Palestine Journal of Mathematics

Vol 14(1)(2025) , 1–14

IMPLICIT CAPUTO TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS WITH RETARDED AND ADVANCED ARGUMENTS IN BANACH SPACES

W. Rahou, A. Salim, J. E. Lazreg and M. Benchohra

Communicated by Ayman Badawi

MSC 2020 Classifications: Primary 26A33; Secondary 34A08.

Keywords and phrases: Caputo tempered fractional derivative, implicit problem, existence, fixed point, measure of noncompactness, retarded argument, advanced argument, Ulam stability.

Corresponding Author: A. Salim

Abstract This article deals with the existence and stability results for a class of implicit fractional differential equations involving the Caputo tempered fractional derivative with retarded and advanced arguments. The results are based on Sadovskii's fixed point theorem. Some examples are given to show the applicability of our results.

1 Introduction

In various fields of research, fractional calculus has recently emerged as a valuable tool for addressing complex issues. Fractional calculus is the extension of differentiation and integration to non-integer orders, and its theory and applications have received significant attention. To gain a comprehensive understanding, we suggest referring to the following resources: monographs like [1–4, 8, 18, 36, 39] and papers such as [5, 10, 11, 15, 31]. Over the past few years, there has been a significant increase in research on fractional calculus, with authors investigating a wide range of outcomes for various forms of fractional differential equations and inclusions under different conditions. Further information can be found in papers such as [1, 14, 22, 23, 34], as well as their respective references.

In [26], the authors considered a class of problems for nonlinear Caputo tempered implicit fractional differential equations with boundary conditions and delay:

$$
\begin{gathered}
{}^{C}_{0}D^{\beta,\gamma}_{\theta}y(\theta) = \Psi\left(\theta, y_\theta, {}^{C}_{0}D^{\beta,\gamma}_{\theta}y(\theta)\right), \quad \theta \in \Theta := [0, \varkappa], \\
y(\theta) = \xi(\theta), \quad \theta \in [-\kappa, 0], \\
\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3,
\end{gathered}
$$

where $0 < \beta < 1$, $\gamma \geq 0$, ${}^{C}_{0}D^{\beta,\gamma}_{\theta}$ is the Caputo tempered fractional derivative, $\Psi : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\xi \in C([-\kappa, 0], R)$, $0 < \varkappa < +\infty$, $\delta_1, \delta_2, \delta_3$ are real constants, and $\kappa > 0$ is the time delay. Their arguments are based on Banach, Schauder and Schaefer fixed point theorems.

One notable class of fractional calculus operators with analytic kernels that has garnered significant attention in recent years is tempered fractional calculus. This class serves as a means of generalizing various types of fractional calculus and is regarded as an extension of the latter due to its capacity to describe the transition between normal and anomalous diffusion. Buschman's pioneering work [13] established the definitions of fractional integration with weak singular and exponential kernels. Further elaboration on this topic can be found in [6,12,16,21,27–30,33,37]. Additionally, the Caputo tempered fractional derivative is a subject that has not been widely explored in literature, providing us with an opportunity to make a significant contribution to this field. Our objective is to deepen our understanding of the characteristics and potential applications of this unique mathematical concept through the study of the Caputo tempered fractional derivative. By doing so, we can play a valuable role in advancing the development of fractional calculus.

In many instances, finding the exact solutions to differential equations is challenging, if not impossible. In such cases, it is common practice to explore approximate solutions. However, it is crucial to note that only stable approximations are considered valid. Consequently, various stability analysis techniques are employed. The issue of stability in functional equations was first introduced by mathematician S. M. Ulam during a 1940 lecture at the University of Wisconsin. In his presentation, Ulam posed the following question: "Under what conditions does an additive mapping exist near an approximately additive mapping?" [38]. The following year, Hyers provided a solution to Ulam's problem for additive functions defined on Banach spaces [20]. In 1978, Rassias extended Hyers' work by proving the existence of unique linear mappings near approximate additive mappings [32]. Since then, numerous studies have investigated the stability of various differential and integral equations. Readers seeking further information can consult [23, 24, 35] and the references therein.

In this paper, we study the existence and stability results for the following implicit problem in a Banach space:

$$ {}_{0}^{C}D_{\theta}^{\alpha,\varrho}\xi(\theta) = \Psi\left(\theta, \xi^{\theta}, {}_{0}^{C}D_{\theta}^{\alpha,\varrho}\xi(\theta)\right), \quad \theta \in \Theta := [0, \varkappa], \tag{1.1} $$

$$ \xi(\theta) = \Lambda_1(\theta), \quad \theta \in [-\varpi, 0], \tag{1.2} $$

$$ \xi(\theta) = \Lambda_2(\theta), \quad \theta \in [\varkappa, \varkappa + \delta], \tag{1.3} $$

where ${}_{0}^{C}D_{\theta}^{\alpha,\varrho}$ represents the Caputo tempered fractional derivative of order $\alpha \in (0,1)$, $\varrho \geq 0$, $\varpi, \delta > 0$, $\Psi : \Theta \times C([-\varpi, \delta], \Xi) \times \Xi \to \Xi$ is a given function, $\Lambda_1 \in C([-\varpi, 0], \Xi)$, and $\Lambda_2 \in C([\varkappa, \varkappa + \delta], \Xi)$. We denote by ξ^{θ} the element of $C([-\varpi, \delta], \Xi)$ defined by

$$ \xi^{\theta} = \xi(\theta + \sigma) : \sigma \in [-\varpi, \delta]. $$

The structure of this paper is as follows: Section 2 presents certain notations and preliminaries about the tempered fractional derivatives used throughout this manuscript. In Section 3, we present an existence result for the problem (1.1)-(1.3) that are based on Sadovskii's fixed point theorem. The Ulam stability is discussed in section 4. Finally, in the last section, illustrative examples are provided in support of the obtained results.

