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Geodesicity and F-geodesicity properties of tangent bundle with Berger-type Cheeger-Gromoll metric

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Abstract *In this paper, we investigate some properties of geodesics and F-geodesics on the tangent bundle TM and on the φ-unit tangent bundle $T_1^{\varphi}M$ with Berger-type Cheeger-Gromoll metric over an anti-paraKähler manifold (M, φ, g).*

1 Introduction

One can define natural Riemannian metrics on the tangent bundle of a Riemannian manifold. Their construction makes use of the Levi-Civita connection. Among them, the so-called Sasaki metric [24] is of particular interest. For this reason, numerous authors have studied it. The rigidity of the Sasaki metric has prompted some researchers to study different deformations of the Sasaki metric. Among them, we mention the Cheeger-Gromoll metric [18] and the Berger-type deformed Sasaki metric [5, 28]. The geometry of tangent bundles remains an affluent area of research in differential geometry

The study of geodesics on tangent bundles is one of the topics many authors have been interested in, especially the study of oblique geodesics, non-vertical geodesics, and their projections onto the base manifold. Sasaki [25] and Sato [26] gave a complete description of the curves and vector fields along them that generated non-vertical geodesics on the tangent bundle and on the unit tangent bundle respectively. They proved that the projected curves have constant geodesic curvatures (Frenet curvatures). Nagy [19] generalized these results to the locally symmetric base manifold case. Yampolsky [28] also did the same studies on the tangent bundle and on the unit the tangent bundle with the Berger-type deformed Sasaki metric over Kählerian manifold, in the cases of the locally symmetric base manifold and of the constant holomorphic curvature base manifold. Also, we refer to [8, 20, 21, 30].

The concept of F-planar curves is a generalization encompassing magnetic curves and, by extension, geodesics, as detailed in references [12]. It is worth noting that the notion of F-geodesics, introduced in [6], presents a variation that slightly differs from that of F-planar curves. We refer to some relevant studies; see [3, 4, 14, 17, 23]. In recent mathematical literature, there has been a series of papers dedicated to the exploration of magnetic curves, F-planar curves, and F-geodesics on tangent bundles and on unit tangent bundles, as evidenced in references [1, 2, 7, 13]. These works have contributed to a deeper understanding of these mathematical concepts and their applications.

This paper aims to study some problems of geodesics and F-geodesics on tangent bundle TM and on φ-unit tangent bundle $T_1^{\varphi}M$ over an anti-paraKähler manifold (M, φ, g). In section 2, we present the preliminary results on the tangent bundle [9, 29] and on the anti-paraKähler manifold [11, 15, 16, 22]. In section 3, we present the Berger-type Cheeger-Gromoll metric *on tangent bundle TM and on φ-unit tangent bundle $T_1^{\varphi}M$ over an anti-paraKähler manifold (M,φ,g) and investigate the Levi-Civita connection of this metric (Theorem 3.3 and Theorem 3.4). In section 4, we are initially interested in studying the necessary and sufficient conditions under which a curve is geodesic on the tangent bundle concerning the Berger-type Cheeger-Gromoll metric (Theorem 4.1, Corollary 4.2 and Corollary 4.3). In the second part of this section, we also establish necessary and sufficient conditions under which a curve on a φ-unit tangent bundle can be a geodesic concerning this metric (Theorem 4.7 and Proposition 4.9). We then study the Frenet curvatures of the projected non-vertical geodesic on $T_1^{\varphi}M$ (Theorem 4.11, Theorem 4.12, and Corollary 4.13). We also study the non-vertical geodesics on $T_1^{\varphi}M$ whose projected curves have vanished the first three Frenet curvatures (Theorem 4.17, Theorem 4.18, Theorem 4.19, Theorem 4.20, and Theorem 4.21). In the last section, we study the F-geodesics and F-planar curves on the φ-unit tangent bundle $T_1^{\varphi}M$ concerning the Levi-Civita connection of the Berger-type Cheeger-Gromoll metric (Theorem 5.1, Theorem 5.3 and Theorem 5.5).*

