R \to E_1^4$ *be an evolute curve whose arc length parameter* s *is such that* $k_1, k_2, k_3 \neq 0$*. And relations (3.1), (3.2) and (3.3) hold for some functions* Φ *and* Ψ *that are differentiable of arc length parameters* s *and real constants* $\Lambda \neq \pm 1$, Ω*. Then the curve* γ^* *can be expressed like this*

$$\gamma^*(s^*) = \gamma(s) + \frac{1}{\kappa_3}(\Phi(s)N_1(s) + \Psi(s)N_3). \tag{3.29}$$

Differentiating (3.29) by using equation (2.2), we obtain

$$\frac{d\gamma^*}{ds} = \left(1 + \frac{\Phi k_2 - \Psi k_3}{k_3}\right)T + \left(\frac{\Phi' k_3 - \Phi k_3'}{k_3^2}\right)N_1 - \left(\frac{\Phi k_1}{k_3}\right)N_2 + \left(\frac{\Psi' k_3 - \Psi k_3'}{k_3^2}\right)N_3.$$

Taking inner product with T *and* N_2

$$\frac{d\gamma^*}{ds} = \left(\frac{\Phi'\kappa_3 - \Phi\kappa_3'}{\kappa_3\kappa_3'}\right)N_1 + \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{\kappa_3\kappa_3'}\right)N_3. \tag{3.30}$$

From (32) and (3), we get

$$\frac{d\gamma^*}{ds} = \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{\kappa_3\kappa_3'}\right)[\Lambda N_1 + N_3]. \tag{3.31}$$

From this

$$g' = \frac{ds^*}{ds} = ||\frac{d\Gamma^*}{ds}|| = c_1\left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{\kappa_3\kappa_3'}\right)\sqrt{c_2(\Lambda^2+1)} > 0, \tag{3.32}$$

such that $c_1(\frac{\Psi'\kappa_3-\Psi\kappa_3'}{\kappa_3\kappa_3'}) > 0$ *where* $c_1 = \pm 1$ *and* $c_2 = \pm 1$ *such that* $c_2(\Lambda^2+1) > 0$. *rewrite equation (32)*

$$T^*g' = \left(\frac{\Psi'\kappa_3 - \Psi\kappa_3'}{\kappa_3\kappa_3'}\right)[\Lambda N_1 + N_3]. \tag{3.33}$$

Using (3.32) in (3.33), we get

$$T^* = \frac{c_1}{\sqrt{c_2(\Lambda^2+1)}}[\Lambda N_1 + N_3], \tag{3.34}$$

which indicates that $g(T^*, T^*) = c_2 = \epsilon_1^*$, *where* $c_1 = \pm 1$.
differentiating equation (3.34) by using (2.2), we acquire

$$\frac{dT^*}{ds^*} = \frac{c_1}{g'\sqrt{c_2(\Lambda^2+1)}}[(\Lambda\kappa_2 - \kappa_3)T - \Lambda\kappa_1 N_2]. \tag{3.35}$$

Using (3.35), we have

$$k_1^* = ||\frac{dT^*}{ds}|| = \frac{\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}{g'\sqrt{c_2(\Lambda^2+1)}} > 0 \tag{3.36}$$

From equation (3.35) and (3.36), we obtain

$$N_1^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{c_1}{\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}[(\Lambda\kappa_2 - \kappa_3)T - \Lambda\kappa_1 N_2], \tag{3.37}$$

which indicates that $g(N_1^*, N_1^*) = 1$.
Let

$$\Delta_3 = \frac{c_1(\Lambda\kappa_2 - \kappa_3)}{\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}, \quad \Delta_4 = \frac{-c_1\Lambda\kappa_1}{\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}, \tag{3.38}$$

we acquire

$$N^* = \Delta_3 T + \Delta_4 N_2. \tag{3.39}$$

differentiating (3.39) by using equation (2.2), we obtain

$$g'\frac{dN_1^*}{ds^*} = \Delta_3' T + (\Delta_3\kappa_1 - \Delta_4\kappa_2)N_1 + \Delta_4' N_2 + \Delta_4\kappa_3 N_3. \tag{3.40}$$

