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A study of $*$-conformal Einstein solitons in trans-Sasakian 3-manifolds

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Abstract In the present work, we examine trans-Sasakian 3-manifolds admitting $*$-conformal Einstein solitons satisfying certain curvature conditions.

1 Introduction

The study of Ricci soliton (in short, RS) is of great importance due to its wide spread usage in quantum field theory, cosmology, general relativity, string theory, etc. In the beginning of 2016, Catino and Mazzieri [4] proposed a new notion on a Riemannian manifold M called "Einstein-soliton", which generates self-similar solution to Einstein flow $\frac{\partial}{\partial t}g = 2(\frac{r}{2}g - S)$, and is governed through the equation

$$(\pounds_K g)(\zeta_1, \zeta_2) + (2\Lambda - r)g(\zeta_1, \zeta_2) + 2S(\zeta_1, \zeta_2) = 0 \tag{1.1}$$

for any vector fields ζ_1, ζ_2 on M, where $\pounds_K$ denotes the Lie derivative operator in the direction of vector field K, S is the Ricci tensor, r is the scalar curvature of the Riemannian metric g and $\Lambda \in \mathbb{R}$ (the set of real numbers). The Einstein soliton is called shrinking, steady or expanding if $\Lambda < 0, = 0$ or > 0, respectively.

In [19], the authors Kaimakamis and Panagiotidou studied $*$-Ricci soliton in real hypersurfaces of complex space forms and is defined by the equation

$$(\pounds_K g)(\zeta_1, \zeta_2) + 2\Lambda g(\zeta_1, \zeta_2) + 2S^*(\zeta_1, \zeta_2) = 0, \tag{1.2}$$

where

$$S^*(\zeta_1, \zeta_2) = \text{Trace}\,\{\varphi \circ R(\zeta_1, \varphi\zeta_2)\},$$

where S^* is a tensor field of type $(0, 2)$; R represents the curvature tensor and φ is a tensor field of type $(1, 1)$. It is to be noted that the notion of $*$-Ricci tensor was first introduced by Tachibana [25] on almost Hermitian manifolds and further studied by Hamada [13] on real hypersurfaces of non-flat complex space forms.

Recently, the authors Gazala, Ahmad and Jamal [9] proposed the notion of $*$-Einstein soliton (in short, $*$-ES) and is defined by the following equation

$$(\pounds_\xi g)(\zeta_1, \zeta_2) + (2\Lambda - r^*)g(\zeta_1, \zeta_2) + 2S^*(\zeta_1, \zeta_2) = 0, \tag{1.3}$$

where $\pounds_\xi$ denotes the Lie derivative operator in the direction of vector field ξ and r^* is the $*$-scalar curvature of the manifold. Likewise Einstein soliton, the nature of $*$-ES depends on the values of Λ such that if $\Lambda > 0, = 0$ or < 0, then the soliton is said to be expanding, steady or shrinking, respectively. We remark that the notions of $*$-Ricci soliton and $*$-ES are different for the manifolds of non-constant scalar curvature and if the scalar curvature is constant, then these notions coincide.

As a generalization of the classical Ricci flow [14], the concept of conformal Ricci flow was introduced by Fischer [8], which is defined on an n-dimensional Riemannian manifold M by the equations

$$\frac{\partial g}{\partial t} = -2(S + \frac{g}{n}) - pg, \quad r(g) = -1,$$

where p defines a time dependent non-dynamical scalar field (also called the conformal pressure) and g is the Riemannian metric. The term $-pg$ plays a role of constraint force to maintain r in the above equation.

In [2], the authors Basu and Bhattacharyya initiated the study of conformal Ricci soliton on M and is defined by

$$(\pounds_K g)(\zeta_1, \zeta_2) + 2S(\zeta_1, \zeta_2) + (2\Lambda - (p + \frac{2}{n}))g(\zeta_1, \zeta_2) = 0,$$

for any vector fields ζ_1, ζ_2 on M.

Now, we introduce a new notion in an n-dimensional Riemannian manifold called $*$-conformal Einstein soliton (in short, $*$-CES) and is defined through the equation

$$(\pounds_\xi g)(\zeta_1, \zeta_2) + 2S^*(\zeta_1, \zeta_2) + (2\Lambda - r^* + (p + \frac{2}{n}))g(\zeta_1, \zeta_2) = 0, \tag{1.4}$$

where the symbols $\pounds_\xi$ and Λ are defined in (1.3).

In [23], Oubina defined a new class of almost contact manifolds called “trans-Sasakian manifold" with the product manifold $M \times \mathbb{R}$ belonging to the class W_4. The local structures of trans-Sasakian manifolds was carried by Marrero [21]. It is to be noticed that the trans-Sasakian structures of kind $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α-Sasakian [18], β-Kenmotsu [18] and cosymplectic [3], respectively.

The study of Ricci solitons and its generalizations has been carried out by many geometers in several ways to a different extent, for instance, we refer the papers [7, 10, 15, 17, 20, 26] and references therein.