2 Preliminaries

In this section, we recall some notations, definitions and previous results which are used throughout this paper. We denote by $C(\Theta, \Xi)$, where $\Theta := [0, \varkappa]$, the Banach space of all continuous functions from Θ into Ξ with the norm

$$ \|\xi\|_{\infty} = \sup\{\|\xi(\theta)\| : \theta \in \Theta\}. $$

Let $C([-\varpi, 0], \Xi)$ the Banach space with the norm

$$ \|\xi\|_{[-\varpi,0]} = \sup\{\|\xi(\theta)\| : \theta \in [-\varpi, 0]\}. $$

Consider $C([\varkappa, \varkappa + \delta], \Xi)$ the Banach space with the norm

$$ \|\xi\|_{[\varkappa,\varkappa+\delta]} = \sup\{\|\xi(\theta)\| : \theta \in [\varkappa, \varkappa + \delta]\}, $$

and $C([-\varpi, \delta], \Xi)$ the Banach space with the norm

$$ \|\xi\|_{[-\varpi,\delta]} = \sup\{\|\xi(\theta)\| : \theta \in [-\varpi, \delta]\}. $$

Let

$$\Upsilon_{\Xi} = \big\{\xi : [-\varpi, \varkappa+\delta] \to \Xi : \xi|_{[0,\varkappa]} \in C(\Theta,\Xi), \xi|_{[-\varpi,0]} \in C([-\varpi,0],\Xi) \text{ and } \xi|_{[\varkappa,\varkappa+\delta]} \in C([\varkappa,\varkappa+\delta],\Xi)\big\}.$$

We note that Υ_{Ξ} is a Banach space with the norm

$$\|\xi\|_{\Upsilon_{\Xi}} = \sup_{\theta\in[-\varpi,\varkappa+\delta]} \|\xi(\theta)\|.$$

Definition 2.1 (The Riemann-Liouville tempered fractional integral [27, 33, 37]). Suppose that the function $\Psi \in C(\Theta,\Xi)$, $\varrho \geq 0$. Then, the Riemann-Liouville tempered fractional integral of order α is defined by

$$ {}_0I_\theta^{\alpha,\varrho}\Psi(\theta) = e^{-\varrho\theta}\, {}_0I_\theta^{\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{1}{\Gamma(\alpha)}\int_0^\theta \frac{e^{-\varrho(\theta-\sigma)}\Psi(\sigma)}{(\theta-\sigma)^{1-\alpha}}d\sigma, \tag{2.1}$$

where ${}_0I_\theta^{\alpha}$ denotes the Riemann-Liouville fractional integral [25], defined by

$$ {}_0I_\theta^{\alpha}\Psi(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^\theta \frac{\Psi(\sigma)}{(\theta-\sigma)^{1-\alpha}}d\sigma. \tag{2.2}$$

Obviously, the tempered fractional integral (2.1) reduces to the Riemann-Liouville fractional integral (2.2) if $\varrho = 0$.

Definition 2.2 (The Riemann-Liouville tempered fractional derivative [27,33]). For $\jmath-1 < \alpha < \jmath$; $\jmath \in \mathbb{N}$, $\varrho \geq 0$. The Riemann-Liouville tempered fractional derivative is defined by

$$ {}_0D_\theta^{\alpha,\varrho}\Psi(\theta) = e^{-\varrho\theta}\,{}_0D_\theta^{\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{e^{-\varrho\theta}}{\Gamma(\jmath-\alpha)}\frac{d^\jmath}{d\theta^\jmath}\int_0^\theta \frac{e^{\varrho\sigma}\Psi(\sigma)}{(\theta-\sigma)^{\alpha-\jmath+1}}d\sigma,$$

where ${}_0D_\theta^{\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right)$ denotes the Riemann-Liouville fractional derivative [25], given by

$$ {}_0D_\theta^{\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{d^\jmath}{d\theta^\jmath}\left({}_0I_\theta^{\jmath-\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right)\right) = \frac{1}{\Gamma(\jmath-\alpha)}\frac{d^\jmath}{d\theta^\jmath}\int_0^\theta \frac{\left(e^{\varrho\sigma}\Psi(\sigma)\right)}{(\theta-\sigma)^{\alpha-\jmath+1}}d\sigma.$$

Definition 2.3 (The Caputo tempered fractional derivative [27, 37]). For $\jmath-1 < \alpha < \jmath$; $\jmath \in \mathbb{N}$, $\varrho \geq 0$. The Caputo tempered fractional derivative is defined as

$$ {}_0^CD_\theta^{\alpha,\varrho}\Psi(\theta) = e^{-\varrho\theta}\,{}_0^CD_\theta^{\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{e^{-\varrho\theta}}{\Gamma(\jmath-\alpha)}\int_0^\theta \frac{1}{(\theta-\sigma)^{\alpha-\jmath+1}}\frac{d^\jmath\left(e^{\varrho\sigma}\Psi(\sigma)\right)}{d\sigma^\jmath}d\sigma,$$

where ${}_0^CD_\theta^{\alpha,\varrho}\left(e^{\varrho\theta}\Psi(\theta)\right)$ denotes the Caputo fractional derivative [25], given by

$$ {}_0^CD_\theta^{\alpha}\left(e^{\varrho\theta}\Psi(\theta)\right) = \frac{1}{\Gamma(\jmath-\alpha)}\int_0^\theta \frac{1}{(\theta-\sigma)^{\alpha-\jmath+1}}\frac{d^\jmath\left(e^{\varrho\sigma}\Psi(\sigma)\right)}{d\sigma^\jmath}d\sigma.$$

Lemma 2.4 ([27]). *For a constant C,*

$$ {}_0D_\theta^{\alpha,\varrho}C = Ce^{-\varrho\theta}\,{}_0D_\theta^{\alpha}e^{\varrho\theta}, \quad {}_0^CD_\theta^{\alpha,\varrho}C = Ce^{-\varrho\theta}\,{}_0^CD_\theta^{\alpha}e^{\varrho\theta}.$$

Obviously, ${}_0D_\theta^{\alpha,\varrho}(C) \neq {}_0^CD_\theta^{\alpha,\varrho}(C)$. And, ${}_0^CD_\theta^{\alpha,\varrho}(C)$ is no longer equal to zero, being different from ${}_0^CD_\theta^{\alpha}(C) \equiv 0$.

Lemma 2.5 ([27,37]). *Let $\Psi \in C^\jmath(\Theta,\Xi)$ and $\jmath-1 < \alpha < \jmath$; $\jmath \in \mathbb{N}$. Then, the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the composite properties*

$$ {}_0I_\theta^{\alpha,\varrho}\left[{}_0^CD_\theta^{\alpha,\varrho}\Psi(\theta)\right] = \Psi(\theta) - \sum_{k=0}^{\jmath-1} e^{-\varrho\theta}\frac{\theta^k}{k!}\left[\frac{d^k\left(e^{\varrho\theta}\Psi(\theta)\right)}{d\theta^k}\bigg|_{\theta=0}\right],$$

and

$$ {}_0^CD_\theta^{\alpha,\varrho}\left[{}_0I_\theta^{\alpha,\varrho}\Psi(\theta)\right] = \Psi(\theta), \text{ for } \alpha \in (0,1).$$

2.1 Measure of Noncompactness

Definition 2.6 ([9]). Let $\digamma$ be a Banach space and let $\Delta_{\digamma}$ be the family of bounded subsets of $\digamma$. The Kuratowski measure of noncompactness is the map $\zeta : \Delta_{\digamma} \longrightarrow [0,\infty)$ defined by

$$\zeta(\Omega) = inf\left\{ \varepsilon > 0 : \Omega \subset \bigcup_{j=1}^{m} \Omega_j, diam(\Omega_j) \leq \varepsilon \right\},$$

where $\Omega \in \Delta_{\digamma}$.