Differentiating (3.2),we will get

$$(\Lambda\kappa_2' - \kappa_3')\Lambda\kappa_1 - (\Lambda\kappa_2 - \kappa_3)\Lambda\kappa_1' = 0. \tag{3.41}$$

differentiating (3.38) with respect to s by using (3.41), we heve

$$\Delta_3' = 0, \quad \Delta_4' = 0. \tag{3.42}$$

Substituting the values (3.38) and (3.42) in (3.40), we get

$$\frac{dN_1^*}{ds^*} = \frac{2c_1\Lambda\kappa_1\kappa_2 - c_1\kappa_1\kappa_3}{g'\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}N_1 - \frac{c_1\Lambda\kappa_1\kappa_3}{g'\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}N_3. \tag{3.43}$$

Using equation (3.34) and (3.36), we obtain

$$\epsilon_1^*\kappa_1^*T^* = \frac{c_1\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2 - \kappa_3)}}{g'(\Lambda^2+1)}[\Lambda N_1 + N_3]. \tag{3.44}$$

From (3.43) and (3.44), we obtain

$$\frac{dN_1^*}{ds^*} + \epsilon_1^* \kappa_1^* T^* = \frac{c_1(\Lambda^2\kappa_1\kappa_3 + 2\Lambda\kappa_1\kappa_2 - \kappa_1\kappa_3)}{g'(\Lambda^2+1)\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2-\kappa_3)}}[N_1 - \Lambda N_3], \tag{3.45}$$

From (3.45), we have

$$k_2^* = \frac{|(\Lambda^2\kappa_1\kappa_3 + 2\Lambda\kappa_1\kappa_2 - \kappa_1\kappa_3)|}{g'(\Lambda^2+1)\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2-\kappa_3)}} > 0. \tag{3.46}$$

Considering (3.45) and (3.46) together, we obtain

$$N_2^* = \frac{\epsilon_3^*}{\kappa_2^*}\left[\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^* T^*\right] = c_2\epsilon_3^*[N_1 - \Lambda N_3], \tag{3.47}$$

where $c_2 = \frac{|(\Lambda^2\kappa_1\kappa_3+2\Lambda\kappa_1\kappa_2-\kappa_1\kappa_3)|}{|(\Lambda^2\kappa_1\kappa_3+2\Lambda\kappa_1\kappa_2-\kappa_1\kappa_3)|} = \pm 1$ *and* $\epsilon_3^* = \pm 1$.
From (3.47), we acquire $g(N_2^*, N_2^*) = c_1 = \epsilon_3^* = -\epsilon_1^*$ *, a unit vector* N_3^* *can be represented like this* $N_3^* = -\Delta_4 T + \Delta_3 N_2$*; that is,*

$$N_3^* = \frac{c_1\epsilon_3}{\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2-\kappa_3)}}\left[(\Lambda\kappa_1)T - \Lambda(\kappa_2-\kappa_3)N_2\right], \tag{3.48}$$

which indicates that $g(N_3^*, N_3^*) = 1$. *In the end we find* κ_3^*

$$\kappa_3^* = H(\frac{dN_2^*}{ds^*}, N_3^*) = \frac{c_1c_2\kappa_1\epsilon_3^*(\Lambda^2+1)\kappa_3}{g'\sqrt{c_3(\Lambda^2+1)}\sqrt{(-2\Lambda\kappa_1)(\Lambda\kappa_2-\kappa_3)}} \neq 0.$$

Hence we find that γ^* *is (1,3)-Evolute curve of the curve* γ *since* $\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}$, $\text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}$.