In the present work, we handle the study of trans-Sasakian 3-manifolds admitting a $*$-CES. The article is structured as follows: Preliminaries on trans-Sasakian 3-manifolds are the focus of Section 2. In Section 3, we confer the $*$-CES in trans-Sasakian 3-manifolds. Section 4 deals with the study of $*$-CES in trans-Sasakian 3-manifolds admitting Codazzi type Ricci tensor and cyclic η-recurrent Ricci tensor. In Section 4, we also study pseudo Ricci symmetric trans-Sasakian 3-manifolds admitting $*$-CES. Section 5 is dedicated to the study of φ-Ricci symmetric trans-Sasakian 3-manifolds admitting $*$-CES. In section 6, we have shown that trans-Sasakian 3-manifolds admitting $*$-CES satisfying $R(\xi, \zeta_1) \cdot S^* = 0$ and $R \cdot E^* = 0$ are Ricci flat manifolds. Section 7 is devoted to the study M-projectively flat and φ-M-projectively semisymmetric trans-Sasakian 3-manifolds admitting $*$-CES.

2 Preliminaries

A manifold M^{2n+1} (dimension $M = 2n + 1$) is said to be an almost contact metric manifold if there is a $(1, 1)$ tensor field φ, a vector field ξ, a 1-form η and g is a Riemannian metric (compatible) such that [3]

$$\varphi^2 \zeta_1 = -\zeta_1 + \eta(\zeta_1)\xi, \quad \eta(\xi) - 1 = 0, \quad \varphi\xi = 0, \quad \eta(\varphi\zeta_1) = 0, \tag{2.1}$$

$$g(\varphi\zeta_1, \varphi\zeta_2) = g(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2), \tag{2.2}$$

$$g(\zeta_1, \varphi\zeta_2) = -g(\varphi\zeta_1, \zeta_2), \quad g(\zeta_1, \xi) = \eta(\zeta_1) \tag{2.3}$$

for all $\zeta_1, \zeta_2 \in \chi(M^{2n+1})$; where $\chi(M^{2n+1})$ is the Lie algebra of vector fields on M^{2n+1}. The fundamental 2-form Φ of M^{2n+1} is defined by

$$\Phi(\zeta_1, \zeta_2) = g(\zeta_1, \varphi\zeta_2) \tag{2.4}$$

for any $\zeta_1, \zeta_2 \in \chi(M^{2n+1})$.

A structure (φ, ξ, η, g) on M^{2n+1} is known as a trans-Sasakian structure [23], if $(M^{2n+1} \times \mathbb{R}, J, G)$ belongs to the class W_4[12], where J is the almost complex structure on $M^{2n+1} \times \mathbb{R}$ demarcated by smooth functions f on $M^{2n+1} \times \mathbb{R}$ and $J(\zeta_1, f\frac{d}{dt}) = (\varphi\zeta_1 - f\xi, \eta(\zeta_1)\frac{d}{dt})$ for all ζ_1 on M^{2n+1}. The condition that might be used to express this is as follows:

$$(\nabla_{\zeta_1}\varphi)\zeta_2 = \alpha(g(\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\zeta_1) + \beta(g(\varphi\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\varphi\zeta_1). \tag{2.5}$$

Here, we assume that α and β are the smooth functions on M^{2n+1} with trans-Sasakian structure of type (α, β). From (2.5), it follows that

$$\nabla_{\zeta_1}\xi = -\alpha\varphi\zeta_1 + \beta(\zeta_1 - \eta(\zeta_1)\xi), \tag{2.6}$$

$$(\nabla_{\zeta_1}\eta)\zeta_2 = -\alpha g(\varphi\zeta_1, \zeta_2) + \beta g(\varphi\zeta_1, \varphi\zeta_2), \tag{2.7}$$

where ∇ stands for the Levi-Civita connection of g.

In a trans-Sasakian 3-manifold (say M^3), we have [6]

$$\begin{aligned} R(\zeta_1, \zeta_2)\xi \quad = \quad & (\alpha^2 - \beta^2)(\eta(\zeta_2)\zeta_1 - \eta(\zeta_1)\zeta_2) \\ & +2\alpha\beta((\eta(\zeta_2)\varphi\zeta_1 - \eta(\zeta_1)\varphi\zeta_2) \\ & +(\zeta_2\alpha)\varphi\zeta_1 - (\zeta_1\alpha)\varphi\zeta_2 \\ & +(\zeta_2\beta)\varphi^2\zeta_1 - (\zeta_1\beta)\varphi^2\zeta_2, \end{aligned} \tag{2.8}$$

$$\begin{aligned} R(\xi, \zeta_1)\zeta_2 \quad = \quad & (\alpha^2 - \beta^2)(g(\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\zeta_1) \\ & +2\alpha\beta(g(\varphi\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\varphi\zeta_1) \\ & +(\zeta_2\alpha)\varphi\zeta_1 + g(\varphi\zeta_2, \zeta_1)(grad\ \alpha) \\ & +(\zeta_2\beta)(\zeta_1 - \eta(\zeta_1)\xi) - g(\varphi\zeta_1, \varphi\zeta_2)(grad\ \beta), \end{aligned} \tag{2.9}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{2.10}$$

$$S(\zeta_1, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(\zeta_1) - \zeta_1\beta - (\varphi\zeta_1)\alpha, \tag{2.11}$$

where R and S represent the curvature tensor and the Ricci tensor of M^3, respectively. Moreover, in an M^3 of type (α, β), we have [6]

$$grad\ \beta = \varphi(grad\ \alpha). \tag{2.12}$$

For constants α and β, we obtain from (2.10) and (2.12) that

$$R(\xi, \zeta_1)\zeta_2 \quad = \quad (\alpha^2 - \beta^2)(g(\zeta_1, \zeta_2)\xi - \eta(\zeta_2)\zeta_1), \tag{2.13}$$