The map ζ satisfies the following properties:

- $\zeta(\Omega) = 0 \Leftrightarrow \overline{\Omega}$ is compact (Ω is relatively compact);
- $\zeta(\Omega) = \zeta(\overline{\Omega})$;
- $\Omega_1 \subset \Omega_2 \Rightarrow \zeta(\Omega_1) \leq \zeta(\Omega_2)$;
- $\zeta(\Omega_1 + \Omega_2) \leq \zeta(\Omega_1) + \zeta(\Omega_2)$;
- $\zeta(c\Omega) = |c|\zeta(\Omega)$, $c \in \mathbb{R}$;
- $\zeta(conv\Omega) = \zeta(\Omega)$.

Lemma 2.7 ([19]). *Let $B \subset \Upsilon_{\Xi}$ be a bounded and equicontinuous set. Then*

$a)$ *The function $\theta \to \zeta(B(\theta))$ is continuous, and*

$$\zeta_{\Upsilon_{\Xi}}(B) = \sup_{\theta \in [-\varpi, \varkappa+\delta]} \zeta(B(\theta)).$$

$b)$ $\zeta\left(\int_0^{\varkappa} \xi(\sigma)d\sigma : \xi \in B\right) \leq \int_0^{\varkappa} \zeta(B(\sigma))d\sigma$, *where*

$$B(\theta) = \{\xi(\theta) : \xi \in B\},\ \theta \in \Theta.$$

Definition 2.8 ([9]). Let $\digamma$ be a Banach space and $\mathcal{H} : \digamma \to \digamma$ a continuous mapping. $\mathcal{H}$ is said to be a condensing mapping if for each bounded set B with $\zeta(B) \neq 0$, we have

$$\zeta(\mathcal{H}(B)) < \zeta(B).$$

Theorem 2.9 (Sadovskii's fixed point Theorem [17]). *Let D be a non-empty, closed, bounded and convex subset of a Banach space $\digamma$, and let $\mathcal{H} : D \to D$ be a condensing mapping. Then $\mathcal{H}$ has a fixed point in D.*

3 Existence Results

Consider the following fractional differential problem:

$$ {}_0^C D_{\theta}^{\alpha,\varrho} \xi(\theta) = \mu(\theta), \quad if\ \theta \in \Theta,\ 0 < \alpha < 1, \varrho \geq 0, \tag{3.1}$$

$$\xi(\theta) = \Lambda_1(\theta), \quad if\ \theta \in [-\varpi, 0],\ \varpi > 0, \tag{3.2}$$

$$\xi(\theta) = \Lambda_2(\theta), \quad if\ \theta \in [\varkappa, \varkappa+\delta],\ \delta > 0, \tag{3.3}$$

where $\mu : \Theta \to \Xi$ is a continuous function, $\Lambda_1 \in C([-\varpi, 0], \Xi)$ and $\Lambda_2 \in C([\varkappa, \varkappa+\delta], \Xi)$.

Lemma 3.1. *Let $\alpha \in (0,1)$ and $\mu : \Theta \to \Xi$ be continuous. Then, the problem (3.1)-(3.3) has a unique solution given by:*

$$\xi(\theta) = \begin{cases} \Lambda_1(0)e^{-\varrho\theta} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{\theta} e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\mu(\sigma)d\sigma,\ \theta \in \Theta, \\ \Lambda_1(\theta), \qquad \theta \in [-\varpi, 0], \\ \Lambda_2(\theta), \qquad \theta \in [\varkappa, \varkappa+\delta]. \end{cases} \tag{3.4}$$

Proof. Suppose that ξ satisfies (3.1)-(3.3). Then, by applying the Riemann-Liouville tempered fractional integral of order α and by Lemma 2.5, we get

$$ {}_0I_\theta^{\alpha,\varrho}\, {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta) = {}_0I_\theta^{\alpha,\varrho}\mu(\theta). $$

This implies that

$$ \xi(\theta) - \xi(0)e^{-\varrho\theta} = \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\mu(\sigma)d\sigma. $$

Then,

$$ \xi(\theta) = \xi(0)e^{-\varrho\theta} + \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\mu(\sigma)d\sigma. $$

Finally, we have

$$ \xi(\theta) = \Lambda_1(0)e^{-\varrho\theta} + \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\mu(\sigma)d\sigma. $$

Conversely, we can easily show by Definition 2.3, Lemma 2.4 and Lemma 2.5 that if ξ verifies (3.4), then it satisfied the problem (3.1)-(3.3). □

Definition 3.2. By a solution of problem (1.1)-(1.3), we mean a function $\xi \in \Upsilon_\Xi$ that satisfies the equation (1.1) and the conditions (1.2)-(1.3).

Lemma 3.3. *Let $\Psi : \Theta \times C([-\varpi,\delta],\Xi) \times \Xi \to \Xi$ be a continuous function. Then, the problem (1.1)-(1.3) is equivalent to the following integral equation:*

$$ \xi(\theta) = \begin{cases} \Lambda_1(0)e^{-\varrho\theta} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\Psi(\sigma,\xi^\sigma,g(\sigma))d\sigma, & if\ \theta \in \Theta, \\ \Lambda_1(\theta), \qquad if\ \theta \in [-\varpi, 0], \\ \Lambda_2(\theta), \qquad if\ \theta \in [\varkappa, \varkappa+\delta], \end{cases} $$

where $g \in C(\Theta,\Xi)$ satisfies the following functional equation

$$ g(\theta) = \Psi(\theta, \xi^\theta, g(\theta)). $$

Let us set the following assumptions:

($A1$) The function $\Psi : \Theta \times C([-\varpi,\delta],\Xi) \times \Xi \to \Xi$ is continuous.