3.2 Theorem

let $\gamma : I \to E_1^4$ *be a Cartan light-like (null) curve by an arc-length parameter* s *such that* $\kappa_1 = 1$ *and* $\kappa_2, \kappa_3 \neq 0$. *Then curve* γ *is a* $(0,2)$*-evolute curve, and its evolute mate curve is a space-like or time-like curve with curvatures not equal to zero iff there exist constant functions* x, y, h, *and* $\mu \pm 1$ *satisfying,*

$$(x\kappa_1 - y\kappa_2) = hy\kappa_3, \tag{3.49}$$

$$-\mu h\kappa_1 = h\kappa_2 - \kappa_3, \tag{3.50}$$

$$h^2\kappa_1\kappa_3 - 2h\kappa_1\kappa_2 - \kappa_1\kappa_3 \neq 0, \tag{3.51}$$

for all $s \in I$.

Proof*. Let* $\gamma : I \to E_1^4$ *be a Cartan light-like curve with an arc-length parameter* s *such that* $\kappa_1, \kappa_2, \kappa_3 \neq 0$. *Let* $\gamma^* : I \to E_1^4$ *be the* $(0,2)$*-evolute curve of* γ. *Let* $\{T^*, N_1^*, N_2^*, N_3^*\}$ *be The Frenet frame along* γ^* *and* κ_1^*, κ_2^*, *and* κ_3^* *are its curvatures of* γ^*. *Then*

$$\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}, \qquad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}.$$

Furthermore, we may express the curve γ^* *as follows:*

$$\gamma^*(s^*) = \gamma(s) + \frac{1}{\kappa_3}[x(s)T(s) + y(s)N_2] \tag{3.52}$$

$\forall\, s^* \in I^*$ *and* $s \in I$ *here* $x(s)$ *and* $y(s)$ *are the* C^∞ *functions on* I.
Differentiating (3.52) by using equation (2.2), we will get

$$T^*g' = \left(1 + \frac{x'\kappa_3 - x\kappa_3'}{\kappa_3\kappa_3'}\right)T(s) + \left(\frac{x\kappa_1 - y\kappa_2}{\kappa_3}\right)N_1 + \left(\frac{y'\kappa_3 - y\kappa_3'}{\kappa_3\kappa_3'}\right)N_2 + yN_3 \tag{3.53}$$

By taking the inner product of (3.53) on both sides with T and N_2, respectively, we acquire $x' = 0$ and $y' = 0$, implying that x and y are constants. As a result, (3.53) turns into

$$T^* g' = \left(\frac{x\kappa_1 - y\kappa_2}{\kappa_3}\right) N_1 + yN_3. \tag{3.54}$$

Multiplying (3.54) by itself, we get

$$\epsilon_1^*(g')^2 = \left(\frac{x\kappa_1 - y\kappa_2}{\kappa_3}\right)^2 + y^2. \tag{3.55}$$

If we denote

$$\alpha = \left(\frac{x\kappa_1 - y\kappa_2}{g'\kappa_3}\right), \beta = \left(\frac{y}{g'}\right). \tag{3.56}$$

So, (3.54) gets the form

$$T^* = \alpha N_1 + \beta N_3. \tag{3.57}$$

Taking derivative of equation (3.57) using equation (2.2) , we acquire

$$\epsilon_1^* g'\kappa_1^* N_1^* = (\alpha\kappa_2 - \beta\kappa_3)T + \alpha' N_1 - \alpha\kappa_1 N_2 + \beta' N_3. \tag{3.58}$$

Taking inner product of (3.58) by N_1 and N_3 respectively, we obtain

$$\alpha' = 0, \beta' = 0. \tag{3.59}$$

Using (3.59) in (3.58), we get

$$\epsilon_2^* g'\kappa_1^* N_1^* = (\alpha\kappa_2 - \beta\kappa_3)T - \alpha\kappa_1 N_2. \tag{3.60}$$

Multiplying (3.60) by itself, we obtain

$$\epsilon_2^*(g')^2(\kappa_1^*)^2 = -2\Lambda\left(\frac{y}{g'}\right)^2 [\Lambda\kappa_1\kappa_2 - \kappa_1\kappa_3]. \tag{3.61}$$