$$R(\xi, \zeta_1)\xi \quad = \quad (\alpha^2 - \beta^2)(\eta(\zeta_1)\xi - \zeta_1), \tag{2.14}$$

$$R(\zeta_1, \zeta_2)\xi \quad = \quad (\alpha^2 - \beta^2)(\eta(\zeta_2)\zeta_1 - \eta(\zeta_1)\zeta_2), \tag{2.15}$$

$$\eta(R(\zeta_1, \zeta_2)\zeta_3) = (\alpha^2 - \beta^2)(g(\zeta_2, \zeta_3)\eta(\zeta_1) - g(\zeta_1, \zeta_3)\eta(\zeta_2)), \tag{2.16}$$

$$S(\zeta_1, \xi) = 2(\alpha^2 - \beta^2)\eta(\zeta_1) \iff Q\xi = 2(\alpha^2 - \beta^2)\xi, \tag{2.17}$$

for all $\zeta_1, \zeta_2, \zeta_3 \in \chi(M^3)$. In the paper, throughout we consider $\alpha = \beta =$constant.

Definition 2.1. [27] A trans-Sasakian 3-manifold M^3 is said to be an η-Einstein if its $S(\neq 0)$ is of the form

$$S(\zeta_1,\zeta_2) = \rho_1 g(\zeta_1,\zeta_2) + \rho_2\eta(\zeta_1)\eta(\zeta_2),$$

where ρ_1 and ρ_2 are smooth functions on M^3. Furthermore, the manifold M^3 is said to be an Einstein if $\rho_2 = 0$.

Definition 2.2. [5, 22] The M-projective curvature tensor $\mathcal{H}$ in a trans-Sasakian 3-manifold M^3 is defined by

$$\begin{aligned}\mathcal{H}(\zeta_1,\zeta_2)\zeta_3 &= R(\zeta_1,\zeta_2)\zeta_3 - \frac{1}{4}[S(\zeta_2,\zeta_3)\zeta_1 - S(\zeta_1,\zeta_3)\zeta_2 \\ &\quad + g(\zeta_2,\zeta_3)Q\zeta_1 - g(\zeta_1,\zeta_3)Q\zeta_2],\end{aligned} \tag{2.18}$$

where $\zeta_1,\zeta_2,\zeta_3 \in \chi(M^3)$.

Now, we recall the following result on a trans-Sasakian 3-manifold admitting $*$-Ricci soliton:

Lemma 2.3. *[16] In a trans-Sasakian 3-manifold M^3, the $*$-Ricci tensor S^* is given by*

$$S^*(\zeta_1,\zeta_2) = S(\zeta_1,\zeta_2) - (\alpha^2-\beta^2)g(\zeta_1,\zeta_2) - (\alpha^2-\beta^2)\eta(\zeta_1)\eta(\zeta_2), \tag{2.19}$$

for any $\zeta_1,\zeta_2 \in \chi(M^3)$.

From (2.19), it follows that

$$r^* = r - 4(\alpha^2-\beta^2). \tag{2.20}$$

3 $*$-CES in trans-Sasakian 3-manifolds

Let a trans-Sasakian 3-manifold M^3 admit a $*$-CES. Then (1.4) holds, and thus we have

$$(\pounds_\xi g)(\zeta_1,\zeta_2) + 2S^*(\zeta_1,\zeta_2) + \{2\Lambda - r^* + (p+\frac{2}{3})\}g(\zeta_1,\zeta_2) = 0. \tag{3.1}$$

As we know that

$$(\pounds_\xi g)(\zeta_1,\zeta_2) = g(\nabla_{\zeta_1}\xi,\zeta_2) + g(\zeta_1,\nabla_{\zeta_2}\xi) = 2\beta g(\zeta_1,\zeta_2) - 2\beta\eta(\zeta_1)\eta(\zeta_2).$$

Thus, (3.1) takes the form

$$S^*(\zeta_1,\zeta_2) = A_1 g(\zeta_1,\zeta_2) + A_2\eta(\zeta_1)\eta(\zeta_2), \tag{3.2}$$

where $A_1 = -\{\beta + \Lambda - \frac{r^*}{2} + \frac{1}{2}(p+\frac{2}{3})\}$ and $A_2 = \beta$.
By putting $\zeta_2 = \xi$ in (3.2), then using (2.1) and (2.3), we have

$$S^*(\zeta_1,\xi) = -\{\Lambda - \frac{r^*}{2} + \frac{1}{2}(p+\frac{2}{3})\}\eta(\zeta_1), \tag{3.3}$$

The equation (3.2) yields

$$Q^*\zeta_1 = A_1\zeta_1 + A_2\eta(\zeta_1)\xi. \tag{3.4}$$

From (2.17), (2.19) and (3.3), we get the following relation

$$\Lambda = \frac{r^*}{2} - \frac{1}{2}(p+\frac{2}{3}). \tag{3.5}$$

In view of (3.5), (3.3) reduces to

$$S^*(\zeta_1,\xi) = 0. \tag{3.6}$$

Also, by contracting (3.2) over ζ_1 and ζ_2, we find

$$\frac{r^*}{2} = 2\beta + 3\Lambda + \frac{3}{2}(p+\frac{2}{3}). \tag{3.7}$$

Thus, from (3.5) and (3.7), it follows that

$$\Lambda = -\beta - \frac{1}{2}(p+\frac{2}{3}). \tag{3.8}$$

Now, we have the following result:

