Palestine Journal of Mathematics

Vol 14(1)(2025) , 195–212

ON LINEAR MAPPINGS AND SEED SETS OF BEIDLEMAN NEAR-VECTOR SPACES

P. Djagba and A.L. Prins

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16Y30; Secondary 12K05.

Keywords and phrases: Near-vector spaces, R-subgroup, Linear mappins.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper. Prudence Djagba was very thankful to Nelson Mandela University for their funding support and facility.

Corresponding Author: P. Djagba

Abstract We studied linear mappings in Beidleman near-vector spaces and explored their matrix representations using R-bases of R-subgroups. In contrast to the theory of vector spaces, the sets of normal linear mappings and linear mappings do not form nearrings. Additionally, we developed algorithms for determining the seed number and seed sets of R-subgroups within finite-dimensional Beidleman near-vector spaces. We introduced the notion of γ-linearly independent of vectors and gave a sufficient condition for a set of vector to be 1-linearly independent after performing the EGE algorithm.

1 Introduction

Nearfields, first studied by Dickson [6] in 1905, found immediate applications in geometry. Despite their close resemblance to fields, the absence of one-sided distributive laws makes the study of nearfields challenging.

Nearfields, also known as skewfields or division rings, lack the distributive law on one side. Dickson's pioneering work in 1905 initiated their exploration, revealing connections to geometry and automata theory [20, 21, 22]. Most finite nearfields are constructed by distorting multiplication in finite fields through Dickson's method, with seven exceptional examples [11]. For a comprehensive overview, consult books by Pilz [8] and Meldrum [3], among others [11, 1, 4, 14, 12, 13]. In 1966, Beidleman introduced the concept of near-vector spaces over nearfields, employing nearring modules and the left distributive law [5]. A different notion of near-vector spaces defined by André in 1974, utilizing automorphisms, results in the right distributive law [17, 18, 24, 16].

More recent contributions to the theory of Beidleman near-vector spaces were made by Djagba and Howell [10, 7, 9]. These contributions delve into subspaces and subgroups of near-vector spaces over nearfield notions like R-dimension, R-basis, seed set, and seed number of an R-subgroup were introduced. Due to the lack of distributivity, near-vector spaces exhibit more anomalous behavior compared to vector spaces over fields. An R-subgroup of a near-vector space is a subset closed under vector addition and vector-scalar multiplication. It can be generated by a set of vectors, with explicit procedures like 'Expanded Gaussian Elimination' [7, 9] characterizing R-subgroups generated by finite sets of vectors. This result implies that a near-vector space R^m over a proper nearfield R can be generated by fewer than m vectors.

This paper primarily delves into the original Beidleman definition, specifically concentrating on the subgroup structure within finite-dimensional Beidleman near-vector spaces, with a special emphasis on the canonical scenario of R^m. Our research involves the derivation of matrix representations for both linear and normal linear mappings between finite-dimensional Beidleman near-vector spaces. It is noteworthy that, unlike vector spaces, the collection of linear mappings originating from near-vector spaces does not exhibit the properties of a nearring. This distinction underscores the unique and intricate mathematical characteristics of near-vector spaces, which

we aim to elucidate and understand further in this study. Furthermore, we have undertaken the development of sophisticated algorithms geared towards efficiently ascertaining both the seed number and seed sets of R-subgroups within finite-dimensional Beidleman near-vector spaces. In our research, we have also introduced the concept of γ-linearity among vectors, and have established a sufficient condition for a set of vectors to exhibit 1-linearity independence, following the execution of the EGE algorithm. This innovation represents a significant advancement in our understanding of near-vector spaces and offers a practical tool for exploring their inherent structural properties.

2 Preliminaries

Let R be a non-empty set.

Definition 2.1. ([3]) The triple $(R, +, \cdot)$ is a (left) nearring if $(R, +)$ is a group, $(R, \cdot)$ is a semigroup, and $a(b + c) = ab + ac$ for all $a, b, c \in R$.

A nearfield is an algebraic structure similar to a skew-field, also known as a division ring. The key distinction is that it has only one of the two distributive laws.

Definition 2.2. ([8]) Let R be nearring. If $\big(R^* = R \setminus \{0\}, \cdot\big)$ is a group then $(R, +, \cdot)$ is called nearfield.

In this paper, we will utilize left nearfields and right nearring modules. Various mathematicians, including Dickson, Zassenhauss, Neumann, Karzel, and Zemmer, have independently demonstrated that the additive group of a nearfield is abelian.

Theorem 2.1. ([8]) The additive group of nearfield is abelian.

To construct finite Dickson nearfields, we require two concepts:

Definition 2.3. ([8]) A pair of numbers $(q, n) \in \mathbb{N}^2$ is called a Dickson pair if q is some power p^l of a prime p, each prime divisor of n divides $q - 1$, $q \equiv 3 \bmod 4$ implies 4 does not divide n.

Definition 2.4. ([8]) Let R be a nearfield and $Aut(R, +, \cdot)$ the set of all automorphisms of N. A map

$$\begin{aligned} \phi : \quad & R^* \to Aut(R, +, \cdot) \\ & n \mapsto \phi_n \end{aligned}$$

is called a coupling map if for all $n, m \in R^*, \phi_n \circ \phi_m = \phi_{\phi_n(m)\cdot n}$.

Dickson's pioneering work in 1905 led to the discovery of the first proper finite nearfield. He achieved this by distorting the multiplication operation of a finite field. For any pair of Dickson numbers (q, n), there exist corresponding finite Dickson nearfields with an order of q^n. These nearfields are obtained by starting with the Galois field $GF(q^n)$ and modifying the multiplication operation. Thus $DN(q, n) = (GF(q^n), +, \cdot)^\phi = \big(GF(q^n), +, \circ\big)$. We will denote a Dickson nearfield arising from the Dickson pair (q, n) as $DN(q, n)$. For more details regarding the construction of the new multiplication operation denoted by '$\circ$', we refer the reader to [6, 8].

Example 2.5. ([8]) Consider the field $(GF(3^2), +, \cdot)$ with

$$GF(3^2) := \{0, 1, 2, x, 1 + x, 2 + x, 2x, 1 + 2x, 2 + 2x\},$$

where x is a zero of $x^2 + 1 \in \mathbb{Z}_3[x]$ with the new multiplication defined as

$$a \circ b := \begin{cases} a \cdot b & \text{if } a \text{ is a square in } (GF(3^2), +, \cdot) \\ a \cdot b^3 & \text{otherwise} \end{cases}$$

This gives the smallest finite Dickson nearfield $DN(3, 2) := (GF(3^2), +, \circ)$, which is not a field. Here is the table of the new operation $\circ$ for $DN(3, 2)$.

$\circ$	0	1	2	x	$1+x$	$2+x$	$2x$	$1+2x$	$2+2x$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	x	$1+x$	$2+x$	$2x$	$1+2x$	$2+2x$
2	0	2	1	$2x$	$2+2x$	$1+2x$	x	$2+x$	$1+x$
x	0	x	$2x$	2	$1+2x$	$1+x$	1	$2+2x$	$2+x$
$1+x$	0	$1+x$	$2+2x$	$2+x$	2	$2x$	$1+2x$	x	1
$2+x$	0	$2+x$	$1+2x$	$2+2x$	x	2	$1+x$	1	$2x$
$2x$	0	$2x$	x	1	$2+x$	$2+2x$	2	$1+x$	$1+2x$
$1+2x$	0	$1+2x$	$2+x$	$1+x$	$2x$	1	$2+2x$	2	x
$2+2x$	0	$2+2x$	$1+x$	$1+2x$	1	x	$2+x$	$2x$	2

We will refer to this example in later sections.

