Palestine Journal of Mathematics

Vol 14(1)(2025) , 217–232

Enumeration of unlabeled series-parallel posets by using the poset matrix

S. U. Mohammad, M. R. Talukder and S. N. Begum

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 06A07; Secondary 05A15, 11Y55, 15B36.

Keywords and phrases: Series-parallel poset, Matrix recognition, Exact enumeration, Poset matrix, Relabeling equivalent, Monotone inter-distant length, Nonisomorphic direct sum and ordinal sum.

This work was supported by the Research Grant 2019-20, SUST Research Center, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh [Grant no. PS/2019/1/20].

Corresponding Author: S. U. Mohammad

Abstract We give an exact enumeration of the unlabeled series-parallel posets according to the number of ordinal terms and according to the number of direct terms of the posets. For every $n \geq 2$, we determine the number of n-element unlabeled connected (analogously, disconnected) series-parallel posets by using the numbers of unlabeled disconnected (connected) series-parallel posets up to $(n-1)$ elements. Here, we use the poset matrix to represent posets. We also give algorithms to determine the values of the parameters involved in the enumeration formulas and to compute the number of unlabeled series-parallel posets. We show that the enumeration algorithm runs in time $\mathcal{O}(n^{m+4})$, where $m+1$ equals the number of ordinal terms of the connected posets.

1 Introduction

Due to the series and parallel constructions, the series-parallel posets are mostly computationally tractable, and consequently, these are well-known as a class of completely decomposable posets, see [15, 20]. Analogously, the notions of numerous combinatorial properties of mathematical structures such as posets [6, 10] and graphs [2, 16] were repeatedly introduced and revealed their significant applications in characterizing of these mathematical structures. Therefore, the recognition and enumeration of series-parallel posets were considered by numerous authors. Bayoumi et al. [1] computed $SP(n)$, the number of n-element unlabeled series-parallel posets, for $n \leq 12$ by recalling the generating function given by Stanley [20]. Later on, El-Zahar et al. [6] computed $SP(n)$ for $n \leq 15$ according to the height of posets by modifying the Stanley's generating function with the height as an additional parameter. Recently, $SP(n)$ for $n \leq 1000$ are included in the integer sequence A003430 in OEIS [19] and noted as computed by J. F. Alcover and A. P. Heinz. In this article, we give an exact enumeration of the unlabeled series-parallel posets according to the number of ordinal terms in the case of connected posets and according to the number of direct terms in the case of disconnected posets.

The notions of several incidence matrices were frequently introduced and applied for certain computational aspects of the concerned structures, particularly see [1, 3, 18] for posets and [7, 17] for graphs. We recall the notion of the poset matrix, a square $(0,1)$-matrix introduced by Mohammad and Talukder [9] to represent posets, where the authors obtained matrix recognitions of the P-graphs, P-series, and series-parallel posets. Here, we recall these results on the matrix recognitions of posets and obtain an exact enumeration of the unlabeled series-parallel posets. We mainly generalize and use the criterion for the nonisomorphic ordinal sum and nonisomorphic direct sum of the poset matrices introduced by Mohammad et al. [14] and applied particularly for an exact enumeration of the unlabeled P-series. A more general setup of the criteria for pairwise nonisomorphic unlabeled disconnected posets was given by Mohammad et al. [11] and used to obtain an exact enumeration of the unlabeled disconnected posets.

Direct algorithmic methods for the recognition and enumeration of some common classes of posets were considered in several literature, see [2, 4, 8]. The algorithms used in most of these cases for the recognition of pairwise nonisomorphic posets work like generate-one and count-one. As a result, the running times of these algorithms grow more rapidly even though the posets under consideration are significantly small in size. Mainly, the recursive process in constructing pairwise nonisomorphic posets makes these algorithms highly time-complex. Therefore, direct algorithmic methods for the enumeration of series-parallel posets were ignored by some authors [6]. Let $CSP(n)$ (analogously, $DSP(n)$) denote the number of n-element unlabeled connected (analogously, disconnected) series-parallel posets. In the proposed exact enumeration method, we use the numbers $CSP(r)$, $1 \le r \le n-1$, to compute $DSP(n)$, $n \ge 2$, according to the number of connected direct terms of posets. Conversely, we use the numbers $DSP(r)$, $1 \le r \le n-1$, to compute $CSP(n)$, $n \ge 2$, according to the number of ordinal terms of posets that are either the singleton or disconnected. See the integer sequences A350772 and A356558 that we contributed to OEIS [19]. Also, we give the algorithms to determine the values of the parameters involved in the enumeration formulas as well as to compute $SP(n)$, $n \ge 2$, which equals the sum of $CSP(n)$ and $DSP(n)$. We show that the overall enumeration algorithm runs in polynomial time with complexity $\mathcal{O}(n^{m+4})$, where $m+1$ equals the number of ordinal terms (either the singleton or disconnected) of the connected posets.

In Section 2, we recall a few definitions and results related to the matrix recognition of series-parallel posets. In Section 3, we establish the criteria for the lengths of the block of 1s (analogously, block of 0s) satisfied by the poset matrices so that they represent pairwise nonisomorphic connected (disconnected) posets. In Section 4, we give the results regarding the enumerations of unlabeled connected and disconnected series-parallel posets. In Section 5, we provide the enumeration algorithms and prove their time complexity. In Section 6, we briefly discuss the implementations of the enumeration algorithms into the computer for numerical results. Here we also include the data corresponding to $CSP(n)$ for $n \le 23$ and $DSP(n)$ for $n \le 24$.

2 Preliminaries

2.1 Posets

A *poset* (*partially ordered set*) is a structure $\mathbf{S} = \langle S, \leqslant \rangle$ consisting of the nonempty set S with the order relation $\leqslant$ on S. A poset $\mathbf{S}$ is called *finite* if the underlying set S is finite. Throughout this paper, we assume that every poset is finite. We use the notations $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n (n \ge 1)$ for the n-element chain poset, and $\mathbf{I}_n (n \ge 1)$ for the n-element antichain poset. We write $\mathbf{R} + \mathbf{S}$ and $\mathbf{R} \oplus \mathbf{S}$, respectively, to mean the direct sum and the ordinal sum of the posets $\mathbf{R}$ and $\mathbf{S}$. Here $\mathbf{R}$ and $\mathbf{S}$ are called the *direct terms* of $\mathbf{R} + \mathbf{S}$ and the *ordinal terms* of $\mathbf{R} \oplus \mathbf{S}$. We briefly write $\sum_{i=1}^{n} \mathbf{S}_i$ for the direct sum and $\bigoplus_{i=1}^{n} \mathbf{S}_i$ for the ordinal sum of the posets $\mathbf{S}_i$, $1 \le i \le n$. A poset having two or more direct terms is called *disconnected*, otherwise, it is called *connected*. For every $n \ge 2$, trivially the poset $\bigoplus_{i=1}^{n} \mathbf{S}_i$ is connected and $\sum_{i=1}^{n} \mathbf{S}_i$ is disconnected. We write $\mathbf{R} \cong \mathbf{S}$ if $\mathbf{R}$ and $\mathbf{S}$ are order isomorphic. Also, by a collection of isomorphic (analogously, nonisomorphic) posets, we mean that they are *pairwise* isomorphic (nonisomorphic). Let the posets $\mathbf{R}_i$, $1 \le i \le n$, and $\mathbf{S}_i$, $1 \le i \le n$, where $n \ge 2$, be given. Then $\sum_{i=1}^{n} \mathbf{R}_i \cong \sum_{i=1}^{n} \mathbf{S}_i$ if $\mathbf{R}_i \cong \mathbf{S}_i$ for all $1 \le i \le n$. Since the direct sum of posets is commutative, the converse of this result is not true. On the other hand, since the ordinal sum of posets is not commutative, $\bigoplus_{i=1}^{n} \mathbf{R}_i \cong \bigoplus_{i=1}^{n} \mathbf{S}_i$ if and only if $\mathbf{R}_i \cong \mathbf{S}_i$ for all $1 \le i \le n$. For further basics of posets, we would like to refer the readers to the classical book by Davey and Priestley [5].