($A2$) There exist constants $\lambda > 0$ and $0 < \widehat{\lambda} < 1$ such that

$$ \|\Psi(\theta,\gamma,\wp) - \Psi(\theta,\bar\gamma,\bar\wp)\| \le \lambda\|\gamma-\bar\gamma\|_{[-\varpi,\delta]} + \widehat{\lambda}\|\wp - \bar\wp\|, $$

for any $\gamma, \bar\gamma \in C([-\varpi,\delta],\Xi)$, $\wp, \bar\wp \in \Xi$ and $\theta \in \Theta$.

($A3$) For each $\theta \in \Theta$ and bounded sets $B_1 \subseteq C([-\varpi,\delta],\Xi)$, $B_2 \subseteq \Xi$, we have

$$ \zeta(\Psi(\theta, B_1, B_2)) \le \lambda \sup_{\sigma\in[-\varpi,\delta]} \zeta(B_1(\sigma)) + \widehat{\lambda}\zeta(B_2). $$

Remark 3.4 ([7]). It is worth noting that the hypotheses $(A2)$ and $(A3)$ are equivalent.

We are now in a position to prove the existence result of the problem (1.1)-(1.3) based on Sadovskii's fixed point theorem.

Theorem 3.5. *Assume that the hypotheses* $(A1)$-$(A2)$ *are verified. If*

$$\frac{\lambda\varkappa^{\alpha}}{(1-\widehat{\lambda})\Gamma(\alpha+1)} < 1, \tag{3.5}$$

then the problem (1.1)-(1.3) *has at least one solution.*

Proof. Transform problem (1.1)-(1.3) into a fixed point problem by considering the operator $A : \Upsilon_{\Xi} \longrightarrow \Upsilon_{\Xi}$ by

$$A\xi(\theta) = \begin{cases} \Lambda_1(0)e^{-\varrho\theta} + \frac{1}{\Gamma(\alpha)}\int_0^{\theta} e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}g(\sigma)d\sigma, & if\ \theta \in \Theta, \\ \Lambda_1(\theta), \quad if\ \theta \in [-\varpi, 0], \\ \Lambda_2(\theta), \quad if\ \theta \in [\varkappa, \varkappa+\delta]. \end{cases}$$

This proof will be given in several steps.

Step 1: The operator $\Bbbk : \Upsilon_{\Xi} \longrightarrow \Upsilon_{\Xi}$ is continuous.
Let $\{\xi_{\jmath}\}_{\jmath\in\mathbb{N}}$ be a sequence such that $\xi_{\jmath} \longrightarrow \xi$ in Υ_{Ξ}. If $\theta \in [-\varpi, 0]$ or $\theta \in [\varkappa, \varkappa+\delta]$, then

$$\|\Bbbk\xi_{\jmath}(\theta) - \Bbbk\xi(\theta)\| = 0.$$

If $\theta \in \Theta$, we have

$$\|\Bbbk\xi_{\jmath}(\theta) - \Bbbk\xi(\theta)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^{\theta} e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\|g_{\jmath}(\sigma) - g(\sigma)\|d\sigma,$$

where $g_{\jmath}$ and g are two functions satisfying the following functional equations:

$$g_{\jmath}(\theta) = \Psi(\theta, \xi_{\jmath}^{\theta}, g_{\jmath}(\theta)),$$

and

$$g(\theta) = \Psi(\theta, \xi^{\theta}, g(\theta)).$$

By $(A2)$, we have

$$\begin{aligned} \|g_{\jmath}(\theta) - g(\theta)\| &= \|\Psi(\theta, \xi_{\jmath}^{\theta}, g_{\jmath}(\theta)) - \Psi(\theta, \xi^{\theta}, g(\theta))\| \\ &\leq \lambda\|\xi_{\jmath}^{\theta} - \xi^{\theta}\|_{[-\varpi,\delta]} + \widehat{\lambda}\|g_{\jmath}(\theta) - g(\theta)\|. \end{aligned}$$

Thus,

$$\|g_{\jmath}(\theta) - g(\theta)\| \leq \frac{\lambda}{1-\widehat{\lambda}}\|\xi_{\jmath}^{\theta} - \xi^{\theta}\|_{[-\varpi,\delta]}.$$

Then,

$$\|\Bbbk\xi_{\jmath}(\theta) - \Bbbk\xi(\theta)\| \leq \frac{\lambda}{(1-\widehat{\lambda})\Gamma(\alpha)}\int_0^{\theta} e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\|\xi_{\jmath}^{\sigma} - \xi^{\sigma}\|_{[-\varpi,\delta]}d\sigma.$$

By applying the Lebesgue dominated convergence theorem, we get

$$\|\Bbbk\xi_{\jmath}(\theta) - \Bbbk\xi(\theta)\| \longrightarrow 0 \quad \text{as} \quad \jmath \longrightarrow \infty,$$

which implies that

$$\|\Bbbk\xi_{\jmath} - \Bbbk\xi\|_{\Upsilon_{\Xi}} \longrightarrow 0 \quad \text{as} \quad \jmath \longrightarrow \infty.$$

Hence, the operator $\Bbbk$ is continuous.

Let $\beta > 0$ such that

$$\beta \geq \max\left\{ \frac{\|\Lambda_1(0)\| + \dfrac{q_1^* \varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)}}{1 - \dfrac{\lambda \varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)}}, \|\Lambda_1\|_{[-\varpi,0]}, \|\Lambda_2\|_{[\varkappa,\varkappa+\delta]} \right\},$$

where $\Psi_1(\theta) = \|\Psi(\theta,0,0)\|$, with $\Psi_1 \in C(\Theta,\Xi)$, such that

$$\Psi_1{}^* = \sup_{\theta\in\Theta} \Psi_1(\theta).$$

Define the ball

$$\Omega_\beta = \{\xi \in \Upsilon_\Xi : \|\xi\|_{\Upsilon_\Xi} \leq \beta\}.$$

It is clear that Ω_β is a bounded, closed and convex subset of Υ_Ξ.