The concept of a ring module can be extended to a more general concept called a nearring module where the set of scalars is taken to be a nearring.

Definition 2.6. An additive group $(M,+)$ is called (right) nearring module over a (left) nearring R if there exists a mapping,

$$\eta : M \times R \to M$$
$$(m,r) \to mr$$

such that $m(r_1+r_2) = mr_1 + mr_2$ and $m(r_1r_2) = (mr_1)r_2$ for all $r_1, r_2 \in R$ and $m \in M$.

We write M_R to denote that M is a (right) nearring module over a (left) nearring R.

Definition 2.7. ([7]) A subset A of a nearring module M_R is called a R-subgroup if A is a subgroup of $(M,+)$, and $AR = \{ar | a \in A, r \in R\} \subseteq A$.

Definition 2.8. ([7]) A nearring module M_R is said to be irreducible if M_R contains no proper R-subgroups. In other words, the only R-subgroups of M_R are M_R and $\{0\}$.

Corollary 2.9. ([7]) Let M_R be a unitary R-module. Then M_R is irreducible if and only if $mR = M_R$ for every non-zero element $m \in M$.

Definition 2.10. ([7])Let M_R be a nearring module. N is a submodule of M_R if :

- $(N,+)$ is normal subgroup of $(M,+)$,
- $(m+n)r - mr \in N$ for all $m \in M, n \in N$ and $r \in R$.

Proposition 2.11. ([7]) Let N be a submodule of M_R. Then N is a R-subgroup of M_R.

Note that the converse of this proposition is not true in general. In his thesis ([7], page 14) Beidleman gives a counter example. However,

Lemma 2.12. If M_R is a ring module, then the notions of R-subgroup and submodule of M_R coincide.

Proof. By Proposition 2.11, every R-submodule is a R-subgroup. Let H be a R-subgroup of M_R. Then $hr \in H$ for all $h \in H$ and $r \in R$. But $hr = (m+h)r - mr$ for all $m \in M$. Hence H is a submodule of M_R. □

Theorem 2.2. ([7]) Let R be a nearring that contains a right identity element $e \neq 0$. R is division nearring if and only if R contains no proper R-subgroups.

Remark 2.13. Let R be a nearfield. By Theorem 2.2, R_R is irreducible R-module. Thus R contains only $\{0\}$ and R as submodules of R_R.

Definition 2.14. ([7]) Let $\{M_i | i \in I\}$ be a collection of submodules of the nearring module M_R. M_R is said to be a direct sum of the submodules M_i, for $i \in I$, if the additive group $(M, +)$ is a direct sum of the normal subgroups $(M_i, +)$, for $i \in I$. In this case we write $M_R = \bigoplus_{i\in I} M_i$.

Proposition 2.15. ([7]) $M_R = \sum_{i\in I} M_i$ and every element of M_R has a unique representation as a finite sum of elements chosen from the submodules M_i if and only if $M_R = \sum_{i\in I} M_i$ and $M_k \cap \sum_{i\in I, i\neq k} M_i = \{0\}$.

We also have that

Proposition 2.16. ([7]) Let $\{M_i |\ i \in I\}$ be a collection of submodules of the nearring module M_R. Then $M_R = \bigoplus_{i\in I} M_i$ implies that $M_R = \sum_{i\in I} M_i$ and the elements of any two distinct submodules permute.

According to the definition of a nearring module, there is no distributivity of elements of R over the elements of M. If we consider M_R as direct sum of the collection of submodules $\{M_i |\ i \in I\}$ of the nearring module M_R, the following result enables us to distribute the elements of R over elements contained in distinct submodules within the direct sum. This result holds significant utility within the concept of Beidleman near-vector spaces.

Lemma 2.17. ([7])Let $M_R = \bigoplus_{i\in I} M_i$, M_i is a submodule of M_R. If $m = \sum_{i\in I} m_i$ where $m_i \in M_i$ and $r \in R$ then

$$mr = (\sum_{i\in I} m_i)r = \sum_{i\in I}(m_i r).$$

Definition 2.18. ([7]) A nearring module M_R is called strictly semi-simple if M_R is a direct sum of irreducible submodules.

We now have,

Definition 2.19. ([7]) Let $(M, +)$ be a group. M_R is called Beidleman near-vector space if M_R is a strictly semi-simple R-module where R is a nearfield.

Theorem 2.3 ([7, 5]). Let R be a (left) nearfield and M_R a (right) nearring module. M_R is a finite dimensional near-vector space if and only if M_R is isomorphic to R^n for some positive integer n.

Definition 2.20 (R-module isomorphism). Let M_R and N_R be two modules. A function $\Phi : M \to N$ is a R-module isomorphism if it is a bijection that respects $\Phi(m + n) = \Phi(m) + \Phi(n)$ and $\Phi(mr) = \Phi(m)r$ for every $m, n \in M$ and $r \in R$.

Definition 2.21 (Linear map). Let M_R and N_R be two near-vector spaces. A function $\Phi : M \to N$ is a linear map if it respects $\Phi(m + n) = \Phi(m) + \Phi(n)$ and $\Phi(mr) = \Phi(m)r$ for every $m, n \in M$ and $r \in R$.

Definition 2.22 (Normal linear map). Let M_R and N_R be two near-vector spaces. A function $\Phi : M \to N$ is a normal linear map if it Φ is linear map and $\Phi(M_R)$ is subspace of N_R.

2.23 Subgroups of R^n

In [7], R-subgroups of finite-dimensional near vector spaces were classified using the Expanded Gaussian Elimination (EGE) algorithm. This algorithm constructs the smallest R-subgroup containing a given finite set of vectors. It's important to note that such an R-subgroup always exists since the intersection of subgroups is also a subgroup.

Definition 2.24. Let V be a set of vectors. Define $gen(V)$ to be the intersection of all R-subgroups containing V.

Let $LC_0(v_1, v_2, \ldots, v_k) := \{v_1, v_2, ..., v_k\}$ and for $n \geq 0$, let LC_{n+1} be the set of all linear combinations of elements in $LC_n(v_1, v_2, \ldots, v_k)$, i.e.

$$LC_{n+1}(v_1, v_2, \ldots, v_k) = \left\{ \sum_{i=1}^{\ell} w_i \lambda_i \mid \ell \geq 0, w_i \in LC_n, \lambda_i \in R \; \forall 1 \leq i \leq \ell \right\}.$$

Theorem 2.4 (Theorem 5.2 in [7]). Let $v_1, v_2, \ldots, v_k \in R^n$. We have

$$gen(v_1, \ldots, v_k) = \bigcup_{i=0}^{\infty} LC_i(v_1, \ldots, v_k).$$

Let M_R be a nearring module. Let $V \subseteq M_R$ and let T be an R-subgroup of M_R.