A poset $\mathbf{P}$ is called a *P-graph* if there exist the singleton or antichain posets $\mathbf{I}_{m_i}, 1 \le i \le n$, such that $\mathbf{P} = \bigoplus_{i=1}^{n} \mathbf{I}_{m_i}$. Obviously, all the nontrivial P-graph, that is, P-graphs except the antichains $\mathbf{I}_n$, $n \ge 2$, are connected. A poset $\mathbf{S}$ is called a *P-series* if there exist the P-graphs $\mathbf{P}_i, 1 \le i \le n$, such that $\mathbf{S} = \sum_{i=1}^{n} \mathbf{P}_i$. All the P-series except the nontrivial P-graphs are disconnected. A poset $\mathbf{R}$ is called *series-parallel* if it can be decomposed into the singleton posets by using only the direct sum and the ordinal sum of posets. For example, the posets $\mathbf{1} \oplus (\mathbf{1} + \mathbf{C}_2)$ and $(\mathbf{1} + \mathbf{C}_2) \oplus \mathbf{1}$ are series-parallel that are neither P-graphs nor P-series. In particular, if there exist the P-series $\mathbf{S}_i, 1 \le i \le n$, such that $\mathbf{R} = \mathbf{S}_1 * \mathbf{S}_2 * \cdots * \mathbf{S}_n$, where $*$ is either the direct sum or the ordinal sum of posets, then $\mathbf{R}$ is series-parallel.

2.2 Poset matrix

Mohammad and Talukder [9] introduced the notion of the poset matrix. See [3, 10, 12, 13] for some recent applications of the poset matrix. A square $(0,1)$-matrix $M_n = [a_{ij}], 1 \leq i, j \leq n$ is called a *poset matrix* if and only if the following conditions hold:

(i) M_n is reflexive: $a_{ii} = 1$ for all $1 \leq i \leq n$,

(ii) M_n is antisymmetric: $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$,

(iii) M_n is transitive: $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$.

To each poset matrix $M_n = [a_{ij}]$, $1 \leq i, j \leq n$, a poset $\mathbf{S} = \langle S, \leqslant \rangle$, where $S = \{s_1, s_2, \ldots, s_n\}$ and s_i corresponds the i-th row (or column) of M_n, is associated by defining the order relation $\leqslant$ on S such that for all $1 \leq i, j \leq n$, we have $s_i \leqslant s_j$ if and only if $a_{ij} = 1$. Then it is said that M_n *represents* the poset $\mathbf{S}$ and vice versa. For some $1 \leq i, j \leq n$, the interchanges of i-th and j-th rows along with the interchanges of i-th and j-th columns in a poset matrix M_n is called *(i,j)-relabeling* of M_n. It was shown in [9] that every poset matrix can be relabeled to an upper (equivalently, lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling. From now on, by a poset matrix, we mean a poset matrix in upper triangular form.

Any two poset matrices M_n and M'_n are called *relabeling equivalent*, or briefly *equivalent*, if the matrix M'_n can be obtained by some relabeling of the matrix M_n and vice versa. We write $M_n \sim M'_n$ if M_n and M'_n are relabeling equivalent. Also, by a collection of equivalent (analogously, nonequivalent) poset matrices, we mean that the matrices are *pairwise* equivalent (nonequivalent). Note that if $M_n \sim M'_n$ (analogously, $M_n \nsim M'_n$) then the posets represented by M_n and M'_n are isomorphic (nonisomorphic).

2.3 Matrix recognition of series-parallel posets

We write $M_m \oplus N_n$ and $M_m \boxplus N_n$, respectively, for the direct sum and the ordinal sum of the poset matrices M_m and N_n. Here we call M_m and N_n the *direct terms* of $M_m \oplus N_n$ and the *ordinal terms* of $M_m \boxplus N_n$. A poset matrix $M_n = [a_{ij}]$, $1 \leq i, j \leq n$, has the property of *block of 0s* (analogously, *block of 1s*) of length r, where $1 \leq r < n$, if and only if $a_{ij} = 0$ ($a_{ij} = 1$) for all $1 \leq i \leq r$ and $r + 1 \leq j \leq n$, see [9] for details. Throughout this paper, we write I_n to denote the identity matrix of order n and C_n to denote the matrix $[c_{ij}], 1 \leq i, j \leq n$, where $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise. Obviously, for every $n \geq 2$, the matrices I_n and C_n satisfy, respectively, the property of block of 0s and the property block of 1s of lengths equal to any subcollection of $1, 2, \ldots, n-1$. Further, in Example 2.1 below, the matrices L and L' satisfy the block of 1s property of length 1 and length 2, respectively, and the matrix L'' satisfies the block of 0s property of length 2.

Example 2.1.

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad L' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad L'' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that for any relabeling, a poset matrix M_n can satisfy either the property of block of 0s or the property of block of 1s at a time, but no poset matrix can satisfy both the properties together. Observe here that $L = 1 \boxplus I_2$, $L' = I_2 \boxplus 1$, and $L'' = C_2 \oplus 1$. This result was proved by Mohammad and Talukder [9] in general as follows:

Theorem 2.2. *[9] Let M_n be any poset matrix. Then for $n \geq 2$,*

(i) $M_n = M_{n_1} \oplus M_{n_2-n_1} \oplus \cdots \oplus M_{n-n_m}$ if and only if M_n satisfies the block of 0s property of lengths $n_1, n_2, \ldots, n_m$.

(ii) $M_n = M_{n_1} \boxplus M_{n_2-n_1} \boxplus \cdots \boxplus M_{n-n_m}$ if and only if M_n satisfies the block of 1s property of lengths $n_1, n_2, \ldots, n_m$.

Then we have the following immediate results that give the matrix recognitions of connected posets and disconnected posets.

Theorem 2.3. *Let M_n represent the poset $\mathbf{S} \ncong \mathbf{1}$. Then*

(i) $\mathbf{S}$ *is connected if M_n can be relabeled in such a form that it satisfies the block of 1s property.*

(ii) $\mathbf{S}$ *is disconnected if and only if M_n can be relabeled in such a form that it satisfies the block of 0s property.*

Proof. The proof follows by Theorem 2.2 and the definitions of connected posets and disconnected posets. □

Note that the converse of the result in the first part of Theorem 2.3 is not true in general. Because, the 4-element N-shaped poset is connected but the poset matrix that represents this poset does not satisfy the block of 1s property for any labeling. However, we find that the converse of this result holds in the case of series-parallel posets. See Theorem 5.7 [9] for the matrix recognition of the series-parallel posets. We use this result to give the matrix recognitions of the connected and disconnected series-parallel posets as follows:

Theorem 2.4. *Let M_n represent the poset $\mathbf{S} \ncong \mathbf{1}$. Then*

(i) $\mathbf{S}$ *is connected series-parallel if and only if M_n can be relabeled in such a form that it satisfies the block of 1s property and every ordinal term until 1 satisfies either the block of 0s property or the block of 1s property.*

(ii) $\mathbf{S}$ *is disconnected series-parallel if and only if M_n can be relabeled in such a form that it satisfies the block of 0s property and every direct term until 1 satisfies either the block of 0s property or the block of 1s property.*

Proof. The proof follows by Theorem 2.3 and Theorem 5.7 [9]. □

3 Nonisomorphic sums and enumeration of posets

For $n \geq 2$, suppose that there exist exactly t, where $t \geq 2$, nonequivalent (pairwise) matrices M_n, poset matrix of order n. Then we say that the matrix M_n can represent t nonisomorphic (pairwise) posets. Since the direct sum of posets is commutative, matrices M_n that satisfy the block of 0s property can represent isomorphic (pairwise) posets. In this section, we establish mainly the criteria for the lengths of the block of 1s (analogously, block of 0s) satisfied by the matrices M_n such that they represent only nonisomorphic connected (disconnected) posets. For $n \geq 2$, let the matrix M_n satisfy the block of 1s (analogously, block of 0s) property for certain lengths. Here, we obtain the formulas giving an enumeration of the nonisomorphic connected (analogously, disconnected) posets that can be represented by M_n.