From (3.56), we obtain

$$(x\kappa_1 - y\kappa_2)\beta = \alpha(y\kappa_3). \tag{3.62}$$

From this, we acquire the result (3.49)

$$(x\kappa_1 - y\kappa_2) = hy\kappa_3, \tag{3.63}$$

where $h = \frac{\alpha}{\beta}$, $\beta \neq 0$.
Using (3.63) in (3.55), we obtain

$$\epsilon_1^* g'^2 = y^2(h^2 + 1). \tag{3.64}$$

Substituting (3.64) in (3.61), we get

$$(g')^2(\kappa_1^*)^2 = \left(\frac{-2h}{h^2+1}\right)[(h\kappa_1\kappa_2 - \kappa_1\kappa_3)]. \tag{3.65}$$

If we denote

$$\Delta_1 = \frac{\alpha\kappa_2 - \beta\kappa_3}{g'\kappa_1^*} = \left(\frac{y\kappa_3}{g'^2\kappa_1^*\kappa_2}\right)[(h\kappa_2 - \kappa_3)], \tag{3.66}$$

$$\Delta_2 = -\frac{\alpha}{g'\kappa_1^*} = -\left(\frac{y\kappa_3}{g'^2\kappa_2\kappa_1^*}\right)h\kappa_1. \tag{3.67}$$

Dividing (3.66) by (3.67), we obtain the result (3.50)

$$-\mu h\kappa_1 = h\kappa_2 - \kappa_3,$$

where $\mu = \frac{\Delta_1}{\Delta_2}$, $\Delta_2 \neq 0$.
Using (3.66) and (3.67) in (3.60), we will get

$$N_1^* = \Delta_1 T + \Delta_2 N_2. \tag{3.68}$$

Differentiating (3.68) by following equation (2.2), we acquire.

$$-\epsilon_1^* g' \kappa_1^* T^* + \epsilon_3^* g' \kappa_2^* N_2^* = \Delta_1^{'} T + (\Delta_1 \kappa_1 - \Delta_2 \kappa_2) N_1 + \Delta_2^{'} N_2 + \Delta_2 \kappa_3 N_3. \tag{3.69}$$

By multiplying equation (3.69) by T and N_2, we acquire.

$$\Delta_1^{'} = 0, \Delta_2^{'} = 0. \tag{3.70}$$

Using the (3.66), (3.67) and (3.70) in (3.69), we obtain

$$\epsilon_3^* g' k_2^* N_2^* = R(s) N_1 + Q(s) N_3, \tag{3.71}$$

where

$$R(s) = \left(\frac{y\kappa_3}{g'^2 (h^2+1) \kappa_1^* \kappa_2} \right) [h^2 \kappa_1 \kappa_3 - 2h\kappa_1 \kappa_2 - \kappa_1 \kappa_3], \tag{3.72}$$

$$Q(s) = \left(\frac{y\kappa_3}{g'^2 (h^2+1) \kappa_1^* \kappa_2} \right) [h^2 \kappa_1 \kappa_3 - 2h\kappa_1 \kappa_2 - \kappa_1 \kappa_3]. \tag{3.73}$$

Since

$$\epsilon_3^* g' k_2^* N_2^* \neq 0.$$

So we get the result (3.51)

$$\Lambda^2 \kappa_1 \kappa_3 - 2\Lambda \kappa_1 \kappa_2 - \kappa_1 \kappa_3 \neq 0. \tag{3.74}$$

In reverse, we assume that $\gamma : I \subset R \to E_1^4$ be a Cartan light-like curve by arc-length parameter s and $\kappa_1, \kappa_2, \kappa_3 \neq 0$, and the relations (3.49), (3.50), (3.51) hold for differentiable scalar functions of arc-length parameter s x, y, Λ, $\Omega \neq 0$. The curve γ^* can be described as follows:

$$\gamma^*(s^*) = \gamma(s) + \frac{1}{\kappa_3} [x(s) T(s) + y(s) N_2]. \tag{3.75}$$

Differentiating equation (3.75) by using equation (2.2), we acquire

$$\frac{d\gamma^*}{ds} = \left(\frac{x\kappa_1 - y\kappa_2}{\kappa_3} \right) N_1 + y N_3. \tag{3.76}$$