Definition 2.25 (Seed set). We say that V *generates* T if $gen(V) = T$. In that case we say that V is a *seed set* of T. We also define the *seed number* $seed(T)$ to be the cardinality of a smallest seed set of T.

In [12], it was demonstrated that each R-subgroup can be expressed as a direct sum of modules $u_i R$ of a special kind:

Theorem 2.5 (Theorem 5.12 in [7]). Let R be a proper nearfield and $\{v_1, \ldots, v_k\}$ be vectors in R^n. Then, $gen(v_1, \ldots, v_k) = \bigoplus_{i=1}^{\ell} u_i R$, where the u_i are rows of some matrix $U = (u_{ij}) \in R^{\ell \times n}$ such that each of its columns has at most one non-zero entry.

The EGE algorithm, is presented below, illustrates the proof of the theorem mentioned above. It is employed to compute the smallest R-subgroup for a given set of vectors.

Proof. Given a particular set of vectors $v_1, \ldots, v_k$, arrange them in a matrix V whose i-th row is composed of the components of v_i, i.e., $V = (v_i^j)$ where $1 \leq j \leq n$. Then $gen(v_1, \ldots, v_k)$ is the R-row space of V, which is a R-subgroup of R^n. We can then do the usual Gaussian elimination on the rows. The gen spanned by the rows will remain unchanged with each operation (swopping rows, scaling rows, adding multiples of a row to another). When the algorithm terminates, we obtain a matrix $W \in R^{k \times n}$ in reduced row-echelon form (denoted by $RREF(V)$). Let the non-zero rows of W be denoted by $w_1, w_2, \ldots, w_t$ where $t \leq k$.

Case 1: Suppose that every column has at most one non-zero entry, then

$$gen(v_1, \ldots, v_k) = gen(w_1, \ldots, w_t) = w_1 R + w_2 R + \cdots + w_t R$$

where the sum is direct. In this case we are done.

Case 2: Suppose that the j-th column is the first column that has two non-zero entries, say $w_r^j \neq 0 \neq w_s^j$ with $r < s$, (we necessarily have $r, s \leq j$) where w_r^j is the j-th entry of row w_r and w_s^j the j-th entry of row w_s. Let $\alpha, \beta, \gamma \in R$ such that $(\alpha + \beta)\lambda \neq \alpha\lambda + \beta\lambda$. We apply what we will call the *distributivity trick*:

Let $\alpha' = (w_r^j)^{-1}\alpha$ and $\beta' = (w_s^j)^{-1}\beta$. Then consider the new row

$$\theta = (w_r\alpha' + w_s\beta')\lambda - w_r(\alpha'\lambda) - w_s(\beta'\lambda).$$

Since $\theta \in LC_2(w_r, w_s)$ we have $\theta \in gen(w_1, \ldots, w_t)$.

For $1 \leq l < j$, either w_r^l or w_s^l is zero because the j-th column is the first column that has two non-zero entries, thus $\theta^l = 0$. Note that by the choice of α, β, λ, we have

$$\begin{aligned}
\theta^j &= (w_r^j)\alpha' + w_s^j\beta')\lambda - (w_r^j\alpha')\lambda - (w_s^j\beta')\lambda \\
&= \big(w_r^j(w_r^j)^{-1}\alpha + w_s^j(w_s^j)^{-1}\beta\big)\lambda - \big(w_r^j(w_r^j)^{-1}\alpha\big)\lambda - \big(w_s^j(w_s^j)^{-1}\beta\big)\lambda \\
&= (\alpha + \beta)\lambda - \alpha\lambda - \beta\lambda \neq 0.
\end{aligned}$$

It follows that $\theta^j \neq 0$. Hence $\theta = (0, \ldots, 0, \theta^j, \theta^{j+1}, \ldots, \theta^n)$. We now multiply the row θ by $(\theta^j)^{-1}$, obtaining the row $\phi = (0, \ldots, 0, 1, \theta^{j+1}(\theta^j)^{-1}, \ldots, \theta^n(\theta^j)^{-1}) \in gen(w_1, \ldots, w_k)$ where $\phi^j = 1$ is the pivot that we have created.

As a next step, we form a new matrix of size $(t+1) \times n$ by adding ϕ to the rows $w_1, \ldots, w_t$. On this augmented matrix we replace the rows w_r, w_s with $y_r = w_r - (w_r^j)\phi, y_s = w_r - (w_s^j)\phi$ respectively. This yields another new matrix composed of the rows $w_1, \ldots, w_{r-1}, y_r, \ldots, y_s, \phi, w_{s+}$ which has only one non-zero entry in the j-th column. By Lemma **??**, the *gen* of the rows of the augmented matrix is the *gen* of the rows of W (which in turn is $gen(v_1, \ldots, v_k)$). Hence

$$\begin{aligned} gen(v_1, \ldots, v_k) &= gen(w_1, \ldots, w_r, \ldots, w_s, \ldots, w_t) \\ &= gen(w_1, \ldots, y_r, \ldots, y_s, \phi, \ldots, w_t). \end{aligned}$$

Continuing this process, we can eliminate all columns with more than one non-zero entry. Let the final matrix have rows $u_1, u_2, \ldots, u_{k'}$. Then

$$gen(v_1, \ldots, v_k) = gen(w_1, \ldots, w_t) = gen(u_1, \ldots, u_{k'}) = u_1 R + u_2 R + \ldots + u_{k'} R,$$

where the sum is direct.

□

3 Developing Algorithms for a construction of a seed set

In contrast to the concept of vector spaces, in which there do not exist k vectors that span the entire finite-dimensional vector space F^m when $k < m$ and F is a field, the case of finite-dimensional Beidleman near-vector spaces exhibits different behavior. Here, there exist vectors that, when combined, span the entire space R^m where R is a proper nearfield. This phenomenon is exemplified in Theorem 5.12 of [7], where the authors classified the R-subgroups of R^m generated from a finite set of vectors.

During the process of explicitly describing the smallest R-subgroup containing a given set of vectors, Theorem 5.12 in [7] demonstrated that the union of p-linear combinations of these finite sets of vectors is utilized. If there exists a finite set of vectors in R^m such that the smallest R-subgroup containing these vectors generates the entire space R^m, then there exists a minimum positive integer p for which the p-linear combinations of these vectors yield the entire space R^m. An intriguing open question pertains to finding tight bounds on positive integers p for which p-linear combinations of a finite set of vectors yield the entire space and investigating potential constructions of seed sets that yield finite-dimensional near-vector spaces. We now introduce the following concepts.

Definition 3.1. A vector u is left multiple of v if there exists $r \in R$ such that $u = rv$.

Definition 3.2. Let $V \in R^{k\times m}$ be a matrix of k rows and m columns for $m \geq 2$. We will say that V is 1-column independent if for all $1 \leq i < j \leq m$, $\alpha \in R$, $v_i \neq \alpha v_j$.

Definition 3.3. Let R be a finite nearfield. A finite set vectors $V = \{v_1, \ldots, v_k\}$ in R^m is called γ-linearly dependent for some positive integer γ if there exists $v_i \in V$ such that $v_i \in LC_\gamma(v_1, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_k)$. We define V to be γ-linearly independent if V is not γ-linearly dependent.