3.1 Ordinal sum and enumeration of connected posets

For $n \geq 2$, let the matrix M_n satisfy the block of 1s property for some lengths. Since the ordinal sum of poset matrices is not commutative, all the posets represented by M_n are nonisomorphic if the posets represented by every ordinal term of M_n are nonisomorphic. Here, we use this result to find the formula that gives an enumeration of the nonisomorphic connected posets, that is, the number of nonisomorphic posets that can be represented by M_n.

Theorem 3.1. *For $n \geq 2$ and $1 \leq m \leq n-1$, let the matrix M_n satisfy the property of block of 1s of lengths $n_1, n_2, \ldots, n_m$ such that for every $1 \leq i \leq m+1$, an ordinal term M_{r_i} of M_n, where $r_i = n_i - n_{i-1}$ (with $n_0 = 0$ and $n_{m+1} = n$), can represent $P(r_i)$ nonisomorphic posets. Then $Q(n)$, the number of nonisomorphic connected posets that can be represented by M_n, is given as follows:*

$$Q(n) = \prod_{i=1}^{m+1} P(r_i), \qquad n \geq 2. \tag{3.1}$$

Proof. Since M_n satisfies the block of 1s property of lengths $n_1, n_2, \ldots, n_m$, by Theorem 2.3, M_n represents connected posets and by Theorem 2.2, $M_n = M_{n_1} \boxplus M_{n_2-n_1} \boxplus \cdots \boxplus M_{n-n_m}$ for some $M_{r_i} = M_{n_i-n_{i-1}}$, $1 \leq i \leq m+1$ (here, $n_0 = 0$ and $n_{m+1} = n$) as the ordinal terms of M_n. Since ordinal sum of poset matrices is not commutative and, for every $1 \leq i \leq m+1$, an ordinal term M_{r_i} can represent $P(r_i)$ nonisomorphic posets, matrix M_n can represent the nonisomorphic posets having ordinal terms as a subcollection of $m+1$ posets each of which is to choose from one of $m+1$ collections of $P(r_i)$ nonisomorphic posets. Therefore, $Q(n)$ equals the number of combinations of $m+1$ items each of which is to choose from one of $m+1$ collections of $P(r_i)$ distinct items. This gives $Q(n)$ as follows:

$$Q(n) = P(r_1) \times P(r_2) \times \cdots \times P(r_{m+1}) = \prod_{i=1}^{m+1} P(r_i), \qquad n \geq 2.$$

□

The following example illustrates the result obtained in Theorem 3.1.

Example 3.2. Consider the matrices M_6 that satisfy the property of block of 1s of length 3, lengths $1, 5$, and lengths $1, 4$. We enumerate the connected posets represented by M_6 in each of these cases as follows:

(i) Let M_6 satisfy the property of block of 1s of length 3. Then $M_6 = M_3 \boxplus M_3$. Since the ordinal term M_3 can represent 5 nonisomorphic posets, in this case, M_6 can represent $5 \times 5 = 25$ nonisomorphic posets all of which are connected.

(ii) Let M_6 satisfy the property of block of 1s of lengths $1, 5$. Then $M_6 = M_1 \boxplus M_4 \boxplus M_1$. Since the ordinal terms M_1 and M_4 can represent 1 and 16 nonisomorphic posets, respectively, in this case, M_6 can represent $1 \times 16 \times 1 = 16$ nonisomorphic connected posets.

(iii) Let M_6 satisfy the property of block of 1s of lengths $1, 4$. Then $M_6 = M_1 \boxplus M_3 \boxplus M_2$. Since the ordinal terms M_1, M_2, and M_3 can represent 1, 2, and 5 nonisomorphic posets, respectively, in this case, M_6 can represent $1 \times 5 \times 2 = 10$ nonisomorphic connected posets.

3.2 Direct sum and enumeration of disconnected posets

For $n \geq 2$, let the matrix M_n satisfy the block of 0s property for some lengths such that every direct term of M_n represents nonisomorphic posets. Then, M_n represents a collection of disconnected posets. We observe that some of these posets can be isomorphic. In particular, we see that M_4 can satisfy the block of 0s property of length 1, length 2, length 3, lengths 1, 2, lengths 1, 3, lengths 2, 3, and lengths 1, 2, 3. Here, the matrices $1 \oplus 1 \oplus C_2$ and $1 \oplus C_2 \oplus 1$ (Example 3.3) satisfy the block of 0s property of lengths 1, 2 and lengths 1, 3, respectively, and represent the posets isomorphic to $\mathbf{C}_2 + \mathbf{1} + \mathbf{1}$.

Example 3.3. Two M_4 that represent the posets isomorphic to $\mathbf{C}_2 + \mathbf{1} + \mathbf{1}$.

$$1 \oplus 1 \oplus C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 1 \oplus C_2 \oplus 1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, to represent nonisomorphic disconnected posets by M_n, we see that matrix M_n must satisfy the block of 0s property of some nondecreasing inter-distant lengths defined as follows:

Definition 3.4. For $n \geq 2$ and $1 \leq m \leq n-1$, the lengths $n_1, n_2, \ldots, n_m$, chosen as a subcollection of the integers $1, 2, \ldots, n-1$ are called

(i) *strictly increasing inter-distant (SIID)* if $n_1 < n_2 - n_1 < \cdots < n - n_m$,

(ii) *equally inter-distant (EQID)* if $n_1 = n_2 - n_1 = \cdots = n - n_m$, and

(iii) *nondecreasing inter-distant (NDID)* if $n_1 \leq n_2 - n_1 \leq \cdots \leq n - n_m$.

Table 1. All SIID, EQID, and NDID lengths $l(m,j)$, $1 \leq m \leq 5$, $1 \leq j \leq p_m$, for some $p_m \leq \binom{5}{m}$.

		$l(m,j)$		
m	j	SIID	EQID	NDID
1	1	1	3	1
1	2	2	–	2
1	3	–	–	3
2	1	1, 3	2, 4	1, 2
2	2	–	–	1, 3
2	3	–	–	2, 4
3	1	–	–	1, 2, 3
3	2	–	–	1, 2, 4
4	1	–	–	1, 2, 3, 4
5	1	–	1, 2, 3, 4, 5	1, 2, 3, 4, 5

For example, all SIID, SQID, and NDID lengths $l(m,j)$, $1 \leq m \leq 5$, $1 \leq j \leq p_m$, for some $p_m \leq \binom{5}{m}$, are given in Table 1.

Further, we see that the matrices $1 \oplus I_2 \oplus L''$ and $I_2 \oplus C_2 \oplus I_2$ (Example 3.5) satisfy the block of 0s property of the nondecreasing inter-distant lengths 1, 3 and lengths 2, 4, respectively, and represent the posets isomorphic to $\mathbf{C}_2 + \mathbf{I}_4$. This happens because, in this case, some of the direct terms M_1, M_2, and M_3 of M_6 represent disconnected posets.

Example 3.5. Two M_6 that represent the posets isomorphic to $\mathbf{C}_2 + \mathbf{I}_4$.

$$1 \oplus I_2 \oplus L'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad I_2 \oplus C_2 \oplus I_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, to represent nonisomorphic disconnected posets by M_n, we find that matrix M_n must satisfy the block of 0s property of some nondecreasing inter-distant lengths such that every direct term of M_n represent nonisomorphic connected posets only. In the following, we establish this result in general.