Step 2: $\Bbbk(\Omega_\beta) \subset \Omega_\beta$.
Let $\xi \in \Omega_\beta$. If $\theta \in [-\varpi, 0]$, then

$$\|\Bbbk\xi(\theta)\| \leq \|\Lambda_1\|_{[-\varpi,0]} \leq \beta,$$

and if $\theta \in [\varkappa, \varkappa+\delta]$, then

$$\|\Bbbk\xi(\theta)\| \leq \|\Lambda_2\|_{[\varkappa,\varkappa+\delta]} \leq \beta.$$

For each $\theta \in \Theta$, we have

$$\|\Bbbk\xi(\theta)\| \leq \|\Lambda_1(0)\|e^{-\varrho\theta} + \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\|g(\sigma)\|d\sigma.$$

From hypothesis $(A2)$, we have

$$\begin{aligned}\|g(\theta)\| &= \|\Psi(\theta,\xi^\theta,g(\theta))\| \\ &\leq \Psi_1(\theta) + \lambda\|\xi^\theta\|_{[-\varpi,\delta]} + \widehat{\lambda}\|g(\theta)\| \\ &\leq \Psi_1{}^* + \lambda\|\xi\|_{\Upsilon_\Xi} + \widehat{\lambda}\|g(\theta)\| \\ &\leq \Psi_1{}^* + \lambda\beta + \widehat{\lambda}\|g(\theta)\|.\end{aligned}$$

Then,

$$\|g(\theta)\| \leq \frac{\Psi_1{}^* + \lambda\beta}{1-\widehat{\lambda}}.$$

Finally, we have

$$\begin{aligned}\|\Bbbk\xi(\theta)\| &\leq \|\Lambda_1(0)\|e^{-\varrho\theta} + \frac{\Psi_1{}^* + \lambda\beta}{(1-\widehat{\lambda})\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}d\sigma \\ &\leq \|\Lambda_1(0)\| + \frac{(\Psi_1{}^* + \lambda\beta)\varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)} \\ &\leq \beta.\end{aligned}$$

Thus, for each $\theta \in [-\varpi, \varkappa+\delta]$, $\|\Bbbk\xi(\theta)\| \leq \beta$, which implies that

$$\|\Bbbk\xi\|_{\Upsilon_\Xi} \leq \beta.$$

Consequently, $\Bbbk(\Omega_\beta) \subset \Omega_\beta$.

Step 3: $\mathbb{k}$ is condensing.
Let B be an equicontinuous subset of Ω_β. If $\theta \in [-\varpi, 0]$, then

$$\begin{aligned}\zeta(\mathbb{k}(B(\theta))) &= \zeta\{\mathbb{k}\xi(\theta), \ \ \xi \in B\} \\ &= \zeta\{\Lambda_1(\theta), \ \ \xi \in B\} \\ &= 0,\end{aligned}$$

and if $\theta \in [\varkappa, \varkappa + \delta]$, then

$$\begin{aligned}\zeta(\mathbb{k}(B(\theta))) &= \zeta\{\mathbb{k}\xi(\theta), \ \ \xi \in B\} \\ &= \zeta\{\Lambda_2(\theta), \ \ \xi \in B\} \\ &= 0.\end{aligned}$$

For each $\theta \in \Theta$, we have

$$\begin{aligned}\zeta(\mathbb{k}(B(\theta))) &= \zeta\{\mathbb{k}\xi(\theta), \ \ \xi \in B\} \\ &= \zeta\left\{\Lambda_1(0)e^{-\varrho\theta} + \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}g(\sigma)d\sigma, \ \ \xi \in B\right\} \\ &\le \zeta\{\Lambda_1(0)e^{-\varrho\theta}, \ \ \xi \in B\} + \zeta\left\{\frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}g(\sigma)d\sigma, \ \ \xi \in B\right\} \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^\theta (\theta-\sigma)^{\alpha-1}\{\zeta(g(\sigma))d\sigma, \ \ \xi \in B\}.\end{aligned}$$

We have by condition $(A3)$

$$\begin{aligned}\zeta(g(\theta)) &= \zeta(\Psi(\theta, \xi^\theta, g(\theta)) \\ &\le \lambda \sup_{\theta \in [-\varpi, \delta]} \zeta(\xi^\theta) + \widehat{\lambda}\zeta(g(\theta)) \\ &\le \lambda \sup_{\theta \in [-\varpi, \varkappa+\delta]} \zeta(\xi(\theta)) + \widehat{\lambda}\zeta(g(\theta)).\end{aligned}$$

Thus,

$$\zeta(g(\theta)) \le \frac{\lambda}{1-\widehat{\lambda}} \sup_{\theta \in [-\varpi, \varkappa+\delta]} \zeta(\xi(\theta)).$$

Then,

$$\begin{aligned}\zeta(\mathbb{k}(B(\theta))) &\le \frac{\lambda}{1-\widehat{\lambda}}\int_0^\theta (\theta-\sigma)^{\alpha-1}\{\sup_{\sigma \in [-\varpi, \varkappa+\delta]} \zeta(\xi(\sigma))d\sigma, \ \ \xi \in B\} \\ &\le \frac{\lambda\varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)}\zeta_{\Upsilon_\Xi}(B).\end{aligned}$$

Therefore,

$$\zeta_{\Upsilon_\Xi}(\mathbb{k}(B)) \le \left[\frac{\lambda\varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)}\right]\zeta_{\Upsilon_\Xi}(B) < \zeta_{\Upsilon_\Xi}(B).$$

Then, we have $\zeta_{\Upsilon_\Xi}(\mathbb{k}(B)) < \zeta_{\Upsilon_\Xi}(B)$, which implies that $\mathbb{k}$ is a condensing operator. As a consequence of Sadovskii's fixed point theorem, the operator $\mathbb{k}$ has at least one fixed point which is solution of the problem (1.1)-(1.3). □

4 Ulam-Hyers Stability

In this section, we will establish the Ulam stability for the problem (1.1)-(1.3). For this, we take inspiration from the following publications [1, 2, 22, 23, 34], and the references therein.

Definition 4.1 ([1]). Problem (1.1)-(1.3) is Ulam-Hyers stable if there exists a real number $C_\Psi > 0$ such that for each $\varepsilon > 0$ and for each solution $\xi \in \Upsilon_\Xi$ of the inequality

$$\left\| {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta) - \Psi\left(\theta, \xi^\theta, {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta)\right)\right\| < \varepsilon, \quad \theta \in \Theta, \tag{4.1}$$

there exists a solution $\bar{\xi} \in \Upsilon_\Xi$ of the problem (1.1)-(1.3) with

$$\|\xi(\theta) - \bar{\xi}(\theta)\| < C_\Psi \varepsilon, \quad \theta \in \Theta.$$

Definition 4.2 ([1]). Problem (1.1)-(1.3) is generalized Ulam-Hyers stable if there exists $\Lambda_{1\Psi} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\Lambda_{1\Psi}(0) = 0$ such that for each solution $\xi \in \Upsilon_\Xi$ of the inequality (4.1) there exists a solution $\bar{\xi} \in \Upsilon_\Xi$ of the problem (1.1)-(1.3) with

$$\|\xi(\theta) - \bar{\xi}(\theta)\| < \Lambda_{1\Psi}\varepsilon, \quad \theta \in \Theta.$$