Theorem 3.1. *If a 3-dimensional trans-Sasakian manifold M^3 admits a ∗-CES, then M^3 is an η-Einstein manifold of the form (3.2) and the soliton constant is given by $\Lambda = -\beta - \frac{1}{2}(p + \frac{2}{3})$. Moreover, the soliton is expanding, steady or shrinking according to $\beta < -\frac{1}{2}(p+\frac{2}{3})$, $= -\frac{1}{2}(p+\frac{2}{3})$ or $> -\frac{1}{2}(p + \frac{2}{3})$.*

We have the following corollary:

Corollary 3.2. *Let the metric of a 3-dimensional trans-Sasakain manifold M^3 be a ∗-CES. Then we have*

Manifold	*Soliton constant*	*Conditions for the ∗-CES to be expanding, shrinking or steady*
Sasakian and Cosymplectic	$\Lambda = -\frac{1}{2}(p + \frac{2}{3})$	*expanding, steady or shrinking if* $p < -\frac{2}{3}$, $= -\frac{2}{3}$ *or* $> -\frac{2}{3}$.
Kenmotsu	$\Lambda = -\frac{(3p+8)}{6}$	*expanding, steady or shrinking if* $p < -\frac{8}{3}$, $= -\frac{8}{3}$ *or* $> -\frac{8}{3}$.

Now, let a 3-dimensional trans-Sasakian manifold M^3 admit a ∗-CES such that K is pointwise collinear with ξ, i.e., $K = f\xi$, where f is a function. Then (1.4) holds, and thus we have

$$(\pounds_{f\xi}g)(\zeta_1, \zeta_2) + 2S^*(\zeta_1, \zeta_2) + (2\Lambda - r^* + (p + \frac{2}{3}))g(\zeta_1, \zeta_2) = 0. \tag{3.9}$$

Applying the properties of the Lie derivative and the Levi-Civita connection in (3.9), we have

$$fg(\nabla_{\zeta_1}\xi, \zeta_2) + (\zeta_1 f)\eta(\zeta_2) + fg(\zeta_1, \nabla_{\zeta_2}\xi) + (\zeta_2 f)\eta(\zeta_1)$$
$$+2S^*(\zeta_1, \zeta_2) + (2\Lambda - r^* + (p + \frac{2}{3}))g(\zeta_1, \zeta_2) = 0,$$

which by using (2.6) takes the form

$$2f\beta[g(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2)] + (\zeta_1 f)\eta(\zeta_2) + (\zeta_2 f)\eta(\zeta_1) \tag{3.10}$$
$$+2S^*(\zeta_1, \zeta_2) + (2\Lambda - r^* + (p + \frac{2}{3}))g(\zeta_1, \zeta_2) = 0.$$

Now, by replacing $\zeta_2 = \xi$ and using (2.1), (2.3) and (3.3), (3.10) reduces to

$$(\zeta_1 f) + (\xi f)\eta(\zeta_1) = 0. \tag{3.11}$$

Again replacing $\zeta_1 = \xi$ and using (2.1), (3.11) yields

$$(\xi f) = 0. \tag{3.12}$$

By combining (3.11) and (3.12), we lead to $\zeta_1(f) = 0$, that is, f is constant. Thus from (3.10), we obtain

$$S^*(\zeta_1, \zeta_2) \quad = \quad -[\Lambda - \frac{r^*}{2} + f\beta + \frac{1}{2}(p + \frac{2}{3})]g(\zeta_1, \zeta_2) + f\beta\eta(\zeta_1)\eta(\zeta_2). \tag{3.13}$$

Now by virtue of (3.5), (3.13) turns to

$$S^*(\zeta_1, \zeta_2) \quad = \quad -f\beta g(\zeta_1, \zeta_2) + f\beta\eta(\zeta_1)\eta(\zeta_2). \tag{3.14}$$

Therefore, we have the following theorem:

Theorem 3.3. *If a 3-dimensional trans-Sasakian manifold M^3 admits a ∗-CES such that K is pointwise collinear with ξ, then K is a constant multiple of ξ and M^3 is an η-Einstein manifold of the form (3.14).*

By contracting (3.14), we lead to $r^* = -2f\beta$. Using this value of r^* in (3.5), we find

$$\Lambda = -f\beta - \frac{1}{2}(p + \frac{2}{3}). \tag{3.15}$$

Thus, we have the following corollary:

Corollary 3.4. *If a 3-dimensional trans-Sasakian manifold M^3 admits a ∗-CES such that K is pointwise collinear with ξ, then the soilton is expanding, steady or shrinking according as $\beta < -\frac{1}{2f}(p + \frac{2}{3})$, $= -\frac{1}{2f}(p + \frac{2}{3})$ or $> -\frac{1}{2f}(p + \frac{2}{3})$.*

4 $*$-CES in trans-Sasakian 3-manifolds admitting certain types of Ricci tensors

Definition 4.1. A trans-Sasakian 3-manifold M^3 with $*$-CES is said to admit
(i) Codazzi type Ricci tensor if [11]

$$(\nabla_{\zeta_1} S^*)(\zeta_2, \zeta_3) = (\nabla_{\zeta_2} S^*)(\zeta_1, \zeta_3), \tag{4.1}$$

(ii) cyclic η-recurrent Ricci tensor if

$$\begin{aligned}&(\nabla_{\zeta_1} S^*)(\zeta_2, \zeta_3) + (\nabla_{\zeta_2} S^*)(\zeta_1, \zeta_3) + (\nabla_{\zeta_3} S^*)(\zeta_1, \zeta_2)\\&= \eta(\zeta_1)S^*(\zeta_2, \zeta_3) + \eta(\zeta_2)S^*(\zeta_1, \zeta_3) + \eta(\zeta_3)S^*(\zeta_1, \zeta_2),\end{aligned} \tag{4.2}$$

for all $\zeta_1, \zeta_2, \zeta_3 \in \chi(M^3)$.