Nonisomorphic direct sum criterion:

Theorem 3.6. *For $n \geq 2$, let the matrices M_n and M'_n satisfy the block of 0s property of different nondecreasing inter-distant lengths such that every direct term of M_n and M'_n represents only connected posets that are nonisomorphic. Then every pair of posets, where one is represented by M_n and another is represented by M'_n, are nonisomorphic.*

Proof. For $n \geq 2$ and $1 \leq m, m' \leq n-1$, let M_n and M'_n satisfy the block of 0s property of the nondecreasing inter-distant lengths $L = \{n_1, n_2, \ldots, n_m\}$ and $L' = \{n'_1, n'_2, \ldots, n'_{m'}\}$, respectively, such that $L \neq L'$. Then we have two different cases as follows:

(i) $m \neq m'$.
In this case, the posets represented by M_n and M'_n have different numbers of direct terms. Then, every pair of posets, where one is represented by M_n and the other is represented by M'_n, are nonisomorphic.

(ii) $m = m'$.
For all $0 \leq i \leq m$, say $r_i = n_{i+1} - n_i$ and $r'_i = n'_{i+1} - n'_i$, where we assign $n_0 = n'_0 = 0$ and

$n_{m+1} = n'_{m+1} = n$. In this case, since both L and L' contain nondecreasing inter-distant lengths, there exist $0 \le s, t \le m$, such that $r_i \ne r'_i$ for all $s \le i \le t$, and $r_i = r'_i$ otherwise (in the simplest case). Then, clearly, $M_{r_i} \ne M_{r'_i}$ for all $s \le i \le t$. Also, $r_i < r_s$ and $r'_i < r'_s$ for all $0 \le i \le s-1$ (when $s > 0$); and $r_i > r_t$ and $r'_i > r'_t$ for all $t+1 \le i \le m$. These show that every pair of posets, where one is represented by M_n and the other is represented by M'_n, has some direct terms of unequal orders. Therefore, these posets are nonisomorphic.

Therefore, in either case, we have every pair of posets, where one is represented by M_n and the other represented by M'_n, are nonisomorphic. □

Enumeration formula in the case of SIID lengths:

Theorem 3.7. *For $n \ge 2$ and $1 \le t \le n-1$, let the matrix M_n satisfy the block of 0s property of the strictly increasing inter-distant lengths $n_1, n_2, \ldots, n_t$ such that for every $1 \le i \le t+1$, a direct term M_{r_i} of M_n, where $r_i = n_i - n_{i-1}$ (with $n_0 = 0$ and $n_{t+1} = n$), can represent $Q(r_i)$ nonisomorphic connected posets. Then $\tilde{R}(n)$, the number of nonisomorphic disconnected posets can be represented by M_n, is given as follows:*

$$\tilde{R}(n) = \prod_{i=1}^{t+1} Q(r_i), \qquad n \ge 2. \tag{3.2}$$

Proof. Since M_n satisfies the block of 0s property of the strictly increasing inter-distant lengths $n_1, n_2, \ldots, n_t$, by Theorem 2.3, M_n represents disconnected posets only and by Theorem 2.2, $M_n = M_{n_1} \oplus M_{n_2-n_1} \oplus \cdots \oplus M_{n-n_t}$ for some $M_{r_i} = M_{n_i - n_{i-1}}$, $1 \le i \le t+1$ (here, $n_0 = 0$ and $n_{t+1} = n$) as the direct terms of M_n. Since $n_1 < n_2 - n_1 < \cdots < n - n_t$, for every $1 \le i \le t+1$, the direct term M_{r_i} represents nonisomorphic connected posets of distinct cardinalities. This shows that M_n represents the nonisomorphic posets having direct terms as a subcollection of $t+1$ posets each of which is chosen from one of the $t+1$ collections of $Q(r_i)$ nonisomorphic posets. Therefore, $\tilde{R}(n)$ equals the number of the combinations of $t+1$ items each of which is chosen from one of the $t+1$ disjoint sets of $Q(r_i)$ distinct items. Then, we have $\tilde{R}(n)$ as follows:

$$\tilde{R}(n) = Q(r_1) \times Q(r_2) \times \cdots \times Q(r_{t+1}) = \prod_{i=1}^{t+1} Q(r_i), \qquad n \ge 2.$$

□

Enumeration formula in the case of EQID lengths:

Theorem 3.8. *For $n \ge 2$ and $1 \le t \le n-1$, let the matrix M_n satisfy the block of 0s property of equally inter-distant lengths $n_1, n_2, \ldots, n_t$ such that for every $1 \le i \le t+1$, the direct term M_r of M_n, where $r = n_i - n_{i-1}$ (with $n_0 = 0$ and $n_{m+1} = n$), can represent $Q(r)$ nonisomorphic connected posets. Then $\bar{R}(n)$, the number of nonisomorphic disconnected posets can be represented by M_n, is given as follows:*

$$\bar{R}(n) = \binom{Q(r)+t}{1+t}, \qquad n \ge 2. \tag{3.3}$$

Proof. Since M_n satisfies the block of 0s property of equally inter-distant lengths $n_1, n_2, \ldots, n_t$, by Theorem 2.3, M_n represents disconnected posets and by Theorem 2.2, $M_n = M_{n_1} \oplus M_{n_2-n_1} \oplus \cdots \oplus M_{n-n_t}$ for some $M_{r_i} = M_{n_i-n_{i-1}}$, $1 \le i \le t+1$ (here, $n_0 = 0$ and $n_{t+1} = n$) as the direct terms of M_n. Since for all $1 \le i \le t+1$, we have $n_i - n_{i-1} = r_i = r$ (say), all $t+1$ direct terms M_r represent nonisomorphic connected posets of the same cardinality. This shows that M_n represents the nonisomorphic posets having direct terms as a subcollection of $t+1$ posets each of which is chosen from one of the same $t+1$ collections of $Q(r)$ nonisomorphic posets. Therefore, $\bar{R}(n)$ equals the number of the combinations of $t+1$ items chosen from $Q(r) + (t+1) - 1$ distinct items. This gives $\bar{R}(n)$ as follows:

$$\bar{R}(n) = \binom{Q(r)+(t+1)-1}{t+1} = \binom{Q(r)+t}{1+t}, \qquad n \ge 2.$$

□

Enumeration formula in the case of NDID lengths:

Theorem 3.9. *For $m \geq 1$ and $n \geq 2$, let the matrix M_n satisfy the block of 0s property of nondecreasing inter-distant lengths $n_1, n_2, \ldots, n_m$, $m \leq n-1$ such that for every $1 \leq i \leq m+1$, the direct term M_{r_i} of M_n, where $r = n_i - n_{i-1}$ (with $n_0 = 0$ and $n_{m+1} = n$), can represent $Q(r_i)$ nonisomorphic connected posets. Then there exist r_k, t_k, $1 \leq k \leq q$, where $q \leq m+1$ such that $R(n)$, the number of nonisomorphic disconnected posets represented by M_n, is given as follows:*

$$R(n) = \prod_{k=1}^{q} \binom{Q(r_k) + t_k}{1 + t_k}, \qquad n \geq 2. \tag{3.4}$$

Proof. Since M_n satisfies the block of 0s property of the nondecreasing inter-distant lengths n_1, $n_2, \ldots, n_m$, we have r_k, t_k, $1 \leq k \leq q$, where $q \leq m+1$ as follows:

$$\begin{aligned}
r_1 &= n_1 - n_0 = n_2 - n_1 = \cdots = n_{t_1+1} - n_{t_1}, \\
r_2 &= n_{t_1+2} - n_{t_1+1} = \cdots = n_{t_1+t_2+2} - n_{t_1+t_2+1}, \\
&\vdots \\
r_q &= n_{t_1+\cdots+t_{q-1}+q} - n_{t_1+\cdots+t_{q-1}+q-1} = \cdots = n - n_m.
\end{aligned}$$