Remark 4.3. A function $\xi \in \Upsilon_\Xi$ is a solution of the inequality (4.1) if and only if there exists a function $\ell \in C(\Theta, \Xi)$ (which depend on ξ) such that

(i) $\|\ell(\theta)\| \leq \varepsilon, \quad for\ each\ \theta \in \Theta.$

(ii) ${}_0^C D_\theta^{\alpha,\varrho}\xi(\theta) = \Psi\left(\theta, \xi^\theta, {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta)\right) + \ell(\theta), \quad for\ each\ \theta \in \Theta.$

Lemma 4.4. *The solution of the following perturbed problem*

$$\begin{aligned} {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta) &= \Psi\left(\theta, \xi^\theta, {}_0^C D_\theta^{\alpha,\varrho}\xi(\theta)\right) + \ell(\theta), \quad \theta \in \Theta := [0, \varkappa], \\ \xi(\theta) &= \Lambda_1(\theta), \quad \theta \in [-\varpi, 0], \\ \xi(\theta) &= \Lambda_2(\theta), \quad \theta \in [\varkappa, \varkappa + \delta], \end{aligned}$$

is given by

$$\xi(\theta) = \begin{cases} \Lambda_1(0)e^{-\varrho\theta} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}g(\sigma)d\sigma \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\ell(\sigma)d\sigma \qquad if\ \theta \in \Theta, \\ \\ \Lambda_1(\theta), \qquad if\ \theta \in [-\varpi, 0], \\ \\ \Lambda_2(\theta), \qquad if\ \theta \in [\varkappa, \varkappa + \delta]. \end{cases}$$

Moreover, the solution satisfies the following inequality

$$\left\| \xi(\theta) - \Lambda_1(0)e^{-\varrho\theta} - \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}g(\sigma)d\sigma \right\| \leq \frac{\varkappa^\alpha \varepsilon}{\Gamma(\alpha+1)}.$$

Proof. The proof can be done in the same steps as in Lemma 3.1. Thus, we omit it. □

Theorem 4.5. *Assume that the conditions* $(A1)$-$(A2)$ *hold and that the condition (3.5) is verified. Then the problem* (1.1)-(1.3) *is Ulam-Hyers stable.*

Proof. Let $\xi \in \Upsilon_\Xi$ be a solution of the inequality (4.1) and $\bar{\xi} \in \Upsilon_\Xi$ the solution of the problem (1.1)-(1.3). Then,

$$\|\xi(\theta) - \bar{\xi}(\theta)\| \leq \frac{\varkappa^\alpha \varepsilon}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1}\|g(\sigma) - h(\sigma)\|d\sigma,$$

where g and h are two functions satisfying the following functional equations:

$$g(\theta) = \Psi(\theta, \xi^\theta, g(\theta)),$$

and

$$h(\theta) = \Psi(\theta, \bar{\xi}^\theta, h(\theta)).$$

From hypothesis $(A2)$, we have

$$\begin{aligned} \|g(\theta) - h(\theta)\| &= \|\Psi(\theta, \xi^\theta, g(\theta)) - \Psi(\theta, \bar{\xi}^\theta, h(\theta))\| \\ &\leq \lambda \|\xi^\theta - \bar{\xi}^\theta\|_{[-\varpi,\delta]} + \widehat{\lambda}\|g(\theta) - h(\theta)\|, \end{aligned}$$

which implies that

$$\|g(\theta) - h(\theta)\| \leq \frac{\lambda}{1-\widehat{\lambda}} \|\xi^\theta - \bar{\xi}^\theta\|_{[-\varpi,\delta]}.$$

Then,

$$\begin{aligned} \|\xi(\theta) - \bar{\xi}(\theta)\| &\leq \frac{\varkappa^\alpha \varepsilon}{\Gamma(\alpha+1)} + \frac{\lambda}{(1-\widehat{\lambda})\Gamma(\alpha)} \int_0^\theta e^{-\varrho(\theta-\sigma)}(\theta-\sigma)^{\alpha-1} \|\xi^\sigma - \bar{\xi}^\sigma\|_{[-\varpi,\delta]} d\sigma \\ &\leq \frac{\varkappa^\alpha \varepsilon}{\Gamma(\alpha+1)} + \frac{\lambda \varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)} \|\xi - \bar{\xi}\|_{\Upsilon_\Xi}. \end{aligned}$$

Thus,

$$\|\xi - \bar{\xi}\|_{\Upsilon_\Xi} \leq \frac{\dfrac{\varkappa^\alpha \varepsilon}{\Gamma(\alpha+1)}}{1 - \dfrac{\lambda \varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)}} := C_f \varepsilon.$$

Consequently, the problem (1.1)-(1.3) is Ulam-Hyers stable. If we take $\Lambda_{1\Psi}(\varepsilon) = C_\Psi \varepsilon$ and $\Lambda_{1\Psi}(0) = 0$, then we get the generalized Ulam-Hyers stability of the problem (1.1)-(1.3). □

5 Examples

Set

$$\Xi = l^1 = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_\jmath, \dots), \sum_{\jmath=1}^{\infty} |\xi_\jmath| < \infty \right\},$$

where Ξ is a Banach space with the norm $\|\xi\| = \sum_{\jmath=1}^{\infty} |\xi_\jmath|$.

Example 5.1. Consider the following implicit problem:

$$ {}_0^C D_\theta^{\frac{1}{2},1} \xi(\theta) = \Psi\left(\theta, \xi^\theta, {}_0^C D_\theta^{\frac{1}{2},1} \xi(\theta)\right), \ \theta \in [0,1], \tag{5.1}$$

$$\xi(\theta) = \Lambda_1(\theta), \quad \theta \in [-1, 0], \tag{5.2}$$

$$\xi(\theta) = \Lambda_2(\theta), \quad \theta \in [1, 2], \tag{5.3}$$

where $\Lambda_1 \in C([-1,0], \Xi)$ and $\Lambda_2 \in C([1,2], \Xi)$.