2 Preliminary Results

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m,g) and let $\pi:TM\to M$ be the natural (bundle) projection. A local chart $(U,x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U),x^i,u^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM=V_{(x,u)}TM\oplus H_{(x,u)}TM$$

of the tangent bundle to TM at any $(x,u)\in TM$ into vertical subspace

$$V_{(x,u)}TM \quad = \quad Ker(d\pi_{(x,u)})=\{a^i\frac{\partial}{\partial u^i}|_{(x,u)},\ a^i\in\mathbb{R}\}$$

and the horizontal subspace

$$H_{(x,u)}TM \quad = \quad \{a^i\frac{\partial}{\partial x^i}|_{(x,u)}-a^iu^j\Gamma_{ij}^k\frac{\partial}{\partial u^k}|_{(x,u)},\ a^i\in\mathbb{R}\}.$$

Let $X=X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$\begin{aligned} {}^VX &= X^i\frac{\partial}{\partial u^i},\\ {}^HX &= X^i(\frac{\partial}{\partial x^i}-u^j\Gamma_{ij}^k\frac{\partial}{\partial u^k}). \end{aligned}$$

We have ${}^H(\frac{\partial}{\partial x^i})=\frac{\partial}{\partial x^i}-u^j\Gamma_{ij}^k\frac{\partial}{\partial u^k}$ and ${}^V(\frac{\partial}{\partial x^i})=\frac{\partial}{\partial u^i}$, then $({}^H(\frac{\partial}{\partial x^i}),{}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

In particular, we have the vertical spray ${}^Vu$ on TM defined by

$${}^Vu=u^i\,{}^V(\frac{\partial}{\partial x^i})=u^i\frac{\partial}{\partial u^i},$$

${}^Vu$ is also called the canonical or Liouville vector field on TM.

The bracket operation of vertical and horizontal vector fields is given by the formulas: [9, 29]

$$\begin{cases} \left[{}^HX,{}^HY\right]={}^H[X,Y]-{}^V(R(X,Y)u)\\ \left[{}^HX,{}^VY\right]={}^V(\nabla_XY)\\ \left[{}^VX,{}^VY\right]=0 \end{cases}\tag{2.1}$$

for all vector fields X and Y on M.

An almost product structure φ on an m-dimensional manifold M is a $(1,1)$-tensor field on M satisfying $\varphi^2=id_M$, where id_M represents the identity tensor field of type $(1,1)$ on M.

Importantly, φ must not be equal to $\pm id_M$. The pair (M, φ) is then referred to as an almost product manifold.

An almost para-complex manifold is essentially an almost product manifold (M, φ) for which the two eigenbundles TM^+ and TM^-, associated with the eigenvalues $+1$ and -1 of φ, respectively, have the same rank. It is essential to note that the dimension of an almost para-complex manifold is always even.

The integrability of an almost para-complex structure φ is determined by the vanishing of the Nijenhuis tensor:

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [X, Y],$$

which must vanish identically on M for all vector fields X and Y on M. Furthermore, an almost para-complex structure φ is integrable if and only if we can introduce a torsion-free linear connection ∇ such that $\nabla\varphi = 0$ [22].

A (pseudo-)Riemannian metric g is considered an anti-paraHermitian metric if it satisfies the condition:

$$g(\varphi X, \varphi Y) = g(X, Y)$$

or equivalently (referred to as the purity condition or B-metric):

$$g(\varphi X, Y) = g(X, \varphi Y),$$

for all vector fields X and Y on M.

If (M, φ) is an almost para-complex manifold with an anti-paraHermitian metric g, we say that (M, φ, g) is an almost anti-paraHermitian manifold or an almost B-manifold, see [11, 15, 16, 22]. If φ is integrable, we say that (M, φ, g) is an anti-paraKähler manifold or B-manifold.