First, we consider a trans-Sasakian 3-manifold that has Codazzi type Ricci tensor and admits a $*$-CES, thus (4.1) holds. By taking the covariant derivative of (3.2) respecting to ζ_1 we lead to

$$(\nabla_{\zeta_1} S^*)(\zeta_2, \zeta_3) = \beta\{(\nabla_{\zeta_1}\eta)(\zeta_2)\eta(\zeta_3) + \eta(\zeta_2)(\nabla_{\zeta_1}\eta)(\zeta_3)\}. \tag{4.3}$$

By using (2.7) in (4.3), we have

$$\begin{aligned}(\nabla_{\zeta_1} S^*)(\zeta_2, \zeta_3) \;=\;& \beta\{-\alpha g(\varphi\zeta_1, \zeta_2)\eta(\zeta_3) - \alpha g(\varphi\zeta_1, \zeta_3)\eta(\zeta_2)\\&+\beta g(\zeta_1, \zeta_2)\eta(\zeta_3) + \beta g(\zeta_1, \zeta_3)\eta(\zeta_2) - 2\beta\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)\}.\end{aligned} \tag{4.4}$$

By virtue of (4.4), the relation (4.1) takes the form

$$\begin{aligned}&\beta\{2\alpha g(\varphi\zeta_1, \zeta_2)\eta(\zeta_3) + \alpha g(\varphi\zeta_1, \zeta_3)\eta(\zeta_2)\\&-\alpha g(\varphi\zeta_2, \zeta_3)\eta(\zeta_1) - \beta g(\zeta_1, \zeta_3)\eta(\zeta_2) + \beta g(\zeta_2, \zeta_3)\eta(\zeta_1)\} = 0,\end{aligned}$$

which by putting $\zeta_2 = \xi$ and using (2.1), (2.2) and (2.3) reduces to

$$\beta\{\alpha g(\varphi\zeta_1, \zeta_3) - \beta g(\varphi\zeta_1, \varphi\zeta_3)\} = 0.$$

By interchanging ζ_1 and ζ_3 in the foregoing equation, we have

$$\beta\{\alpha g(\varphi\zeta_3, \zeta_1) - \beta g(\varphi\zeta_1, \varphi\zeta_3)\} = 0.$$

On adding the last two equations and using (2.3), we find $\beta = 0$, where $g(\varphi\zeta_1, \varphi\zeta_3) \neq 0$. Thus, we have the following theorem:

Theorem 4.2. *Let a 3-dimensional trans-Sasakian manifold M^3 admit a $*$-CES. If the manifold M^3 has a Codazzi type Ricci tensor, then M^3 reduces to an α-Sasakian manifold.*

In view of (3.2) and (4.4), (4.2) takes the form

$$\begin{aligned}&2\beta A_2[g(\zeta_1, \zeta_2)\eta(\zeta_3) + g(\zeta_1, \zeta_3)\eta(\zeta_2) + g(\zeta_2, \zeta_2)\eta(\zeta_1) - 3\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)]\\=\;&\eta(\zeta_1)[A_1 g(\zeta_2, \zeta_3) + A_2\eta(\zeta_2)\eta(\zeta_3)] + \eta(\zeta_2)[A_1 g(\zeta_1, \zeta_3) + A_2\eta(\zeta_1)\eta(\zeta_3)]\\+\;&\eta(\zeta_3)[A_1 g(\zeta_1, \zeta_2) + A_2\eta(\zeta_1)\eta(\zeta_2)],\end{aligned}$$

which by putting $\zeta_2 = \zeta_3 = \xi$, and using (2.2) and (2.3) it follows that $A_1 + A_2 = 0$. This implies that $\Lambda = \frac{r^*}{2} - \frac{1}{2}(p + \frac{2}{3})$. Consequently, (3.2) turns to

$$S^*(\zeta_1, \zeta_2) = -\beta g(\zeta_1, \zeta_2) + \beta\eta(\zeta_1)\eta(\zeta_2). \tag{4.5}$$

By contracting (4.5), we have $r^* = -2\beta$. By using this value of r^* in (3.5), we find $\Lambda = -\beta - \frac{1}{2}(p + \frac{2}{3})$. Thus, we have the following theorem:

Theorem 4.3. *Let a 3-dimensional trans-Sasakian manifold M^3 admit a $*$-CES. If M^3 has a cyclic η-recurrent Ricci tensor, then M^3 is an η-Einstein manifold of the form (4.5). Moreover, the soliton is expanding, steady or shrinking according to $\beta < -\frac{1}{2}(p + \frac{2}{3})$, $= -\frac{1}{2}(p + \frac{2}{3})$ or $> -\frac{1}{2}(p + \frac{2}{3})$.*

Definition 4.4. A 3-dimensional trans-Sasakian manifold M^3 admitting a $*$-CES is called pseudo Ricci symmetric and is denoted by $(PRS)_3$ if its Ricci tensor $S^*(\neq 0)$ of type $(0,2)$ satisfies the condition [1]

$$(\nabla_{\zeta_1} S^*)(\zeta_2,\zeta_3) = 2A(\zeta_1)S^*(\zeta_2,\zeta_3) + A(\zeta_2)S^*(\zeta_1,\zeta_3) + A(\zeta_3)S^*(\zeta_1,\zeta_2), \tag{4.6}$$

where A is a non-zero 1-form such that $g(\zeta_1,\sigma) = A(\zeta_1)$, for all vector fields $\zeta_1;\sigma$ being the vector field corresponding to the associated 1-form A. In particular, if $A = 0$, then the manifold is called Ricci symmetric.