From (3.49), we get

$$\frac{d\gamma^*}{ds} = \left(\frac{hy\kappa_3}{\kappa_3} \right) N_1 + \left(\frac{y\kappa_3}{\kappa_3} \right) N_3.$$

$$\frac{d\gamma^*}{ds} = y[hN_1 + N_3]. \tag{3.77}$$

From this

$$g' = \frac{ds^*}{ds} = ||\frac{d\gamma^*}{ds}|| = c_1 y \sqrt{c_2(h^2+1)} > 0, \tag{3.78}$$

such that $c_1(\frac{y\kappa_3}{\kappa_2}) > 0$ where $c_1 = \pm 1$ and $c_2 = \pm 1$ such that $c_2(h^2 - 1) > 0$. Rewrite equation (3.77)

$$T^* g' = y[hN_1 + N_3]. \tag{3.79}$$

Substituting (3.78) in (3.79), we get

$$T^* = \frac{c_1}{\sqrt{c_2(h^2+1)}} [hN_1 + N_3], \tag{3.80}$$

which shows that $H(T^*, T^*) = c_2 = \epsilon_1^*$.
Differentiating equation (3.80) s by using equation (2.2), we acquire

$$\frac{dT^*}{ds^*} = \frac{c_1}{g' \sqrt{c_2(h^2+1)}} [(h\kappa_2 - \kappa_3) T - h\kappa_1 N_2]. \tag{3.81}$$

Using (3.81), we get

$$k_1^* = ||\frac{dT^*}{ds}|| = \frac{\sqrt{-2h\kappa_1(h\kappa_2 - \kappa_3)}}{f' \sqrt{c_2(h^2+1)}} > 0 \tag{3.82}$$

From (3.81) and (3.82), we have

$$N_1^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{c_1}{\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}[(h\kappa_2-\kappa_3)T - h\kappa_1 N_2], \tag{3.83}$$

which indicate that $g(N_1^*, N_1^*) = 1$.

If we denote

$$\Delta_3 = \frac{c_1(h\kappa_2-\kappa_3)}{\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}, \quad \Delta_4 = -\frac{c_1 h\kappa_1}{\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}. \tag{3.84}$$

Using (3.84) in (3.83), we get

$$N_1^* = \Delta_3 T + \Delta_4 N_2. \tag{3.85}$$

Taking derivative of (3.85) using equation (2.2), we acquire

$$g'\frac{dN_1^*}{ds^*} = \Delta_3' T + (\Delta_3\kappa_1 - \Delta_4\kappa_2)N_1 + \Delta_4' N_2 + \Delta_4\kappa_3 N_3. \tag{3.86}$$

Differentiating (3.50), we get

$$h\kappa_1'(h\kappa_2-\kappa_3) + h\kappa_1(h\kappa_2' - \kappa_3') = 0. \tag{3.87}$$

Differentiating equation (3.84) with respect to s *by using (3.87), we acquire*

$$\Delta_3' = 0, \quad \Delta_4' = 0. \tag{3.88}$$

Substituting (3.84) and (3.88) in (3.86), we get

$$\frac{dN_1^*}{ds^*} = \frac{c_1(2h\kappa_1\kappa_2 - \kappa_1\kappa_3)}{g'\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}N_1 + \frac{c_1 h\kappa_1\kappa_3}{g'\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}N_3 \tag{3.89}$$

From (3.80) and (3.82), we get

$$\epsilon_1^*\kappa_1^* T^* = \frac{c_1\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}{g'(h^2+1)}[hN_1 + N_3]. \tag{3.90}$$

Adding (3.89),(3.90), we get

$$\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^* T^* = \frac{c_1(2h\kappa_1\kappa_2 + h^2\kappa_1\kappa_3 - \kappa_1\kappa_3)}{g'(h^2+1)\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}[N_1 + hN_3], \tag{3.91}$$

From (3.91), we have

$$k_2^* = \frac{|(2h\kappa_1\kappa_2 + h^2\kappa_1\kappa_3 - \kappa_1\kappa_3)|}{g(h^2+1)\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}} > 0. \tag{3.92}$$