Here, $r_1 < r_2 < \cdots < r_q$ and $m = t_1 + \cdots + t_q + q - 1$. Also, $n_0 = 0$ and $n_{m+1} = n$. Then, we have $n_i - n_{i-1} = r_i = r_k$, where $1 \leq k \leq q$ and $t_1 + \cdots + t_{k-1} + k \leq i \leq t_1 + \cdots + t_k + k$. This shows that, for every $1 \leq k \leq q$, all the $t_k + 1$ consecutive direct terms become equivalent to M_{r_k} that represents $Q(r_k)$ nonisomorphic connected posets of the same order. Then by Theorem 3.8, for every $1 \leq k \leq q$, we have $\bar{R}((t_k+1)r_k)$, the number of nonisomorphic disconnected posets represented by the poset matrix consisting of $t_k + 1$ consecutive direct terms of order r_k, as follows:

$$\bar{R}((t_k+1)r_k) = \binom{Q(r_k) + t_k}{1 + t_k}.$$

Therefore, since $r_1 < r_2 < \cdots < r_q$, by Theorem 3.7, we have $R(n)$ as follows:

$$R(n) = \prod_{k=1}^{q} \bar{R}((t_k+1)r_k) = \prod_{k=1}^{q} \binom{Q(r_k) + t_k}{1 + t_k}, \qquad n \geq 2.$$

□

The following example illustrates the result obtained in Theorem 3.9.

Example 3.10. Consider the matrices M_6 that satisfy the block of 0s property of the nondecreasing inter-distant length 2, length 3, and lengths $1, 2, 4$. We enumerate the nonisomorphic disconnected posets represented by M_6 in each of these cases as follows:

(i) Let M_6 satisfy the block of 0s property of length 2. Then $M_6 = M_2 \oplus M_4$. Since the direct terms M_2 and M_4 represent 1 and 10 connected posets, respectively, in this case, M_6 can represent $\binom{1+0}{1+0} \times \binom{10+0}{1+0} = 1 \times 10 = 10$ disconnected posets all of which are nonisomorphic.

(ii) Let M_6 satisfy the block of 0s property of length 3. Then $M_6 = M_3 \oplus M_3$. Since the direct term M_3 can represent 3 nonisomorphic connected posets, in this case, M_6 can represent $\binom{3+1}{1+1} = 6$ nonisomorphic disconnected posets.

(iii) Let M_6 satisfy the block of 0s property of lengths $1, 2, 4$. Then $M_6 = (M_1 \oplus M_1) \oplus (M_2 \oplus M_2)$. Since both the direct terms M_1 and M_2 represent only 1 connected poset, in this case, M_6 can represent $\binom{1+1}{1+1} \times \binom{1+1}{1+1} = 1$ disconnected poset only.

4 Exact enumeration of unlabeled series-parallel posets

For $n \geq 1$, let $CSP(n)$ be the number of unlabeled connected series-parallel posets and $DSP(n)$ be the number of unlabeled disconnected series-parallel posets represented by M_n, a poset matrix of order n. Since M_1 represents the singleton poset $\mathbf{1}$ only, in particular, we have $CSP(1) = 1$ and $DSP(1) = 0$. For computational purposes, we assume $DSP(1) = 1$. Let $n \geq 2$ be given. Here, we give the enumeration formulas to determine the numbers $CSP(n)$ and $DSP(n)$ provided that for all $1 \leq r \leq n-1$ the numbers $CSP(r)$ and $DSP(r)$ are given. By using Theorem 2.4, we give the enumeration of n-element unlabeled series-parallel posets as follows:

(i) To determine $CSP(n)$, we compute the number of nonequivalent matrices M_n that satisfy the block of 1s property of all possible lengths such that for every length every ordinal term of M_n represents nonisomorphic series-parallel posets that are either the singleton or disconnected.

(ii) To determine $DSP(n)$, we compute the number of nonequivalent matrices M_n that satisfy the block of 0s property of all nondecreasing inter-distant lengths such that for every length every direct term M_n represents nonisomorphic connected series-parallel posets.

4.1 Enumeration of connected series-parallel posets

Theorem 4.1. *For $n \geq 2$, let the matrix M_n satisfy the block of 1s property of all possible lengths $l(m, j)$, $1 \leq m \leq n-1$, $1 \leq j \leq \binom{n-1}{m}$. Also let for every $1 \leq m \leq n-1$, $1 \leq j \leq \binom{n-1}{m}$, $1 \leq i \leq m+1$, the number $DSP(r_{mji})$ (the number of unlabeled disconnected series-parallel posets that can be represented by an ordinal term $M_{r_{mji}} \neq M_1$ of M_n) be given. Then we have $CSP(n)$ as follows:*

$$CSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} \prod_{i=1}^{m+1} DSP(r_{mji}), \qquad n \geq 2. \tag{4.1}$$

Proof. For every $1 \leq m \leq n-1$ and $1 \leq j \leq \binom{n-1}{m}$, let $S(m, j)$ be the number of M_n that satisfies the block of 1s property of lengths $l(m, j)$: $n_{1j}, n_{2j}, \ldots, n_{mj}$ and represents nonisomorphic connected series-parallel posets. Let $r_{mji} = n_{ij} - n_{(i-1)j}$, $1 \leq i \leq m+1$, where we assume $n_{0j} = 0$ and $n_{(m+1)j} = n$. Then the ordinal terms of M_n are the matrices $M_{r_{mji}}$, $1 \leq i \leq m+1$. By hypothesis, for every $1 \leq i \leq m+1$, the matrix $M_{r_{mji}}$ represents $DSP(r_{mji})$ nonisomorphic disconnected series-parallel posets. Then by Theorem 3.1, we have $S(m, j)$ as follows:

$$S(m, j) = \prod_{i=1}^{m+1} DSP(r_{mji}). \tag{4.2}$$

Since the equation (4.2) holds for all lengths $l(m, j)$, $1 \leq m \leq n-1$, $1 \leq j \leq \binom{n-1}{m}$, we have $CSP(n)$ as follows:

$$CSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} S(m, j) = \sum_{m=1}^{n-1} \sum_{j=1}^{\binom{n-1}{m}} \prod_{i=1}^{m+1} DSP(r_{mji}), \qquad n \geq 2.$$

□

The following example illustrates the result established in the above theorem.

Example 4.2. Enumeration of the 5-element unlabeled connected series-parallel posets, that is, determination of $CSP(5)$. We have $DSP(r)$, $1 \leq r \leq 4$, (the number of unlabeled disconnected series-parallel posets up to $r = 4$ elements) as follows:

r	1	2	3	4
$DSP(r)$	1	1	2	6

For all $1 \leq m \leq 4$ and $1 \leq j \leq p_m$, where $p_m = \binom{4}{m}$, we compute $S(m, j)$ considering the lengths $l(m, j)$, as follows:

Number of 5-element unlabeled connected series-parallel posets with 2 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
1	1	1	$1,4$	$1\times 6=6$
1	2	2	$2,3$	$1\times 2=2$
1	3	3	$3,2$	$2\times 1=2$
1	4	4	$4,1$	$6\times 1=6$
				Total: 16

Number of 5-element unlabeled connected series-parallel posets with 3 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
2	1	$1,2$	$1,1,3$	$1\times 1\times 2=2$
2	2	$1,3$	$1,2,2$	$1\times 1\times 1=1$
2	3	$1,4$	$1,3,1$	$1\times 2\times 1=2$
2	4	$2,3$	$2,1,2$	$1\times 1\times 1=1$
2	5	$2,4$	$2,2,1$	$1\times 1\times 1=1$
2	6	$3,4$	$3,1,1$	$2\times 1\times 1=2$
				Total: 9

Number of 5-element unlabeled connected series-parallel posets with 4 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
3	1	$1,2,3$	$1,1,1,2$	1
3	2	$1,2,4$	$1,1,2,1$	1
3	3	$1,3,4$	$1,2,1,1$	1
3	4	$2,3,4$	$2,1,1,1$	1
				Total: 4

Number of 5-element unlabeled connected series-parallel posets with 5 disconnected ordinal terms (possibly, including the singleton poset):

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
4	1	$1,2,3,4$	$1,1,1,1,1$	1
				Total: 1

Thus, $CSP(5)=16+9+4+1=30$.