Set

$$\Psi_\jmath\left(\theta, \xi_\jmath^\theta, {}_0^C D_\theta^{\frac{1}{2},1} \xi_\jmath(\theta)\right) = \frac{7 + \|\xi^\theta\|_{[-\varpi,\delta]} + \left| {}_0^C D_\theta^{\frac{1}{2},1} \xi_\jmath(\theta) \right|}{90 e^{\theta+2} \left(1 + \|\xi^\theta\|_{[-\varpi,\delta]} + \left\| {}_0^C D_\theta^{\frac{1}{2},1} \xi(\theta) \right\| \right)},$$

for $\theta\in[0,1],\ \xi\in C([-1,1],\Xi)$, where

$$\xi=(\xi_1,\xi_2,\ldots,\xi_\jmath,\ldots),$$

$$\Psi=(\Psi_1,\Psi_2,\ldots,\Psi_\jmath,\ldots),$$

and

$${}_0^C D_\theta^{\frac{1}{2},1}\xi=\left({}_0^C D_\theta^{\frac{1}{2},1}\xi_1\ ,\ {}_0^C D_\theta^{\frac{1}{2},1}\xi_2\ ,\ldots,\ {}_0^C D_\theta^{\frac{1}{2},1}\xi_\jmath,\ldots\right).$$

Clearly, Ψ is a continuous function, then the hypothesis $(A1)$ is satisfied.
For any $\gamma,\bar{\gamma}\in C([-1,1],\Xi)$, $\wp,\bar{\wp}\in\Xi$ and $\theta\in[0,1]$, we have

$$\|\Psi(\theta,\gamma,\wp)-\Psi(\theta,\bar{\gamma},\bar{\wp})\|\le\frac{1}{90e^2}\left[\|\gamma-\bar{\gamma}\|_{[-\varpi,\delta]}+\|\wp-\bar{\wp}\|\right].$$

Then, the hypothesis $(A2)$ is satisfied with $\lambda=\widehat{\lambda}=\frac{1}{90e^2}$. Also we have

$$\begin{aligned}\frac{\lambda\varkappa^\alpha}{(1-\widehat{\lambda})\Gamma(\alpha+1)}&=\frac{\frac{1}{90e^2}}{\left(1-\frac{1}{90e^2}\right)\frac{\sqrt{\pi}}{2}}\\&=\frac{2}{(90e^2-1)\sqrt{\pi}}\\&\approx 0.0016993277019224\\&<1,\end{aligned}$$

for $\varkappa=1$, $\alpha=\frac{1}{2}$. It follows from Theorem 3.5 that the problem (5.1)-(5.3) has at least one solution. Moreover, it verifies the Ulam-Hyers stability.

Example 5.2. Consider the following implicit problem:

$${}_0^C D_\theta^{\frac{1}{2},2}\xi(\theta)=\Psi\left(\theta,\xi^\theta,\ {}_0^C D_\theta^{\frac{1}{2},2}\xi(\theta)\right),\ \theta\in[0,1],\tag{5.4}$$

$$\xi(\theta)=\Lambda_1(\theta),\qquad\theta\in[-1,0],\tag{5.5}$$

$$\xi(\theta)=\Lambda_2(\theta),\qquad\theta\in[1,2],\tag{5.6}$$

where $\Lambda_1\in C([-1,0],\Xi)$ and $\Lambda_2\in C([1,2],\Xi)$.

Set

$$\Psi_\jmath\left(\theta,\xi_\jmath^\theta,\ {}_0^C D_\theta^{\frac{1}{2},2}\xi_\jmath(\theta)\right)=\frac{4\sin(\theta)+\|\xi^\theta\|_{[-\varpi,\delta]}+\frac{1}{2}\left|{}_0^C D_\theta^{\frac{1}{2},2}\xi_\jmath(\theta)\right|}{183e^{\sqrt{\theta+2}}},$$

for $\theta\in[0,1],\ \xi\in C([-\varpi,\delta],\Xi)$, where

$$\xi=(\xi_1,\xi_2,\ldots,\xi_\jmath,\ldots),$$

$$\Psi=(\Psi_1,\Psi_2,\ldots,\Psi_\jmath,\ldots),$$

and

$${}_0^C D_\theta^{\frac{1}{2},2}\xi=\left({}_0^C D_\theta^{\frac{1}{2},2}\xi_1\ ,\ {}_0^C D_\theta^{\frac{1}{2},2}\xi_2\ ,\ldots,\ {}_0^C D_\theta^{\frac{1}{2},2}\xi_\jmath,\ldots\right).$$

Obviously, Ψ is a continuous function, then the hypothesis $(A1)$ is satisfied.
For any $\gamma,\bar{\gamma}\in C([-\varpi,\delta],\Xi)$, $\wp,\bar{\wp}\in\Xi$ and $\theta\in[0,1]$, we have

$$\|\Psi(\theta,\gamma,\wp)-\Psi(\theta,\bar{\gamma},\bar{\wp})\|\le\frac{1}{183e^{\sqrt{2}}}\left[\|\gamma-\bar{\gamma}\|_{[-\varpi,\delta]}+\frac{1}{2}\|\wp-\bar{\wp}\|\right].$$

Then, the hypothesis $(A2)$ is satisfied with $\lambda = \frac{1}{183e^{\sqrt{2}}}$ and $\widehat{\lambda} = \frac{1}{366e^{\sqrt{2}}}$. Also we have

$$\begin{aligned}\frac{\lambda \varkappa^{\alpha}}{(1-\widehat{\lambda})\Gamma(\alpha+1)} &= \frac{\frac{1}{183e^{\sqrt{2}}}}{\left(1-\frac{1}{366e^{\sqrt{2}}}\right)\frac{\sqrt{\pi}}{2}} \\ &= \frac{4}{(366e^{\sqrt{2}}-1)\sqrt{\pi}} \\ &\approx 0.00150005575210831 \\ &< 1,\end{aligned}$$

for $\varkappa = 1$, $\alpha = \frac{1}{2}$. By Theorem 3.5, the problem (5.4)-(5.6) has at least one solution. Moreover, it verifies the Ulam-Hyers stability.

6 Conclusion

In this paper, we have made a substantial contribution to the study of certain classes of implicit fractional differential equations involving the Caputo tempered fractional derivative with retarded and advanced arguments. The methodologies utilized are primarily Sadovskii's fixed point theorem as well as the technique of measure of noncompactness. We have investigated Ulam's stability of the problem, advancing the understanding of fractional differential equations under various conditions. In future research, we aim to explore additional classes of fractional differential equations and inclusions, including problems with infinite delays, as well as impulsive problems, focusing on both instantaneous and non-instantaneous impulses.

References

[1] S. Abbas, M. Benchohra, J.R. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations*: Existence and Stability, de Gruyter, Berlin, 2018.

[2] S. Abbas, M. Benchohra, J.E. Lazreg, J. J. Nieto and Y. Zhou, *Fractional Differential Equations and Inclusions: Classical and Advanced Topics*, World Scientific, Singapore, 2023.