Considering (3.91) and (3.92) together, we obtain

$$N_2^* = \frac{\epsilon_3^*}{\kappa_2^*}[\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^* T^*] = c_2 c_3 \epsilon_3^*[N_1 + hN_3], \tag{3.93}$$

where $c_3 = \frac{|(2h\kappa_1\kappa_2 + h^2\kappa_1\kappa_3 - \kappa_1\kappa_3)|}{|(2h\kappa_1\kappa_2 + h^2\kappa_1\kappa_3 - \kappa_1\kappa_3)|} = \pm 1$ *and* $\epsilon_3^* = \pm 1$. *From (3.93)* $g(N_2^*, N_2^*) = c_1 = \epsilon_3^* = -\epsilon_1^*$ *, also unit vector* N_3^* *can be expressed like this* $N_3^* = -\Delta_4 T + \Delta_3 N_2$*; i.e,*

$$N_3^* = \frac{c_2(h\kappa_2-\kappa_3)}{\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}T + \frac{c_2 h\kappa_1}{\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}}N_2, \tag{3.94}$$

which indicates that $g(N_3^*, N_3^*) = 1$. *In the end, we find* κ_3^* *as,*

$$\kappa_3^* = g(\frac{dN_2^*}{ds^*}, N_3^*) = \frac{c_3\epsilon_3^*\kappa_3}{g'\sqrt{-2h\kappa_1(h\kappa_2-\kappa_3)}} \neq 0.$$

Thus, we examine that γ^* *is space-like or time-like curve and a (1,2)-evolute curve of the curve* Γ *assuming* $\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}$, $\quad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}$.

4 Conclusion Remarks

Null curves have different properties than the other curves, such as time-like, space-like,or Euclidean curves. So, the partner curves of a null curve are also interesting and fascinating. The popular examples of a couple of curves are Bertrand curves, quaternionic Bertrand curves, mannheim curves, and involute and evolute curves. Of course, in 4-dimensional Minkowski space-time, the partner curves of a null Curves have different types and characterizations. This paper gives two new types of involute curves of a null Cartan curve in E_1^4 called (1,2)-type and (1,3)-type generalized involutes. Sufficient and required conditions for a curve to be an evolute of a null Cartan curve are introduced as two systems of equations, one of which is differentiable while the other is not.

References

[1] *Altinok, Mesut, and Levent Kula. "Slant helices generated by plane curves in Euclidean 3-space." Palestine Journal of Mathematics 5.2 (2016).*

[2] *Bonnor, W. B.* Curves with null normals in Minkowski space-time*, In A random walk in relativity and cosmology, 33-47, (1985).*

[3] *Bonnor. W.B.* Null curves in a Minkowski space-time*, Tensor,* **20(2)***, 229-242, (1969.*

[4] *Bukcu. B. and Karacan. M.K.* On the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3-space. *Int. J. Contemp. Math, Sciences,* **2(5)** *221-232, (2007).*

[5] *ken, A. Ceylan, and ver ft "On the Cartan curvatures of a null curve in Minkowski spacetime." Geometriae Dedicata 114 (2005): 71-78.*

[6] *Hanif, Muhammad, and Zhong Hua Hou.* Generalized involute and evolute curve-couple in Euclidean space*, Int. J. Open Problems Compt. Math* **11(2)** *28-39, (2018).*

[7] *Hanif, M., Hou, Z.H. and Nisar, K.S* On Special Kinds of Involute and Evolute Curves in 4-Dimensional Minkowski Space*, Symmetry* **10(8)** *p.317, (2018)*

[8] *Hanif, Muhammad, and Zhong Hua Hou.* A new approach to find a generalized evolute and involute curve in 4-dimensional Minkowski space-time*, Palestine Journal of Mathematics* **8(1)** *(2019).*

[9] *Ilarslan. K. and Neovic. E.* Spacelike and timelike normal curves in Minkowski space-time*, Publications de l'Institut Mathematique,* **85(99)***, 111-118, (2009).*

[10] *Izumiya. S.and Takahashi. M.* Spacelike parallels and evolutes in Minkowski pseudo-spheres*, Journal of Geometry and Physics,* **57(8)***, 1569-1600, (2007).*