Definition 3.4. Let $m \geq 3$. Let R be a finite nearfield and $v_1, \ldots, v_k \in R^m$ be a finite set of vectors such that $k \geq 2$. The set $LC_p(v_1, \ldots, v_k)$ will be called the p-linear combinations of the vectors $v_1, \ldots, v_k$. We define the index of R-linearity of $v_1, \ldots, v_k \in R^m$ to be

$$I(v_1, \ldots, v_k) = \min\{p \in \mathbb{N} \ : \ LC_p(v_1, \ldots, v_k) = R^m\}$$

the smallest positive integer for which the p-linear combination of the vectors $v_1, \ldots, v_k$ yields the whole space R^m.

Suppose that there exists $V = \{v_1, \ldots, v_k\}$ a finite set of vectors in R^m such that $gen(V) = R^m$. Since $\mathbb{N}$ is an ordered set then $I(V)$ is well-defined.

Example 3.5. Taking $m = 3$, it has been shown by Theorem 5.12 in [7] that there exists $v_1 = (1, 0, 1)$ and $v_2 = (1, 1, 0)$ in R^3 such that $gen(v_1, v_2) = R^3$. Note that $LC_2(v_1, v_2) = R^3$ and $LC_1(v_1, v_2) \neq R^3$. Hence $I(v_1, v_2) = 2$.

We have the following interesting observation.

Theorem 3.1. Let $V^{(t)}$ be the matrix after t steps of the EGE algorithm on the columns vectors $\{(a_1, \ldots, a_l), \ldots, (b_l, \ldots, b_l)\}$. We have the following:

(i) If $V^{(t)}$ is 1-column dependant then $V^{(t+1)}$ is also 1-column dependant.

(ii) If V^t is 1-column independent then V^{t+1} is also 1-column independent.

Proof. We prove the two statements.

1. We want to show that if $V^{(t)}$ is 1-column dependent then $V^{(t+1)}$ is also 1-column dependent. Consider, Without loss of generality after t steps of EGE, the following vectors columns $(v_1^1, \ldots, v_k^1)$ and $(v_1^2, \ldots, v_k^2)$. In the EGE process, we take the linear combinations of the rows to form new rows. For example, let $\alpha_1, \ldots, \alpha_k, \lambda \in R$. Consider $z_1 = (\sum_{i=1}^k v_i\alpha_i)\lambda$. Then $z_1^1 = (\sum_{i=1}^k v_i^1\alpha_i)\lambda$. Assume that $z_1^2 = sz_1^1$ for some $s \in R$ i.e, the columns $(z_1^1, 0, \ldots, 0)$ and $(z_1^n, 0, \ldots, 0)$ are not 1-column independent. Then

$$(\sum_{i=1}^k v_i^2\alpha_i)\lambda = s(\sum_{i=1}^k v_i^1\alpha_i)\lambda = (\sum_{i=1}^k sv_i^1\alpha_i)\lambda$$

It follows that $v_i^2 = sv_i^1$ for $i = 1, \ldots, k$. Thus $(v_1^1, \ldots, v_k^1)$ and $(v_1^2, \ldots, v_k^2)$ are 1-column dependent.

2. Let's assume that there are no $s \in R$ such that $a_i = sb_i$ for all i. In the process of EGE we take the linear combination of the rows to form the new rows. Our additional row is of the form $(\sum_{i=1}^{l} a_i\alpha_i)\lambda, (\sum_{i=1}^{l} b_i\alpha_i)\lambda$. The update matrix of size $(l+1) \times 2$ will be constituted of the columns

$(a_1 - (\sum_{i=1}^{l} a_i\alpha_i)\lambda, a_2, \ldots, a_l, (\sum_{i=1}^{l} a_i\alpha_i)\lambda)$ and $(b_1 - (\sum_{i=1}^{l} b_i\alpha_i)\lambda, b_2, \ldots, b_l, (\sum_{i=1}^{l} b_i\alpha_i)\lambda)$. Let's assume that there exists s such that for $1 \leq i \leq l+1$ we have $u_i = sv_i$. For $i = l+1$ we have, $a_{l+1} = sb_{l+1}$ which implies that $(\sum_{i=1}^{l} a_i\alpha_i)\lambda = s(\sum_{i=1}^{l} b_i\alpha_i)\lambda$. Hence $\sum_{i=1}^{l} a_i\alpha_i = s(\sum_{i=1}^{l} b_i\alpha_i)$. Furthermore,

$$a_1 - (a_1 - \sum_{i=1}^{l} a_i\alpha_i)\lambda = s(b_1 - (\sum_{i=1}^{l} b_i\alpha_i)\lambda) = sb_1 - s(\sum_{i=1}^{l} b_i\alpha_i) = sb_1 - (\sum_{i=1}^{l} a_i\alpha_i)\lambda.$$

It follows that $a_1 = sb_1$ which leads to contradiction.

□

We also have.

Lemma 3.6. Let R be a finite nearfield and $V = \{v_1, \ldots, v_k\}$ be a finite set of vectors in R^m and $|R| = t$. Then $|LC_1(V)| \leq t^k$. Furthermore, if V is 2-linearly independent every element of $LC_1(V)$ is unique, $|LC_1(V)| = t^k$ and $k \leq m$.

Proof. Let $u, v \in LC_1(V)$ such that $u = \sum_{i=1}^{k} v_i\alpha_i$ and $v = \sum_{j=1}^{k} v_i\beta_j$ where $(\alpha_1, \ldots, \alpha_k) \neq (\beta_1, \ldots, \beta_k)$. Without loss of generality, we can assume that $\alpha_1 \neq \beta_1$. Suppose that $u = v$. Then $v_1\alpha_1 - v_1\beta_1 = \sum_{i=2}^{k} v_i\alpha_i - \sum_{j=2}^{k} v_i\beta_j$. It follows that

$$v_1 = \big(\sum_{j=2}^{k} v_i(\beta_i - \alpha_i)\big)(\alpha_1 - \beta_1)^{-1}.$$

Thus $v_1 \in LC_2(v_2, \ldots, v_k)$. So V is 2-linearly dependent. We reach to contradiction. Therefore if V is 2-linearly independent then $|LC_1(V)| = t^k$. Suppose that V is 2-linearly independent and $k > m$, we have $|LC_1(V)| = t^k > t^m$ and all the t^k are distinct. It contradicts the fact that we have at most t^m vectors in the space. Hence $t \leq m$. □

In analogy to the notion of a basis of a subspace in the theory of vector spaces, we also have R-basis and R-dimension of an R-subgroup of the finite-dimensional Beidleman near-vector spaces R^m. In the following, we count the R-subgroups of R-dimension k of R^m.

Definition 3.7. The number of R-subgroups of R^m of dimension k of R^m up to the reordering of coordinate is the number of matrix obtained after EGE without reordering the column.

Proposition 3.8. Let R be a finite nearfield. The number of R-subgroups of R-dimension k of R^m up to the reordering of coordinates is

$$\sum_{t=k}^{m} p_k(t)(|R| - 1)^{t-k},$$

where $p_k(t)$ is the number of partitions of t into k parts and t is the total number of non-zero entries in all the rows.