4.2 Enumeration of disconnected series-parallel posets

Theorem 4.3. *For $n\geq 2$, let the matrix M_n satisfy the block of 0s property of nondecreasing inter-distant lengths $l(m,j)$, $1\leq m\leq n-1$, $1\leq j\leq p_m$ for some $p_m\leq\binom{n-1}{m}$. Also let for every $1\leq m\leq n-1$, $1\leq j\leq p_m$, $1\leq k\leq q_{mj}$ for some $q_{mj}\leq m+1$, the numbers t_{mjk} (the number of the k-th consecutive direct terms $M_{r_{mjk}}$ of M_n) and $CSP(r_{mjk})$ be given. Then we have $DSP(n)$ as follows:*

$$DSP(n)=\sum_{m=1}^{n-1}\sum_{j=1}^{p_m}\prod_{k=1}^{q_{mj}}\binom{CSP(r_{mjk})+t_{mjk}}{1+t_{mjk}},\qquad n\geq 2. \tag{4.3}$$

Proof. For every $1\leq m\leq n-1$ and $1\leq j\leq p_m$ where $p_m\leq\binom{n-1}{m}$, let $S(m,j)$ be the number of nonisomorphic posets represented by M_n that satisfies the block of 0s property of the nondecreasing inter-distant lengths $l(m,j)$: $n_{1j}, n_{2j}, \ldots, n_{mj}$ and represents nonisomorphic disconnected series-parallel posets. Then we have r_{mjk}, t_{mjk}, $1 \le k \le q_{mj}$, where $q_{mj} \le m+1$ as follows:

$$\begin{aligned} r_{mj1} &= n_{ij} - n_{(i-1)j}, 1 \le i \le t_{mj1} + 1, \\ r_{mj2} &= n_{ij} - n_{(i-1)j}, t_{mj1} + 2 \le i \le t_{mj2} + 1, \\ &\vdots \\ r_{mjq_{mj}} &= n_{ij} - n_{(i-1)j}, t_{mj(q_{mj}-1)} + 2 \le i \le t_{mjq_{mj}} + 1. \end{aligned}$$

Here, $r_{mj1} < r_{mj2} < \cdots < r_{mjq_{mj}}$ and we assume $n_{0j} = 0$ and $n_{(m+1)j} = n$. Then the direct terms of M_n are the matrices $M_{r_{mjk}}$, $1 \le i \le t_{mjk} + 1$, $1 \le k \le q_{mj}$. By hypothesis, for every $1 \le i \le t_{mjk} + 1$ and $1 \le k \le q_{mj}$, matrix $M_{r_{mjk}}$ represents $CSP(r_{mjk})$ nonisomorphic connected series-parallel posets. Then by Theorem 3.9, we have $S(m,j)$ as follows:

$$S(m,j) = \prod_{k=1}^{q_{mj}} \binom{CSP(r_{mjk}) + t_{mjk}}{1 + t_{mjk}}. \tag{4.4}$$

Since the equation (4.4) holds for all nondecreasing inter-distant lengths $l(m,j)$, $1 \le m \le n-1$, $1 \le j \le p_m$, we have $DSP(n)$ as follows:

$$DSP(n) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} S(m,j) = \sum_{m=1}^{n-1} \sum_{j=1}^{p_m} \prod_{k=1}^{q_{mj}} \binom{CSP(r_{mjk}) + t_{mjk}}{1 + t_{mjk}}, \qquad n \ge 2.$$

□

The following example illustrates the result established in the above theorem.

Example 4.4. In this example, we enumerate the 6-element unlabeled disconnected series-parallel posets, that is, we determine the number $DSP(6)$. We have $CSP(r)$, $1 \le r \le 5$, (the number of unlabeled connected series-parallel posets up to $r = 5$ elements) as follows:

r	1	2	3	4	5
$CSP(r)$	1	1	3	9	30

We now compute $S(m,j)$, as in Equation 4.4, by using the nondecreasing inter-distant lengths $l(m,j)$, as in Table 1, obtained for all $1 \le m \le 5$ and $1 \le j \le p_m$, where $p_m \le \binom{5}{m}$. Recall that we compute the number of unlabeled disconnected posets according to the number of connected direct terms of the posets. Here $m+1$ equals the number of connected direct terms of a poset.
Number of 6-element unlabeled series-parallel posets with 2 connected direct terms:

m	j	$l(m,j)$	$r_{mj1}, \ldots, r_{mj(m+1)}$	$S(m,j)$
1	1	1	$1, 5$	$\binom{1}{1}\binom{30}{1} = 30$
1	2	2	$2, 4$	$\binom{1}{1}\binom{9}{1} = 9$
1	3	3	$3, 3$	$\binom{3+1}{1+1} = 6$
				Total: 45

Number of 6-element unlabeled series-parallel posets with 3 connected direct terms:

m	j	$l(m,j)$	$r_{mj1}, \ldots, r_{mj(m+1)}$	$S(m,j)$
2	1	$1, 2$	$1, 1, 4$	$\binom{1+1}{1+1}\binom{9}{1} = 9$
2	2	$1, 3$	$1, 2, 3$	$\binom{1}{1}\binom{1}{1}\binom{3}{1} = 3$
2	3	$2, 4$	$2, 2, 2$	$\binom{1+2}{1+2} = 1$
				Total: 13

Number of 6-element unlabeled series-parallel posets with 4 connected direct terms:

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
3	1	$1,2,3$	$1,1,1,3$	$\binom{1+2}{2+1}\binom{3}{1}=3$
3	2	$1,2,4$	$1,1,2,2$	$\binom{1+1}{1+1}\binom{1+1}{1+1}=1$

Total: 4

Number of 6-element unlabeled series-parallel posets with 5 connected direct terms:

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
4	1	$1,2,3,4$	$1,1,1,1,2$	$\binom{1+3}{1+3}\binom{1}{1}=1$

Total: 1

Number of 6-element unlabeled series-parallel posets with 6 connected direct terms:

m	j	$l(m,j)$	$r_{mj1},\dots,r_{mj(m+1)}$	$S(m,j)$
5	1	$1,2,3,4,5$	$1,1,1,1,1,1$	$\binom{1+5}{1+5}=1$

Total: 1

Thus, $DSP(6) = 45 + 13 + 4 + 1 + 1 = 64$.

5 Enumeration algorithms

Recall that we do not specify the values of the parameters p_m, $1 \le m \le n-1$, and q_{mj}, $1 \le m \le n-1$, $1 \le j \le p_m$, as in the equation (4.3), explicitly. Therefore, for given $n \ge 2$, the computation of $DSP(n)$ depends mainly on determining the values of these parameters. By inspection, we have $p_m \le n^2 \le \binom{n-1}{m}$ for all $n \ge 2$ and $1 \le m \le n-1$. We also have $q_{mj} \le m+1$ for all $1 \le j \le p_m$. By using Algorithm 5.3 below, we determine the aforementioned parameters and compute ultimately the numbers $DSP(n)$, $n \ge 2$. Also, we use the equation (4.1) in Algorithm 5.1 to compute the numbers $CSP(n)$, $n \ge 2$. Finally, by using Algorithm 5.5, for $n \ge 2$ we compute $SP(n)$, the number of unlabeled series-parallel posets with n elements.

Algorithm 5.1. To compute $CSP(n)$ for $n \ge 2$.