Let the manifold M^3 admitting a $*$-CES be a pseudo Ricci symmetric. Therefore, (4.6) holds. By using (3.2) and (4.4), (4.6) transforms to

$$\begin{aligned}&A_2\{-\alpha g(\varphi\zeta_1,\zeta_2)\eta(\zeta_3) - \alpha g(\varphi\zeta_1,\zeta_3)\eta(\zeta_2) + \beta g(\zeta_1,\zeta_2)\eta(\zeta_3) + \beta g(\zeta_1,\zeta_3)\eta(\zeta_2)\\ &-2\beta\eta(\zeta_1)\eta(\zeta_2)\eta(\zeta_3)\} = 2A(\zeta_1)[A_1 g(\zeta_2,\zeta_3) + A_2\eta(\zeta_2)\eta(\zeta_3)]\\ &+A(\zeta_2)[A_1 g(\zeta_1,\zeta_3) + A_2\eta(\zeta_1)\eta(\zeta_3)] + A(\zeta_3)[A_1 g(\zeta_1,\zeta_2) + A_2\eta(\zeta_1)\eta(\zeta_2)],\end{aligned}$$

which by putting $\zeta_1 = \zeta_3 = \xi$ and using (2.1) and (2.3) reduces to

$$(A_1 + A_2)(3A(\xi)\eta(\zeta_2) + A(\zeta_2)) = 0. \tag{4.7}$$

Again putting $\zeta_2 = \xi$ in the foregoing equation, we lead to

$$A(\xi)(A_1 + A_2) = 0.$$

Thus, either we have $A(\xi) = 0$, or $(A_1 + A_2) = 0$. By using the first case in (4.7), we find $A(\zeta_2) = 0$, where $A_1 + A_2 \neq 0$ and hence the manifold M^3 reduces to the Ricci symmetric manifold. Now, from the second case it follows that $\Lambda = \frac{r^*}{2} - \frac{1}{2}(p + \frac{2}{3})$, thus (3.2) turns to $S^*(\zeta_1,\zeta_2) = -\beta g(\zeta_1,\zeta_2) + \beta\eta(\zeta_1)\eta(\zeta_2)$. This gives $r^* = -2\beta$. By using this value of r^*, we have $\Lambda = -\beta - \frac{1}{2}(p + \frac{2}{3})$. Now, we state the following theorem:

Theorem 4.5. *A $(PRS)_3$ admitting a $*$-CES is either Ricci symmetric, or is an η-Einstein manifold. Moreover, the soliton is expanding, steady or shrinking according to $\beta < -\frac{1}{2}(p + \frac{2}{3})$, $= -\frac{1}{2}(p + \frac{2}{3})$ or $> -\frac{1}{2}(p + \frac{2}{3})$.*

5 φ-Ricci symmetric trans-Sasakian 3-manifolds admitting $*$-CES

Definition 5.1. A trans-Sasakian 3-manifold M^3 is said to be φ-Ricci symmetric if [24]

$$\varphi^2(\nabla_{\zeta_2} Q^*)\zeta_1 = 0, \tag{5.1}$$

for all ζ_1, ζ_2 on M^3.

Let a trans-Sasakian 3-manifold M^3 be a φ-Ricci symmetric, therefore (5.1) holds. By the covariant differentiation of (3.4) with respect to ζ_2, we have

$$(\nabla_{\zeta_2} Q^*)\zeta_1 + Q^*(\nabla_{\zeta_2}\zeta_1) = A_1(\nabla_{\zeta_2}\zeta_1) + A_2[(\nabla_{\zeta_2}\eta)(\zeta_1)\xi + \eta(\nabla_{\zeta_2}\zeta_1)\xi + \eta(\zeta_1)\nabla_{\zeta_2}\xi],$$

which by using (2.6), (2.7) and (3.4) transforms to

$$(\nabla_{\zeta_2} Q^*)\zeta_1 = A_2[-\alpha g(\varphi\zeta_2,\zeta_1)\xi - \alpha\eta(\zeta_1)\varphi\zeta_2 + \beta g(\zeta_1,\zeta_2)\xi + \beta\eta(\zeta_1)\zeta_2 - 2\beta\eta(\zeta_1)\eta(\zeta_2)\xi].$$

By operating φ^2 on both the sides of the foregoing equation and using (2.1), we have

$$\varphi^2(\nabla_{\zeta_2} Q^*)\zeta_1 = A_2[\alpha\eta(\zeta_1)\varphi\zeta_2 + \beta\eta(\zeta_1)\varphi^2\zeta_2]. \tag{5.2}$$

From (5.1) and (5.2) it follows that

$$A_2\eta(\zeta_1)[\alpha\varphi\zeta_2 + \beta\varphi^2\zeta_2] = 0. \tag{5.3}$$