[11] *ken. A. Ceylan and nver.* On the Cartan curvatures of a null curve in Minkowski spacetime*, Geometriae Dedicata* **114(1)***, 71-78 (2005).*

[12] *Nolasco. B. and Pacheco. R.* Evolutes of plane curves and null curves in Minkowski 3-space*, Journal of Geometry,* **108(1)***, 195-214, (2017).*

[13] *O'neill. B.* Semi-Riemannian geometry with applications to relativity (Vol. 103)*. Academic press, (1983).*

[14] *yilmaz, Emin, and Sha Yilmaz,* Involute-evolute curve couples in the euclidean 4-Space*, Int. J. Open Problems Compt. Math,* **2(2)** *168-174, (2009).*

[15] *Sato, T.* Pseudo-spherical evolutes of curves on a spacelike surface in three dimensional Lorentz Minkowski space*, Journal of geometry,,* **103(2)***, 319-331, (2012).*

[16] *Sakaki, M.* Notes on null curves in Minkowski spaces. Turkish Journal of Mathematics*,* **34(3)***, 417-424, (2010)*

[17] *Shaikh, A. A. and Ghosh, P. R., Rectifying curves on a smooth surfaces immersed in the Euclidean space, Indian J. Pure Appl. Math.,* **50(4)***, 883-890, (2019).*

[18] *Shaikh, A. A. and Ghosh, P. R., Curves on a smooth surface with position vectors lie in the tangent plane, Indian J. Pure Appl. Math.,* **51(3)** *1097-1104, (2020).*

[19] *Shaikh, A. A. and Ghosh, P. R., Rectifying and Osculating curves on a smooth surface, Indian J. Pure Appl. Math.,* **51(1)***, 67-75, (2020).*

[20] *Shaikh, A. A., Lone, M. S. and Ghosh, P. R., Conformal image of an osculating curves on a smooth immersed surface, J. Geom. Phy.,* **151***, (2020).*

[21] *Shaikh, A. A., Lone, M. S. and Ghosh, P. R., Normal curves on a smooth immersed surface, Indian J. Pure Appl. Math.,* **51(4)***, 1343-1355, (2020).*

[22] *Shaikh, A. A., Lone, M. S. and Ghosh, P. R., Rectifying curves under conformal transformation, J. Geom. Phy.,* **163***, (2021).*

[23] *Shaikh, A. A., Kim, Y. H. and Ghosh, P. R., Some characterizations of rectifying and osculating curves on a smooth immersed surface, J. Geom. Phy.,* **171**, *(2022).*

[24] *Shaikh, A. A. and Ghosh, P. R., On the position vector of surface curves in the Eulclidean space, Publ. Inst. Math (Beograd) (N.S.),* **112(126)**, *95-101, (2022).*

[25] *Yerlikaya, F., S. Karaahmetoglu, and I. Aydemir. "ON THE PAIR OF SPACE LIKE BERTR AND-BCURVE SWITH TIME LIKE PRINCIPAL NORMAL IN R 1 ." Palestine Journal of Mathematics 9.2 (2020).*

[26] *Yoder, Joella G,* book on the pendulum clock (1673) , *Sive de Motu Pendulorum ad Horologia Aptato Demonstrationes Geometrical, Translated into English: Ames: Iowa State Univ. Press, (1673).*

Author information

M. Hanif, Department of Mathematics, University of Narowal, Pakistan.
E-mail: muhammad.hanif@uon.edu.pk

A. Ali, Department of Mathematics, University of Narowal, Pakistan.
E-mail: 22uon1155@uon.edu.pk

A.Ahmed, Department of Mathematics, University of Narowal, Pakistan.
E-mail: ambreen.ahmed@uon.edu.pk

M.Ilyas, K.H. Shah, Institute of Physics, Gomal University, Dera Ismail Khan, Department of Physics, University of Narowal, Pakistan.
E-mail: ilyas_mia@yahoo.com, khalid.shah@uon.edu.pk

Received: 2024-01-12
Accepted: 2024-06-26