1. Initialize $CSP(n)$ as $CSP(n) = 0$.
2. Repeat (a) for $m = 1$ to $n-1$.
   a. Repeat (i) to (iv) up to $p = \binom{n-1}{m}$ times for every distinct lengths $l(m,j)$ as is constructed in (i).
      i. Construct j-th lengths $l(m,j)$ consisting of m integers chosen from the integers less than or equal to $n-1$.
      ii. Initialize $S(m,j)$ as $S(m,j) = 1$ (the equation (4.2)).
      iii. Compute t_{mjk} and repeat (α) for every $m+1$ distinct r_{mjk} in the lengths $l(m,j)$ (Theorem 4.1).
         α. Update $S(m,j)$ with $S(m,j) \times DSP(r_{mjk})$.
      iv. Increase $CSP(n)$ by $S(m,j)$.
3. Return $CSP(n)$.

Lemma 5.2. *Algorithm 5.1 runs in time $\mathcal{O}(n^{m+3})$, where $m+1$ equals the number of ordinal terms of the connected series-parallel posets.*

Proof. The constructions of the lengths $l(m,j)$ in the (i)-th step have complexity equal to $m(n-1)$. Since $1 \leq t_{mjk} \leq m+1$, the computations of $S(m,j)$ in the (iii)-th step have complexity equal to $m+1$. Then $m \leq n-1$ implies that the complexity $m(n-1) \approx \mathcal{O}(n(n-1)) \approx \mathcal{O}(n^2)$ and the complexity $m+1 \approx \mathcal{O}(n-1+1) \approx \mathcal{O}(n)$. Since $p = \binom{n-1}{m} \leq (\frac{en}{m})^m$, the repetitions in the step (a) increase the complexities to $(\frac{en}{m})^m\{\mathcal{O}(n^2)+\mathcal{O}(n)\} \approx \mathcal{O}(n^m)\times\mathcal{O}(n^2) \approx \mathcal{O}(n^{m+2})$. Finally, the repetitions in the step (2) increase the complexities to $(n-1)\{\mathcal{O}(n^{m+2})\} \approx \mathcal{O}(n^{m+3})$. □

Algorithm 5.3. To compute $DSP(n)$ for $n \geq 2$.

1. Initialize $DSP(n)$ as $DSP(n) = 0$.
2. Repeat (a) for $m = 1$ to $n-1$.
    a. Repeat (i) to (iv) for every distinct nondecreasing inter-distant lengths $l(m,j)$ as is constructed in (i). (Here, the total number of repetitions equals the parameter p_m in Theorem 4.3).
        i. Construct j-th nondecreasing inter-distant lengths $l(m,j)$ consisting of m integers chosen from the integers less than or equal to $n-1$.
        ii. Initialize $S(m,j)$ as $S(m,j) = 1$ (the equation (4.4)).
        iii. Compute t_{mjk} and repeat (α) for every distinct r_{mjk} in the lengths $l(m,j)$. (Here, the total number of distinct r_{mjk} equals the parameter q_{mj} in Theorem 4.3).
            α. Update $S(m,j)$ with $S(m,j) \times \binom{CSP(r_{mjk})+t_{mjk}}{1+t_{mjk}}$.
        iv. Increase $DSp(n)$ by $S(m,j)$.
3. Return $DSP(n)$.

Lemma 5.4. *Algorithm 5.3 runs in time $\mathcal{O}(n^5)$.*

Proof. The constructions of the lengths $l(m,j)$ in the (i)-th step have complexity equal to $m(n-1)$. Since $1 \leq t_{mjk}, q \leq m+1$ and $t_{mjk} \propto \frac{1}{q}$, the computations of $S(m,j)$ in the step (iii) have complexity equal to $m+1$. Then $m \leq n-1$ implies that the complexity $m(n-1) \approx \mathcal{O}((n-1)(n-1)) \approx \mathcal{O}(n^2)$ and the complexity $m+1 \approx \mathcal{O}(n-1+1) \approx \mathcal{O}(n)$. Since $1 \leq p \leq n^2$, the repetitions in the step (a) increase the complexity to $n^2\{\mathcal{O}(n^2) + \mathcal{O}(n)\} \approx \mathcal{O}(n^4)$. Finally, the repetitions in the step (2) increase the complexity to $(n-1)\{\mathcal{O}(n^4)\} \approx \mathcal{O}(n^5)$. □

Algorithm 5.5. To compute $SP(n)$ for $n \geq 2$.

1. Initialize the arrays $CSP(n)$ and $DSP(n)$ as $CSP(1) = 1$ and $DSP(1) = 1$.
2. Repeat (a) to (c) for $r = 2$ to n.
    a. Compute $CSP(r)$ as in Algorithm 5.1.
    b. Compute $DSP(r)$ as in Algorithm 5.3.
    c. Preserve the numbers $CSP(r)$ and $DSP(r)$.
3. Return the sum of $CSP(n)$ and $DSP(n)$.

Lemma 5.6. *Algorithm 5.5 runs in time $\mathcal{O}(n^{m+4})$, where $m+1$ equals the number of ordinal terms of the connected series-parallel posets.*

Proof. The computations in the steps (a) and (b) have complexities equivalent to $\mathcal{O}(n^{m+3})$ and $\mathcal{O}(n^5)$, respectively (Lemma 5.2 and Lemma 5.4). Then, the repetitions in the step (2) increase the complexity to $n(\mathcal{O}(n^{m+3})+\mathcal{O}(n^5)) \approx \mathcal{O}(n^{m+4})$ for all $m \geq 2$. □

6 Numerical results

We implemented the enumeration algorithms on an Intel CORE-i7 (3.6 GHz) personal computer. To determine $SP(n)$, the machine took about 1 second for $n \leq 15$, 1 minute for $n \leq 20$, 1 hour for $n \leq 26$, and 1 day for $n \leq 33$. The results for $SP(n)$, $n \leq 33$, agree the numerical results obtained by El-Zahar and Khamis [6] and included in the integer sequences A003430, A007453, and A007454 in OEIS [19]. We include the numerical results on $CSP(n)$ for $n \leq 23$ according to the number of ordinal terms of posets and $DSP(n)$ for $n \leq 24$ according to the number of direct terms of posets in the following Table 2, Table 3, Table 4, and Table 5. See also the integer sequences A350772 and A356558 that we contributed to OEIS [19].

Table 2. $CSP(n)$ for $2 \leq n \leq 16$ according to the number of ordinal terms (singleton or disconnected) $t = m + 1$, where $2 \leq t \leq 16$.

$t \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	2	5	16	52	188	690	2638	10272	40782	164114	668544	2750025	11409144	47678552
3		1	3	9	31	108	402	1523	5934	23505	94539	384732	1581819	6559179	27400899
4			1	4	14	52	193	744	2908	11580	46716	190664	785596	3263860	13656666
5				1	5	20	80	315	1261	5085	20730	85260	353525	1476257	6203590
6					1	6	27	116	483	2010	8363	34938	146616	618178	2618268
7						1	7	35	161	707	3059	13132	56253	241003	1033949
8							1	8	44	216	998	4488	19876	87328	382121
9								1	9	54	282	1368	6390	29187	131544
10									1	10	65	360	1830	8872	41780
11										1	11	77	451	2398	12056
12											1	12	90	556	3087
13												1	13	104	676
14													1	14	119
15														1	15
16															1
$CSP(n)$	1	3	9	30	103	375	1400	5380	21073	83950	338878	1383576	5702485	23696081	99163323

Table 3. $CSP(n)$ for $17 \leq n \leq 23$ according to the number of ordinal terms (singleton or disconnected) $t = m + 1$, where $2 \leq t \leq 23$.