Thus, we have $A_2 = 0$. By using $A_2 = 0$ and (3.5), (3.2) reduces $S^*(\zeta_1,\zeta_2) = 0$. Thus, we have the following theorem:

Theorem 5.2. *A trans-Sasakian 3-manifold M^3 admitting a $*$-CES is φ-Ricci symmetric if and only if the manifold is $*$-Ricci flat.*

6 Trans-Sasakian 3-manifolds admitting $*$-CES satisfying $R(\xi,\zeta_1)\cdot S^* = 0$ and $R\cdot E^* = 0$

In this section, first we consider a trans-Sasakian 3-manifold admitting a $*$-CES that satisfies the condition $R(\xi,\zeta_1)\cdot S^* = 0$. Therefore, we have

$$S^*(R(\xi,\zeta_1)\zeta_2,\zeta_3) + S^*(\zeta_2, R(\xi,\zeta_1)\zeta_3) = 0, \tag{6.1}$$

for any ζ_1,ζ_2,ζ_3 on M^3.
By using (2.13) in (6.1), we have

$$S^*(\xi,\zeta_3)g(\zeta_1,\zeta_2) - \eta(\zeta_2)S^*(\zeta_1,\zeta_3) + S^*(\zeta_2,\xi)g(\zeta_1,\zeta_3) - \eta(\zeta_3)S^*(\zeta_1,\zeta_2) = 0,$$

where $\alpha^2 - \beta^2 \neq 0$, which in view of (3.6) reduces to

$$\eta(\zeta_2)S^*(\zeta_1,\zeta_3) + \eta(\zeta_3)S^*(\zeta_1,\zeta_2) = 0. \tag{6.2}$$

Now by putting $\zeta_2 = \xi$ in (6.2) then using (2.1) and (3.6) we obtain $S^*(\zeta_1,\zeta_3) = 0$, from which it follows that $r^* = 0$. Thus (3.5) gives

$$\Lambda = -\frac{1}{2}(p + \frac{2}{3}). \tag{6.3}$$

Now, we have the following result:

Theorem 6.1. *If a 3-dimensional trans-Sasakian manifold M^3 admits a $*$-CES and satisfies the condition $R(\xi,\zeta_1)\cdot S^* = 0$, then the manifold is $*$-Ricci flat and the soliton constant is given by $\Lambda = -\frac{1}{2}(p + \frac{2}{3})$. Moreover, the soliton is expanding, steady or shrinking according to $p < -\frac{2}{3}$, $= -\frac{2}{3}$ or $> -\frac{2}{3}$.*

Next, we consider a 3-dimensional trans-Sasakian manifold admitting a $*$-CES that satisfies the condition $R\cdot E^* = 0$, where E^* is the $*$-Einstein tensor given by

$$E^*(\zeta_1,\zeta_2) = S^*(\zeta_1,\zeta_2) - \frac{r^*}{3}g(\zeta_1,\zeta_2). \tag{6.4}$$

Thus, the condition $R\cdot E^* = 0$ is expressed as

$$S^*(R(\zeta_1,\zeta_2)\zeta_3,\zeta_4) + S^*(\zeta_3,R(\zeta_1,\zeta_2)\zeta_4) = \frac{r^*}{3}[g(R(\zeta_1,\zeta_2)\zeta_3,\zeta_4) + g(\zeta_3,R(\zeta_1,\zeta_2)\zeta_4)],$$

which by putting $\zeta_1 = \zeta_3 = \xi$ and using (2.1), (2.3), (2.13) and (2.14) takes the form

$$\eta(\zeta_2)S^*(\xi,\zeta_4) - S^*(\zeta_2,\zeta_4) + g(\zeta_2,\zeta_4)S^*(\xi,\xi) - \eta(\zeta_4)S^*(\xi,\zeta_2) = 0,$$

where $\alpha^2 - \beta^2 \neq 0$.
In view of (3.6), the foregoing equation reduces to $S^*(\zeta_2,\zeta_4) = 0$, from which it follows that $r^* = 0$. Thus (3.5) gives

$$\Lambda = -\frac{1}{2}(p + \frac{2}{3}). \tag{6.5}$$

Thus, we have the following theorem:

Theorem 6.2. *If a 3-dimensional trans-Sasakian manifold M^3 admits a $*$-CES and satisfies the condition $R\cdot E^* = 0$, then the manifold is $*$-Ricci flat and the soliton constant is given by $\Lambda = -\frac{1}{2}(p + \frac{2}{3})$. Moreover, the soliton is expanding, steady or shrinking according to $p < -\frac{2}{3}$, $= -\frac{2}{3}$ or $> -\frac{2}{3}$.*

7 M-projectively flat and φ-M-projectively semisymmetric trans-Sasakian 3-manifolds admitting $*$-CES

In this section, first we consider an M-projectively flat trans-Sasakian 3-manifold admitting $*$-CES, that is, $\mathcal{H}(\zeta_1,\zeta_2)\zeta_3 = 0$. Thus, from (2.18) it follows that

$$R(\zeta_1,\zeta_2)\zeta_3 = \frac{1}{4}[S(\zeta_2,\zeta_3)\zeta_1 - S(\zeta_1,\zeta_3)\zeta_2 + g(\zeta_2,\zeta_3)Q\zeta_1 - g(\zeta_1,\zeta_3)Q\zeta_2]. \tag{7.1}$$