Proof. Let $t = \sum_{i=1}^{k} \#u_i$ where $\#u_i$ is the number of non-zero entries in the row u_i. It is clear that $t \leq n$. For $\#u_i = 1$ for all $i = 1, \ldots, k$ then $t = k$. Hence $k \leq t \leq n$. Note that t is the total number of non-zero entries in all columns or in all the rows.

Given t, we can partition t into k parts where each part (containing some non-zeros entry) will represent each row vector. We have $(|R| - 1)^t$ possible choice of non-zero elements from R^* to fill in the t places of each partition.

Since the gen is unchanged for any permutations of row vectors and is unchanged for any scalar multiplies to others rows, then, for a given t, $N = \frac{1}{(|R|-1)^k} p_k(t)(|R|-1)^t$ is the number of R-subgroups of dimension k up to reordering of coordinate. For $k \leq t \leq m$ yield to $\sum_{t=k}^{m} p_k(t)(|R|-1)^{t-k}$ is the number of R-subgroups of R-dimension k of R^m up to the reordering □

The form of the seed set of $DN(3,2)^m$ for $2 \leq m \leq 9$, as described in [7], motivates us to seek a possible general construction of seed sets for R^m, where R is a nearfield and m is a natural number. Consequently, we have developed Algorithm 1 to provide such a construction. The following algorithm derives the explicit expression of the seed number k to generate a given R-subgroups.

Algorithm 1 Create a seed set V_m of R^m and give its number of row

Require: $m \in \mathbb{N}$, R is a finite nearfield with unity 1, the indexing of matrices, vector start from 0

Ensure: V_m and k such that $gen(V_m) = R^m$ and $V - m$ has k columns.

if $m = 1$ **then**
  return (1)
else
  $K \leftarrow (1)$
  $S \leftarrow R - \{0, 1\}$
  $k \leftarrow 1$ {number of row}
  $numcol \leftarrow 1$ { the index of column to fill next time/number of column already filled}
  while $numcol < m$ **do**
    $k \leftarrow k + 1$ {need to add one more row}
    $Kprim \leftarrow I_k(R)$ {Identity matrix of order k}
    $numcol \leftarrow column\_number(Kprim)$
    if $numcol = m$ **then**
      show k
      return $Kprim$
    end if
    $K \leftarrow$ sub-matrix of k from column $k - 1$ { Matrices column of K}
    if $K \neq ()$ **then**
      $K \leftarrow \overline{\begin{matrix} K \\ 0 \ldots 0 \end{matrix}}$ {add a row of zero in K}
      $K \leftarrow Kprim|(K)$ {Merge $Kprim$ and K by column}
    else
      $K \leftarrow Kprim$
    end if
    $numcol \leftarrow column\_number(K)$
    if $numcol = m$ **then**
      return K
    end if
    $counter \leftarrow 0$
    $S1 \leftarrow copy(S)$
    while $counter < k - 1$ **do**
      while $numcol < m$ and $S1 \neq \emptyset$ **do**
        $newcolumn \leftarrow []$
        for i in $[0; counter]$ **do**
          append $[R[2]]$ to new column {fill the first counter rows with 1}
        end for
        for i in $[counter + 1, k - 1]$ **do**
          new column.append([S1[0]]) {Complete the other rows with S[0]}
        end for
        remove $S1[0]$ from $S1$
        $newcolumn \leftarrow matrix(newcolumn)$
        $K \leftarrow K.augment(newcolumn)$
        $numcol \leftarrow numcol + 1$
        if $numcol = m$ **then**
          show k
          return K
        end if
      end while
      $counter \leftarrow counter + 1$
      $S1 \leftarrow copy(S)$
    end while
  end while
end if
show k
return K

We will use the notation V_m to represent the matrix output of Algorithm 1 for a given input value of m, and S_m to denote the ordered set of its row vectors, with ordering determined by their respective row numbers in V_m.

First, we found the following:

Theorem 3.2. Let $k \geq 1$ be an integer. Then, the maximal value of integer m such that V_m has k-rows is given by the sequence $(u_k)_{k\geq 1}$ defined by:

$$\begin{aligned} u_1 &= 1 \\ u_{k+1} &= u_k + (|R| - 2)k + 1 \end{aligned}$$

The range of values of integer m such that V_m has k-rows is given by:
$[u_{k-1} + 1, u_k]$

Proof. Let u_k be the maximal value of integer m such that V_m has k-rows.

We see that the construction of V_m is obtained by adding eventual row/column to V_{m-1}. Also, we only add an additional column when the maximum number of columns that can be created with a given number of rows is complete. Thus, if we find V_m has a number of rows greater than k, then $m > k$. And if we find V_m has a number of rows less than k, then $m < k$.

We have $u_1 = 1$ because, for $m = 1$, Algorithm 1 return (1), and for $m = 2$, as we enter already in the while loop, we add a new row so $1 \leq u_1 < 2$. It implies that $u_1 = 1$ as u_1 is an integer.

Let us now prove that $u_{k+1} = u_k + (|R| - 2)k + 1$. From the construction, to construct a matrix of $k + 1$ rows, we need to complete the maximal number of columns that can have a matrix of k rows.

So first, we replace the identity matrix of range k with a new identity matrix of range $k + 1$ and complete the row with 0. This means that the minimum number of column for $k+1$ is u_k+1. Then, we will be able to add a new column to this new construction as long as the $counter$ does not reach $k - 1$ which means the $counter$ can take k values. Then, for a given $counter$, we can add $|R| - 2$ columns as we remove 0 and 1 to R. So, we have $u_{k+1} = u_k + 1 + (|R| - 2)k$.

□

Here we give an explicit form of u_k in function of k.

$$\begin{aligned} u(0) &= 0 \\ u(1) &= u(0) + (|R| - 2) * 0 + 1 \\ u(2) &= u(1) + (|R| - 2) * 1 + 1 \\ &\vdots \\ u(k) &= u(k-1) + (|R| - 2) * (k-1) + 1 \\ u(k) &= (|R| - 2) * \sum_{i=0}^{k-1} + k \\ u(k) &= (|R| - 2)\frac{(k-1) * k}{2} + k \\ u(k) &= \frac{(|R| - 2)(k-1) * k + 2k}{2} \\ u(k) &= \frac{((|R| - 2)(k-1) + 2)\, k}{2} \end{aligned}$$

Solving this second degree equality in regard to k and taking the only one positive value, we have:

Lemma 3.9. From Theorem 3.2, the explicit expression of k is given by

$$k = \frac{|R| + \sqrt{|R|^2 + 8\,(|R| - 2)u_k - 8\,|R| + 16} - 4}{2\,(|R| - 2)}$$

.

Then we notice that the function $f : m \mapsto \frac{|R|+\sqrt{|R|^2+8\,(|R|-2)m-8\,|R|+16}-4}{2\,(|R|-2)}$ is strictly increasing (because $|R|^2 + 8\,(|R| - 2)m - 8\,|R| + 16$ is an affine function of m of positive leading coefficient, square root is an increasing function and addition of number followed by multiplication with positive number does not change the variation of a function) such that $f(u_k) = k$ for any integer $k \geq 1$.