[3] S. Abbas, M. Benchohra, G.M.N' Guérékata, *Topics in Fractional Differential Equations*. Springer, New York, 2012.

[4] S. Abbas, M. Benchohra and G M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2014.

[5] S. Abdelkebir, Existence and uniqueness results for a nonlinear fractional Volterra integro-differential equation with non-local boundary conditions, *Palest. J. Math.* **13** (3) (2024), 312-319.

[6] R. Almeida, M. L. Morgado, Analysis and numerical approximation of tempered fractional calculus of variations problems, *J. Comput. Appl. Math.* **361** (2019), 1-12.

[7] J. Appell, Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator, *J. Math. Anal. Appl.* **83** (1) (1981), 251-263.

[8] D. Baleanu, Z. B. Güvenç, and J. A. T. Machado *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[9] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces*. Marcel Dekker, New York, 1980.

[10] M. Benchohra, F. Bouazzaoui, E. Karapinar and A. Salim, Controllability of second order functional random differential equations with delay. *Mathematics.* **10** (2022), 16pp. https://doi.org/10.3390/math10071120

[11] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra and C. Tunc, Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses, *Appl. Anal. Optim.* **6** (2022), 79-94.

[12] A. Ben Makhlouf, D. Boucenna, A. M. Nagy and L. Mchiri, Some weakly singular integral inequalities and their applications to tempered fractional differential equations, *J. Math.* **2022** (1) (2022), 1682942. https://doi.org/10.1155/2022/1682942

[13] R. G. Buschman, Decomposition of an integral operator by use of Mikusinski calculus, *SIAM J. Math. Anal.* **3** (1972), 83-85.

[14] C. Derbazi, H. Hammouche, A. Salim and M. Benchohra, Weak solutions for fractional Langevin equations involving two fractional orders in banach spaces. *Afr. Mat.* **34** (2023), 10 pages. https://doi.org/10.1007/s13370-022-01035-3

[15] M. Ferhat and T. Blouhi, Existence for coupled Random system fractional order functional differential with infinite delay, *Palest. J. Math.* **13** (1) (2024), 232-241.

[16] H. Gassara, D. Kharrat, A. Ben Makhlouf, L. Mchiri, and M. Rhaima, SOS approach for practical stabilization of tempered fractional-order power system, *Mathematics.* **11** (13) (2023), 3024. https://doi.org/10.3390/math11133024

[17] K. Goebel, *Concise Course on Fixed Point Theorems*. Yokohama Publishers, Japan, 2002.

[18] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[19] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*. Kluwer Academic Publishers, Dordrecht, 1996.

[20] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941), 222-224.

[21] O. Kahouli, D. Boucenna, A. Ben Makhlouf and Y. Alruwaily, Some new weakly singular integral inequalities with applications to differential equations in frame of tempered χ-fractional derivatives, *Mathematics.* **10** (20) (2022), 3792. https://doi.org/10.3390/math10203792

[22] N. Laledj, A. Salim, J. E. Lazreg, S. Abbas, B. Ahmad and M. Benchohra, On implicit fractional q-difference equations: Analysis and stability. *Math. Methods Appl. Sci.* **45** (2022), 10775-10797. https://doi.org/10.1002/mma.8417

[23] J. E. Lazreg, M. Benchohra and A. Salim, Existence and Ulam stability of k-generalized Λ_2-Hilfer fractional problem. *J. Innov. Appl. Math. Comput. Sci.* **2** (2022), 1-13.

[24] D. Luo, Z. Luo, H. Qiu, Existence and Hyers-Ulam stability of solutions for a mixed fractional-order nonlinear delay difference equation with parameters. *Math. Probl. Eng.* **2020**, 9372406 (2020).

[25] A.A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Amsterdam, 2006.

[26] S. Krim, A. Salim and M. Benchohra, On implicit Caputo tempered fractional boundary value problems with delay. *Lett. Nonlinear Anal. Appl.* **1** (2023), 12-29.

[27] C. Li, W. Deng and L. Zhao, Well-posedness and numerical algorithm for the tempered fractional differential equations, *Discr. Contin. Dyn. Syst. Ser. B.* **24** (2019), 1989-2015.

[28] M. Medved and E. Brestovanska, Differential equations with tempered ψ-Caputo fractional derivative, *Math. Model. Anal.* **26** (2021), 631-650.

[29] N. A. Obeidat, D. E. Bentil, New theories and applications of tempered fractional differential equations, *Nonlinear Dyn.* **105** (2021), 1689-1702.

[30] M. D. Ortigueira, G. Bengochea and J. T. Machado, Substantial, tempered, and shifted fractional derivatives: Three faces of a tetrahedron, *Math. Methods Appl. Sci.* **44** (2021), 9191-9209.

[31] Priscila. S. Ramos, J. Vanterler da C. Sousa and E. Capelas de Oliveira, Controllability of fractional impulsive integro-differential control system, *Palest. J. Math.* **13** (4) (2024), 102-115.

[32] T. M. Rassias, On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* **72** (1978), 297-300.

[33] F. Sabzikar, M. M. Meerschaert and J. Chen, Tempered fractional calculus, *J. Comput. Phys.* **293** (2015), 14-28.

[34] A. Salim, M. Benchohra, J. E. Lazreg and G. N'Guérékata, Existence and k-Mittag-Leffler-Ulam-Hyers stability results of k-generalized Λ_2-Hilfer boundary value problem. *Nonlinear Stud.* **29** (2022), 359–379.

[35] A. Salim, J. E. Lazreg, B. Ahmad, M. Benchohra and J. J. Nieto, A study on k-generalized ψ-Hilfer derivative operator, *Vietnam J. Math.* (2022). https://doi.org/10.1007/s10013-022-00561-8

[36] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.

[37] B. Shiri, G. Wu and D. Baleanu, Collocation methods for terminal value problems of tempered fractional differential equations, *Appl. Numer. Math.* **156** (2020), 385-395.

[38] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

[39] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

Author information

W. Rahou, Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, Algeria.
E-mail: wafaa.rahou@yahoo.com

A. Salim, Faculty of Technology, Hassiba Benbouali University of Chlef, Algeria.
E-mail: salim.abdelkrim@yahoo.com

J. E. Lazreg, Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, Algeria.
E-mail: lazregjamal@yahoo.fr

M. Benchohra, Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, Algeria.
E-mail: benchohra@yahoo.com

Received: 2023-03-26
Accepted: 2025-01-28