$t \diagdown n$	17	18	19	20	21	22	23
2	200523288	848079588	3604696476	15389640287	65966258818	283779863972	1224797039140
3	115204380	487115119	2069995539	8835884304	37868209637	162882764373	702919507509
4	57499516	243423630	1035562696	4424662736	18979499816	81701017310	352832260716
5	26214600	111328615	474906920	2034031171	8743566945	37710179635	163133278430
6	11137278	47563411	203876406	876872208	3783262364	16370149059	71023158084
7	4444629	19149403	82698658	357984578	1553201218	6753847884	29430083844
8	1668912	7284896	31806528	138969836	607801416	2661416220	11668455336
9	588033	2615422	11597202	51332706	227004264	1003509306	4436346375
10	193150	882205	3996880	18008635	80836620	361931465	1617718520
11	58509	277420	1294722	5975926	27365635	124602578	564987973
12	16080	80384	390828	1863699	8764276	40796310	188459664
13	3913	21099	108589	541242	2636907	12634440	59786831
14	812	4893	27286	144501	738138	3673635	17933580
15	135	965	6045	34833	189710	992835	5046810
16	16	152	1136	7388	43952	246040	1318752
17	1	17	170	1326	8942	54876	315571
18		1	18	189	1536	10728	67860
19			1	19	209	1767	12768
20				1	20	230	2020
21					1	21	252
22						1	22
23							1
$CSP(n)$	417553252	1767827220	7520966100	32135955585	137849390424	593407692685	2562695780058

Table 4. $DSP(n)$ for $2 \leq n \leq 17$ according to the number of direct terms (connected) $d = m+1$, where $2 \leq d \leq 17$.

$d \diagdown n$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	1	1	4	12	45	160	613	2354	9297	37118	150369	615092	2540061	10569575	44286415	186666608
3		1	1	4	13	48	175	680	2646	10566	42628	174190	718126	2985818	12499982	52657256
4			1	1	4	13	49	178	695	2713	10873	43987	180270	745092	3105499	13029939
5				1	1	4	13	49	179	698	2728	10940	44294	181650	751307	3133365
6					1	1	4	13	49	179	699	2731	10955	44361	181957	752687
7						1	1	4	13	49	179	699	2732	10958	44376	182024
8							1	1	4	13	49	179	699	2732	10959	44379
9								1	1	4	13	49	179	699	2732	10959
10									1	1	4	13	49	179	699	2732
11										1	1	4	13	49	179	699
12											1	1	4	13	49	179
13												1	1	4	13	49
14													1	1	4	13
15														1	1	4
16															1	1
17																1
$DSP(n)$	1	2	6	18	64	227	856	3280	12885	51342	207544	847886	3497384	14541132	60884173	256480895

Table 5. $DSP(n)$ for $18 \leq n \leq 24$ according to the number of direct terms (connected) $d = m + 1$, where $2 \leq d \leq 24$.

$d \diagdown n$	18	19	20	21	22	23	24
2	790997237	3367700038	14399128769	61801658911	266177276692	1150041480293	4983225458906
3	223021245	949144261	4056825706	17407092435	74953129228	323772409451	1402671363141
4	55003554	233403698	995091525	4260196175	18307596324	78942369641	341454986527
5	13154855	55562641	235904703	1006273218	4310176483	18530969703	79940691015
6	3139608	13182910	55688862	236471653	1008819448	4321606312	18582272604
7	752994	3140988	13189153	55716953	236598126	1009388183	4324163571
8	182039	753061	3141295	13190533	55723196	236626217	1009514701
9	44380	182042	753076	3141362	13190840	55724576	236632460
10	10959	44380	182043	753079	3141377	13190907	55724883
11	2732	10959	44380	182043	753080	3141380	13190922
12	699	2732	10959	44380	182043	753080	3141381
13	179	699	2732	10959	44380	182043	753080
14	49	179	699	2732	10959	44380	182043
15	13	49	179	699	2732	10959	44380
16	4	13	49	179	699	2732	10959
17	1	4	13	49	179	699	2732
18	1	1	4	13	49	179	699
19		1	1	4	13	49	179
20			1	1	4	13	49
21				1	1	4	13
22					1	1	4
23						1	1
24							1
$DSP(n)$	1086310549	4623128656	19759964149	84784735379	365066645854	1576927900803	6831518134251

References

[1] B. I. Bayoumi, M. H. El-Zahar, and S. M. Khamis, *Asymptotic enumeration of N-free partial orders*, Order, **6**, 219–232, (1989).

[2] B. I. Bayoumi, M. H. El-Zahar, and S. M. Khamis, *Algorithmic counting of types of UPO graphs and posets*, Congressus Numerantium, **127**, 117–122, (1997).

[3] G. S. Cheon, B. Curtis, G. Kwon, and A. M. Mwafise, *Riordan posets and associated incidence matrices*, Linear Algebra and its Applications, **632**, 308–331, (2022).

[4] C. Culberson and G. J. E. Rawlins, *New results from an algorithm for counting posets*, Order, **7**, 361–374, (1991).

[5] B. A. Davey and H. A. Priestly, *Introduction to Lattices and Order*, Second Edition, Cambridge University Press, Cambridge, (2002).

[6] M. H. El-Zahar and S. M. Khamis, *Enumeration of series-parallel posets according to heights*, J. Egypt. Math. Soc., **8**, 1–7, (2000).

[7] M. J. Karama and A. J. Qudaimat, *Generating Primitive Triples Using a Matrix Upon Pythagorean Triples*, Palestine Journal of Mathematics, **11(1)**, 57–65, (2022).

[8] S. M. Khamis, *Recognition of prime posets and one of its applications*, J. Egypt. Math. Soc., **14(1)**, 5–13, (2006).

[9] S. U. Mohammad and M. R. Talukder, *Poset matrix and recognition of series-parallel posets*, International Journal of Mathematics and Computer Science, **15(1)**, 107–125, (2020).

[10] S. U. Mohammad and M. R. Talukder, *Interpretations of Kronecker product and ordinal product of poset matrices*, International Journal of Mathematics and Computer Science, **16(4)**, 1665-1681, (2021).

[11] S. U. Mohammad, M. S. Noor, and M. R. Talukder, *An Exact Enumeration of the Unlabeled Disconnected Posets*, Journal of Integer Sequences, **25(5)**, 22.5.4, (2022).

[12] S. U. Mohammad, M. R. Talukder, and S. N. Begum, *Recognition of decomposable posets by using the poset matrix*, Italian Journal of Pure and Applied Mathematics, **50**, 455–466, (2023).

[13] S. U. Mohammad, M. S. Noor, and M. R. Talukder, *Recognition of N-free posets by using the poset matrix*, Asia Pacific Journal of Mathematics, **11**, 72, (2024).

[14] S. U. Mohammad, M. R. Talukder, and S. N. Begum, *Enumeration of unlabeled P-series by using the poset matrix*, Asia Pacific Journal of Mathematics, To appear.

[15] R. H. Möhring, *Computationally Tractable Classes of Ordered Sets*, In I. Rival (ed.), Algorithm and Order, Kluwer Acad. Publ., Dordrecht, 105–194, (1989).

[16] R. Rajendra, K. Bhargava, D. Shubhalakshmi, and P. S. K. Reddy, *Peripheral Harary Index of Graphs*, Palestine Journal of Mathematics, **11(3)**, 323–336, (2022).

[17] H. S. Ramane, G. A. Gudodagi, and K. C. Nandeesh, *On Degree Product Eigenvalues and Degree Product Energy of Graphs*, Palestine Journal of Mathematics, **12(4)**, 314–320, (2023).

[18] M. S. Rhee, *A matrix representation of poset and its applications*, J. Korean Math. Soc., **27(2)**, 223–230, (1990).

[19] N. J. A. Sloane and S. Plouffe, The On-Line Encyclopedia of Integer Sequences, (2022). Available at https://oeis.org.

[20] R. P. Stanley, *Enumeration of posets generated by disjoint unions and ordinal sums*, Proc. Am. Math. Soc., **45**, 295–299, (1974).

Author information

S. U. Mohammad, Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh.
E-mail: salahuddin-mat@sust.edu

M. R. Talukder, Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh.
E-mail: r.talukder-mat@sust.edu

S. N. Begum, Department of Mathematics, Shahjalal University of Science and Technology, Sylhet-3114, Bangladesh.
E-mail: snbegum-mat@sust.edu

Received: 2024-01-17
Accepted: 2024-07-24