As we have $f(u_{k-1}) = k - 1$ and $f(u_k) = k$, for any element m in $[u_{k-1} + 1; k]$ we have $k - 1 < f(m) \leq k$. It means that any element m is in $[u_{k-1} + 1; k]$, $k = ceil(f(m))$. So we have:

Lemma 3.10. Let m be a positive integer, then the number of rows of V_m is given by:
$k = \left\lceil \frac{|R|+\sqrt{|R|^2+8\,(|R|-2)m-8\,|R|+16}-4}{2\,(|R|-2)} \right\rceil$

Now, we are ready to give the Theorem:

Theorem 3.3. For any V_m obtained by the Algorithm 1, we have that the set of its row vectors S_m is R-linearly independent and $gen(S_m) = R^m$ (i.e., S_m is a seed set of R^m).

Proof. Note that S_m is R-linearly independent since by the construction of V_m it is in reduced row echelon form. We will do the proof of $gen(S_m) = R^m$ by induction in m For $m = 1$, we know that $\{1\}$ generates R^m. Let us assume that $gen(S_m) = R^m$ for $m \geq 1$. We have 2 cases:

Case 1: $m = u(k)$ for a certain value of k and $m+1 = u(k)+1$. From Algorithm 1 described above, the first k rows of V_{m+1} are obtained by inserting one 0 in the $(k+1)^{th}$ position of every row of V_m of the same position. When we perform the EGE Algorithm on the matrix V_{m+1}, the first k columns and they will behave bijectively as for V_m. The new rows will have the component 0 in the $k + 1^{th}$ column since the $k + 1^{th}$ components of each above vector are all 0 and the last row will not be involved in the distributivity trick (by Theorem 5.12 in [7], we see that the first two non-zero elements of the column j will all have number of row less than j). For the $k+1^{th}$ column, we don't need to add any additional row since it has only one zero component which is already the only one in its column. The algorithm will work on the remaining columns with the same behavior as for V_m for its sub-matrix from column k. It means that the m columns will generate $m - k$ additional rows (By our hypothesis $gen(S_m)$) and at the end of the EGE Algorithm we have $k + 1 + m - k = m + 1$ rows. Thus, $gen(S_{m+1}) = R^{m+1}$.

Case 2: m and $m - 1$ is in the same interval $[u(k - 1) + 1, u(k)]$. From the algorithm described in the Appendix, each row vector of V_{m+1} is obtained by adding only one 1 or an element s of S in its last column. It means that the column 1 to m generates $m - k$ additional vectors in the running of the EGE algorithm. The last column generates only one new non-zero row $(0, \ldots, 1)$ by definition of the EGE algorithm, and we will not delete any of the previous rows because the other rows have already one non-zero element in one of the previous column (by our hypothesis, it generate already R^m). Thus, the number of rows we obtain is $k + m - k + 1 = m + 1$.

In both cases we have $gen(S_{m+1}) = R^{m+1}$. Thus, for any positive integer m, $gen(S_m) = R^m$.

□

4 Representation of linear maps

In vector spaces, matrix representation is motivated by the fact that, for a given basis, a linear mapping is well-defined when we specify the images of all elements in the basis. In a vector space M_R with a basis $X = x_i, i \in I$, we can define a linear mapping T by specifying the images of each element of X. This can be expressed as $\left(\sum_{i\in I} x_i r_i\right) T = \left(\sum_{i\in I} x_i T\right) r_i$. Any

matrix can serve as a representation of such a linear mapping, where each column of the matrix corresponds to the image of an element in the basis.

In a vector space, a linear mapping is uniquely determined when we specify its behavior on a basis. However, in a near-vector space, setting the image of the elements of a basis and following the same rules as in vector spaces may lead to a function that is not a linear mapping.

We consider the mapping T from R^2 to itself, where R is the Dickson Nearfield $DN(3,2)$. It is defined as follows: $(0,1)T = (1,1)$, $(1,0)T = (1,2)$, and for all (a,b) in R^2, $(a,b)T = (0,1)Tb + (1,0)Ta$. Under this mapping, we find that $(1,X)TX = (X+2, X+1)$ but $(1,X)XT = (2X+1, 2X+2)$ for X satisfying $X^2 + 1 = 0$.

4.1 Case for $DN(3,2)^2$

In this section, we will examine the case of $DN(3,2)$ and its finite-dimensional near-vector spaces.

We will start by considering R^2, and let $\mathcal{M}$ be the set of all mappings from R^2 to itself. These mappings are defined by specifying the images of $(1,0)$ and $(0,1)$, and for all (x_1,x_2) in R^2, $(x_1,x_2)T = (1,0)Tx_1 + (0,1)Tx_2$. It's worth noting that $|\mathcal{M}| = 6561$. We see that.

$$\begin{aligned}(x_1,x_2)T + (x_1',x_2')T &= (1,0)Tx_1 + (0,1)Tx_2 + (1,0)Tx_1' + (0,1)Tx_2' \\ &= (1,0)Tx_1 + (1,0)Tx_1' + (0,1)Tx_2 + (0,1)Tx_2' \\ &= (1,0)T(x_1 + x_1') + (0,1)T(x_2 + x_2') \text{ (Left distributivity)},\end{aligned}$$

but

$$\begin{aligned}((x_1,x_2)T)r &= ((1,0)Tx_1 + (0,1)Tx_2)r \\ ((x_1,x_2)r)T = (x_1r, x_2r)T &= (1,0)Tx_1r + (0,1)Tx_2r\end{aligned}$$

which can be different because of the lack of right distributivity. We want to explore the characterizations of linear mappings.

Proposition 4.2. If T in is defined by $(1,0)T = a$ and $(0,1)T = b$ is a linear mapping then all mappings T' of defined by $(0,1)T' = ar$ and $(0,1)T' = br'$ is a linear mapping also.

Proof.

$$\begin{aligned}(x_1,x_2) &= (1,0)x_1 + (0,1)x_2 \\ (x_1,x_2)T' &= arx_1 + br'x_2 = (rx_1, r'x_2)T \\ ((x_1,x_2) + (x_1',x_2'))T' &= (x_1 + x_1', x_2 + x_2')T' \\ &= (r(x_1 + x_1'), r'(x_2 + x_2'))T \\ &= (rx_1, r'x_2)T + (rx_1', r'x_2')T \\ &= (x_1,x_2)T' + (x_1',x_2')T' \\ ((x_1,x_2)T')\lambda &= (rx_1, r'x_2)T\lambda \\ &= (rx_1\lambda, r'x_2\lambda)T \\ &= (x_1\lambda, x_2\lambda)T' \\ &= (x_1,x_2)\lambda T'\end{aligned}$$

□

We will define the representation matrix of a linear mapping as it is used in vector space, and we see that for R^2, the representation matrix of a linear mapping is has at most one non-zero element in each row.

Next, we determine the normal linear mappings, and we find that their representation matrices have at most one non-zero element in each row and each column. There are a total of 161 normal linear mappings. We find that the number of linear mappings in R^2 is 289.

4.3 Generalization

Let R be a nearfield and n an integer. We will consider the near vector space R^n with a basis $X = \{x_1, \ldots, x_n\}$.

Let $\mathcal{M}$ be the set of all mappings of R^n to itself such that $x_iT = a_i$ for $i = 1, \ldots, n$ where a_i is an element in R . For any element m of R^n such that $m = \sum_{i=1}^{n} x_i r_i$, r_i in R, we have $mT = \sum_{i=1}^{n} a_i r_i$.