By putting $\zeta_1 = \xi$ in (7.1) and using (2.3), (2.13) and (2.17), we have

$$4(\alpha^2-\beta^2)(g(\zeta_2,\zeta_3)\xi - \eta(\zeta_3)\zeta_2) \;=\; S(\zeta_2,\zeta_3)\xi - S(\xi,\zeta_3)\zeta_2 + g(\zeta_2,\zeta_3)Q\xi - \eta(\zeta_2)Q\zeta_2.$$

Taking the inner product of the foregoing equation with ξ and using (2.1), (2.3) and (2.17), we obtain

$$S(\zeta_2,\zeta_3) = 2(\alpha^2-\beta^2)g(\zeta_2,\zeta_3). \tag{7.2}$$

By contracting (7.2) over ζ_2 and ζ_3, we obtain $r = 6(\alpha^2-\beta^2)$. Thus, from (2.20) and (3.5), we obtain

$$\Lambda = (\alpha^2-\beta^2) - \frac{1}{2}(p+\frac{2}{3}).$$

Thus, we have the following theorem:

Theorem 7.1. *An M-projectively flat trans-Sasakian 3-manifold admitting $*$-CES is an Einstein manifold and the soliton constant is given by $\Lambda = (\alpha^2-\beta^2) - \frac{1}{2}(p+\frac{2}{3})$.*

Next, we consider a φ-M-projectively semisymmetric trans-Sasakian 3-manifold admitting $*$-CES, that is, $\mathcal{H}\cdot\varphi = 0$. Thus, it follows that

$$(\mathcal{H}(\zeta_1,\zeta_2)\cdot\varphi)\zeta_3 = \mathcal{H}(\zeta_1,\zeta_2)\varphi\zeta_3 - \varphi\mathcal{H}(\zeta_1,\zeta_2)\zeta_3 = 0, \tag{7.3}$$

for any ζ_1,ζ_2,ζ_3 on M^3.

From (2.18), we find

$$\begin{aligned}\mathcal{H}(\zeta_1,\zeta_2)\varphi\zeta_3 \;&=\; R(\zeta_1,\zeta_2)\varphi\zeta_3 - \frac{1}{4}[S(\zeta_2,\varphi\zeta_3)\zeta_1 - S(\zeta_1,\varphi\zeta_3)\zeta_2 \\ &\quad + g(\zeta_2,\varphi\zeta_3)Q\zeta_1 - g(\zeta_1,\varphi\zeta_3)Q\zeta_2],\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}\varphi\mathcal{H}(\zeta_1,\zeta_2)\zeta_3 \;&=\; \varphi R(\zeta_1,\zeta_2)\zeta_3 - \frac{1}{4}[S(\zeta_2,\zeta_3)\varphi\zeta_1 - S(\zeta_1,\zeta_3)\varphi\zeta_2 \\ &\quad + g(\zeta_2,\zeta_3)\varphi Q\zeta_1 - g(\zeta_1,\zeta_3)\varphi Q\zeta_2].\end{aligned} \tag{7.5}$$

Thus, from (7.3)-(7.5), we arrive at

$$\begin{aligned}R(\zeta_1,\zeta_2)\varphi\zeta_3 - \varphi R(\zeta_1,\zeta_2)\zeta_3 \;&=\; -\frac{1}{4}[S(\zeta_2,\zeta_3)\varphi\zeta_1 - S(\zeta_1,\zeta_3)\varphi\zeta_2 + g(\zeta_2,\zeta_3)\varphi Q\zeta_1 \\ &\quad - g(\zeta_1,\zeta_3)\varphi Q\zeta_2 - S(\zeta_2,\varphi\zeta_3)\zeta_1 + S(\zeta_1,\varphi\zeta_3)\zeta_2 \\ &\quad - g(\zeta_2,\varphi\zeta_3)Q\zeta_1 + g(\zeta_1,\varphi\zeta_3)Q\zeta_2],\end{aligned}$$

which by putting $\zeta_1 = \xi$ and using (2.1), (2.3), (2.13) and (2.17), we arrive at

$$2(\alpha^2-\beta^2)g(\zeta_2,\varphi\zeta_3)\xi + 2(\alpha^2-\beta^2)\eta(\zeta_3)\varphi\zeta_2 - \eta(\zeta_3)\varphi Q\zeta_2 = S(\zeta_2,\varphi\zeta_3)\xi. \tag{7.6}$$

By taking the inner product of (7.6) with ξ and using (2.1), we find

$$S(\zeta_2,\varphi\zeta_3) = 2(\alpha^2-\beta^2)g(\zeta_2,\varphi\zeta_3). \tag{7.7}$$

By replacing ζ_3 by $\varphi\zeta_3$ in (7.7) then using (2.1) and (2.17), we obtain

$$S(\zeta_2,\zeta_3) = 2(\alpha^2-\beta^2)g(\zeta_2,\zeta_3). \tag{7.8}$$

Contracting (7.8) over ζ_2 and ζ_3 gives $r = 6(\alpha^2-\beta^2)$. Thus, from (2.20) and (3.5), we obtain

$$\Lambda = (\alpha^2-\beta^2) - \frac{1}{2}(p+\frac{2}{3}).$$

Thus, we have the following theorem:

Theorem 7.2. *Let a trans-Sasakian 3-manifold admitting a $*$-CES be a φ-M-projectively semisymmetric, then the manifold is an Einstein manifold and the soliton constant is given by* $\Lambda = (\alpha^2-\beta^2) - \frac{1}{2}(p+\frac{2}{3})$.

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