Such mapping can be represented as a matrix $n \times n$ over the nearfield R by setting as columns the a_i as it is in vector space. Let us first prove that the element of $\mathcal{M}$ are homomorphisms of the additive group $(R^n, +)$.

Lemma 4.4. Let T be an element of $\mathcal{M}$, then T is an homomorphism of the additive group $(R^n, +)$.

Proof. Let m_1, m_2 be in R^n,

$$\begin{aligned} m_1T + m_2T &= \sum_{i=1}^{n} a_{i1}r_{i1} + \sum_{i=1}^{n} a_{i2}r_{i2} \\ &= \sum_{i=1}^{n} (a_{i1}r_{i1} + a_{i2}r_{i2}) && \text{(Commutativity of the additive group)} \\ &= \sum_{i=1}^{n} a_{i1}(r_{i1} + r_{i2}) && \text{(Left distributivity)} \\ &= (m_1 + m_2)T \end{aligned}$$

□

Now we want to see which of those mapping are linear. Then, let us denote M to be the matrix representation of T.

Theorem 4.1. Let T be an element of M, then b T is linear if and only if the matrix representation associated with T has at most one non-zero element in each row.

Proof. Let T be in , $m = \sum_{i=1}^{n} x_i r_i$ in R^n and r in R. Let $M = \begin{pmatrix} M_{11} & \ldots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \ldots & M_{nn} \end{pmatrix}$.

Then,

$$mTr = \begin{pmatrix} (\sum_{j=1}^{n} M_{1j}r_j)r \\ \vdots \\ (\sum_{j=1}^{n} M_{nj}r_j)r \end{pmatrix} \text{ and } mrT = \begin{pmatrix} \sum_{j=1}^{n} T_{1j}r_j r \\ \vdots \\ \sum_{j=1}^{n} T_{nj}r_j r \end{pmatrix}$$

So by definition of linear mapping, T is linear if and only if, $mMr = mrM$ which means $\sum_{j=1}^{n} T_{ij}r_j r = (\sum_{j=1}^{n} T_{ij}r_j)r$ for all i in $\{1, \ldots, n\}$.

If the M has only one non-zero element, then it is obvious that the equality is true.

Conversely, let us suppose that T is a linear mapping, i.e $\sum_{j=1}^{n} T_{ij}r_j r = (\sum_{j=1}^{n} T_{ij}r_j)r$ for all i in $\{1, \ldots, n\}$ and there exist one row containing more than one element.

Let i be such row and let s, t the be the first two columns such that the component is not 0. We know that because of the lack of right distributivity, we have elements α, β and γ in R satisfying $(\alpha + \beta)\gamma \neq \alpha\gamma + \beta\gamma$.

Let m be the element of R^n which component are 0 except at the s^{th} and t^{th} the component which are are respectively $r_s = M_{is}^{-1}\alpha$ and $r_t = M_{it}^{-1}\beta$.

Then the i^{th} component of $(mT)\gamma$ is

$$\begin{aligned} ((mT)\gamma)_i &= \left(\sum_{j=1}^{n} M_{ij}r_j \right) \gamma \\ &= (\alpha + \beta)\gamma \end{aligned}$$

since $M_{ij}r_j = 0$ for all $j \notin \{s,t\}$ by definition of m and we have

$$M_{is}r_s = M_{is}M_{is}^{-1}\alpha = \alpha$$
$$M_{it}r_t = M_{it}M_{it}^{-1}\beta = \beta.$$

and the i^{th} component of $(m\gamma T)$ is

$$((m\gamma T))_i = \left(\sum_{j=1}^{n} M_{ij}r_j\gamma\right)$$
$$= (\alpha\gamma + \beta\gamma)$$

since $M_{ij}r_j\gamma = 0$ for all $j \notin \{s,t\}$ by definition of m and we have

$$M_{is}r_s\gamma = M_{is}M_{is}^{-1}\alpha\gamma = \alpha\gamma$$
$$M_{it}r_t\gamma = M_{it}M_{it}^{-1}\beta\gamma = \beta\gamma.$$

By the choice of α, β and γ, we have $m\gamma T \neq (mT)\gamma$ which is a contradiction. So all rows have at most one non-zero component. □

Theorem 4.2. Let T be a linear mapping from R^n to R^n. Then T is normal if and only if, the matrix representation M of T has at most one non-zero element in each row and column.

Proof. Let T be a linear mapping with matrix representation $M = \begin{pmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \dots & M_{nn} \end{pmatrix}$.

We know from (Theorem 6.1, [7]) that all subspace of R^n are in form $e_1R \times \ldots \times u_kR$ where e_i have only one non-zero component which is 1.

Then $R^nT = \sum_{i=1}^{n} a_iR$ and $a_iR \cap \sum_{j\neq i} a_jR = \{0\}$ since a row contains at most one non-zero element.

Suppose that all a_i has at most one non-zero component. Then the image of T is $\bigoplus a_iR = \bigoplus_{a_k \neq (0,0,\ldots,0)} e_kR$ where e_k is obtained by multiplying a_k with the inverse of its non-zero component. So R^nT is clearly a subspace. Thus T is normal.

Conversely, let us suppose that $R^nT = \sum_{i=1}^{n} a_iR$ is a subspace and suppose that the matrix representation M of T has a column j with more than one non zero element say at position $j_1, \ldots, j_k$.

We know that a_jR is an R-subgroup of R^n which is contained in the subspace $A = e_{j_1}R + \ldots + e_{j_k}R$. where e_{j_i} is a vector with only one non-zero components 1 in position j_i . Then as A is a subspace, it is a near vector space and a_jR is a proper R-subgroup of A.

Hence, there exists a non-zero element m in A, a in a_jR and r in R such that $(m+a)r - mr$ is not in a_jR.

But this element is not as well in $\sum_{i\neq j} a_iR$ since a_i has 0 as component in row j_l for each l.

Thus, there is a contradiction and the column has at most one non-zero component. □

The sets of normal linear mappings and linear mappings are not nearrings but are closed under multiplication. In contrast to the theory of vector spaces and Andre near vector spaces, we make the following observations. Let's consider
$L(R^n) = \{T : R^n \to R^n | T \text{ is normal linear mapping }\}$, and $Hom(R^n) = \{T : R^n \to R^n | T \text{ is linear mapping }\}$.

Proposition 4.5. We have

(1) $L(R^n, R^n)$ and $Hom(R^n)$ are not nearrings and do not form Beidleman near vector spaces

(2) $L(R^n)$ and $Hom(R^n)$ are closed under multiplication.

Proof. (1) We take our example of $DN(3,2)$ and the normal linear mapping T and T' : $R^2 \to R^2$ defined by their respective matrices $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then the matrix of $T + T'$ will be $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. So $T + T'$ is not a linear mapping by our characterization in Proposition 4.1. So $L(R^n)$ and $Hom(R^n)$ are not closed under addition. Hence they do not form groups under addiction.

(2) Let M and M' be two representative matrices of linear mappings T and T'.

We have $(MM')_{ij} = \sum_{k=1}^{n} M_{ik}M'_{kj}$, if the row i of M has only zero elements, then $(MM')_{ij} = 0$ for all j. If the row i of M has one non-zero element in column k, then $(MM')_{ij}$ will be non-zero if the element of row k of column j is also non-zero but such a situation occurs at most once since the row k of M' has at most one non zero elements.

So when we do the multiplication MM', we will always have at most one non-zero element in each row. Hence $T \circ T'$ is a linear mapping.

Then, suppose they are both normal, if the column j has only zero elements, then $(MM')_{ij} = 0$ for all i. if at most one row has non-zero element, then the product will be obviously normal. Let us suppose there are two rows i_1, i_2 with non-zero element at columns k_1, k_2. So we have $k_1 \neq k_2$. Else if the column j of T' has one non-zero element in column k, then MM'_{ij} will be non-zero if the element of column k of the row i of M is also non-zero but such situation occurs at most once since the column k of M has at most one non zero element.

□

5 Conclusion

In this paper, we provided representations of linear and normal linear maps between finite-dimensional Beidleman near-vector spaces, and we derived algorithms for constructing seed sets for such spaces.

Let R be a finite nearfield of size q. A rough upper bound on the size of $LC_2(v_1, \ldots, v_k)$ is q^{q^k}. We propose the following open questions:

Question 5.1. Does there exist any example of a near-vector space where $I(v_1, \ldots, v_k) > 2$ for some $v_1, \ldots, v_k$?

More generally, can we find an explicit expression or at least some nontrivial bounds for $I(v_1, \ldots, v_k)$?

Acknowledgments This work was supported by the Council-funded from the Office of Research Development funding at Nelson Mandela University.

6 Appendix

Note that the Algorithm 2 is a recursive version of Algorithm 1.

Algorithm 2 Create a seed set V_m of R^m

Require: $m \in \mathbb{N}$, R is a finite nearfield with unity 1
Ensure: V_m such that $gen(V_m) = R^m$
$k \leftarrow \left\lceil \frac{|R|+\sqrt{|R|^2+8\,(|R|-2)m-8\,|R|+16}-4}{2\,(|R|-2)} \right\rceil$ {upper bound of the seed number of R^m}
if m=1 **then**

return (1)
else
$K \leftarrow 0_{k\times m}$ { The array we will complete, a zero matrix of k rows and m columns. }
$prevm \leftarrow \frac{((|R|-2)(k-1-1)+2)(k-1)}{2}$ {The biggest value of n which have a seed set of cardinality $k-1$. }
$V \leftarrow V_{prevm}$

for i in [1;k] **do**
$K[i][i] \leftarrow 1$ {Copy the identity matrix in K in the first k columns.}
end for

for j in [k; prevm] **do**
for i in [1,k] **do**
$K[i][j+1] \leftarrow V[i][j]$ {Copy the submatrix of V_{prevm} from the kth column in K in the same row number but the column number shifted by 1. (just after the identity matrix) }
$K[k][j+1] \leftarrow 0$ { Add 0 element in the last rows from column $k+1$ to column $prevm$}
end for
end for
{Now I will complete the first row with 1}
$S \leftarrow R\,\{0,1\}$ {The non zero element of R different to 1}
counter=1
$numcol \leftarrow prevm+2$ {The number of the column to be filled, note that we already fill the first $prevm+1$ columns.}

if numcol=m+1 **then**

return K {We return K as matrix because we already completed the required number of columns.}
end if

while $counter < k$ **do**

for i in $[numcol, m]$ **do**
$K[counter][i] \leftarrow 1$ {Fill with 1 the row counter}
end for

for j in S **do**

for i in [counter ,k] **do**
$K[i][numcol] \leftarrow j$ { Put a copy of S}
end for
$numcol \leftarrow numcol+1$ {Change the number of the column after setting all number in the column}

if $numcol = m+1$ **then**

return K {We return K as matrix because, we manage to complete every columns of our matrix K}
end if
end for
$counter \leftarrow counter+1$
end while
end if
return K

References

[1] E. Ellers and H. Karzel, Endliche Inzidenzgruppen, *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg,* 27(1964), 250-264.

[2] J. Dauns, Prime modules, *J. Reine Angew. Math,* 298 (1978), 156-181.

[3] J. DP Meldrum, Near-rings and their links with groups, *Pitman Advanced Publishing Program,* 134, 1985.

[4] H. Wähling, Theorie der Fastkörper *Thales Verlag,* 1987.

[5] James C. Beidleman, On near-rings and near-ring modules, PhD thesis, Pennsylvanian State University, 1966.

[6] L.E. Dickson, On finite algebras, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse,* 358-393, 1905.

[7] P.Djagba and K-T. Howell, The subspace structure of finite dimensional Beidleman near-vector spaces, *Linear and Multilinear Algebra,* 2019.

[8] Pilz, Günter, Near-rings: the theory and its applications *North-Holland Publishing Company,* 23 (1983)

[9] P. Djagba and J. Hazła, Combinatorics of subgroups of Beidleman near-vector spaces *preprint arXiv:2306.16421,* 2023.

[10] P. Djagba, Contributions to the theory of Beidleman near-vector spaces *Stellenbosch University,* 2019.

[11] H. Zassenhaus , über endliche Fastkörper, *Abhandlungen aus dem mathematischen Seminar der Universität Hamburg,* 11 (1935), 187-220.

[12] P. Djagba , On the generalized distributive set of a finite nearfield, *Journal of Algebra,* 542 (2020), 130-161.

[13] P. Djagba , On the center of a finite Dickson nearfield, *arXiv preprint:2003.08306,* 2020.

[14] P. Djagba , Construction of a finite Dickson nearfield, *IJNAA,* 2023.

[15] E. A. Hussein and S. O. Alsalihi, Some new results on near fields, *Tikrit Journal of Pure Science,* 27 (2022), 101-104.

[16] S. P. Sogo, On counting subspaces of Near-vector Spaces, *Discrete Mathematics,* 343 (2020).

[17] J. André, Lineare algebra über Fastkörpern, *Mathematische Zeitschrift,* 136 (1974), 295-313

[18] K.-T. Howell , Contributions to the theory of near-vector spaces, *Free State University,* 2008.

[19] J.L. Zemmer , Near-fields, planar and non-planar, *The Mathematics Student,* 38, 1964, 145-150.

[20] Veblen, Oswald and Maclagan-Wedderburn, Joseph H , Non-Desarguesian and non-Pascalian geometries, *Transactions of the American Mathematical Society,* 8, 1907, 379-388.

[21] Karzel, Helmut and Kist, Günter, Some applications of nearfields, *Proceedings of the Edinburgh Mathematical Society,* 23, 1980, 129-139.

[22] Boykett, Tim and Wendt, Gerhard, J_2 radical in automata nearrings, *International Journal of Foundations of Computer Science,* 25, 2014, 585-595.

[23] Sloane, The on-line encyclopedia of integer sequences, https://oeis.org/A003580

[24] Dorfling, S and Howell, K-T and Sanon, SP, The decomposition of finite-dimensional near-vector spaces, *Communications in Algebra,* 46, 2018, 3033-3046.

Author information

P. Djagba, Department of Mathematics, Nelson Mandela University, South Africa.
E-mail: prudence@aims.ac.za

A.L. Prins, Department of Mathematics, University of Fort Hare, South Africa.
E-mail: aprins@ufh.ac.za

Received: 2024-01-15
Accepted: 2024-11-21