It is well-established that in an anti-paraKähler manifold (M, φ, g), the Riemannian curvature tensor has a specific property, where:

$$\begin{cases} R(\varphi Y, Z) & = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z) \\ R(\varphi Y, \varphi Z) & = R(Y, Z) \end{cases}$$

for all vector fields Y and Z on M [22].

3 The Berger-type Cheeger-Gromoll metric

Definition 3.1. Let (M, φ, g) be an almost anti-paraHermitian manifold and TM its tangent bundle. A Berger-type Cheeger-Gromoll metric on TM is defined as follows: For all vector fields X and Y on M

$$\begin{aligned} \tilde{g}({}^HX, {}^HY) &= g(X, Y), \\ \tilde{g}({}^VX, {}^HY) &= \tilde{g}({}^HX, {}^VY) = 0, \\ \tilde{g}({}^VX, {}^VY) &= \frac{1}{\alpha}(g(X, Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)), \end{aligned}$$

where $\alpha = 1 + \delta^2 g(u, u) = 1 + \delta^2 |u|^2$ and $|.|$ represents the norm with respect to the metric g.

Lemma 3.2. *Let (M, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. Then we have the following formulas:*

$$\begin{aligned} 1)\ {}^HX\big(\tilde{g}({}^HY, {}^HZ)\big) &= X(g(Y, Z)), \\ 2)\ {}^VX\big(\tilde{g}({}^HY, {}^HZ)\big) &= 0, \\ 3)\ {}^HX\big(\tilde{g}({}^VY, {}^VZ)\big) &= \tilde{g}({}^V(\nabla_X Y), {}^VZ) + \tilde{g}({}^VY, {}^V(\nabla_X Z)), \\ 4)\ {}^VX\big(\tilde{g}({}^VY, {}^VZ)\big) &= \frac{\delta^2}{\alpha}\big(g(X, \varphi Y) g(Z, \varphi u) + g(Y, \varphi u) g(X, \varphi Z)\big) \\ &\quad - \frac{2\delta^2}{\alpha} g(X, u) \tilde{g}({}^VY, {}^VZ), \end{aligned}$$

for all vector fields X, Y and Z on M.

The Levi-Civita connection $\widetilde{\nabla}$ for the tangent bundle TM endowed with the Berger-type Cheeger-Gromoll metric $\tilde{g}$ is defined by the Koszul formula, which expresses the metric compatibility of the connection:

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) &= \widetilde{X}\big(\tilde{g}(\widetilde{Y},\widetilde{Z})\big)+\widetilde{Y}\big(\tilde{g}(\widetilde{Z},\widetilde{X})\big)-\widetilde{Z}\big(\tilde{g}(\widetilde{X},\widetilde{Y})\big)+\tilde{g}(\widetilde{Z},[\widetilde{X},\widetilde{Y}]) \\ &\quad +\tilde{g}(\widetilde{Y},[\widetilde{Z},\widetilde{X}])-\tilde{g}(\widetilde{X},[\widetilde{Y},\widetilde{Z}]), \end{aligned} \tag{3.1}$$

for all vector fields $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ on TM. Using (2.1), Koszul formula (3.1), Lemma 3.2 and usual direct calculations, we find the following result:

Theorem 3.3. *Let (M,φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ its tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. Then $\widetilde{\nabla}$ satisfies the following:*

$$\begin{aligned} 1.\ \widetilde{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_XY)-\frac{1}{2}{}^V(R(X,Y)u), \\ 2.\ \widetilde{\nabla}_{{}^HX}{}^VY &= \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big)+{}^V(\nabla_XY), \\ 3.\ \widetilde{\nabla}_{{}^VX}{}^HY &= \frac{1}{2\alpha}{}^H\big(R(u,X)Y\big), \\ 4.\ \widetilde{\nabla}_{{}^VX}{}^VY &= -\frac{\delta^2}{\alpha}(g(X,u){}^VY+g(Y,u){}^VX)+\delta^2\tilde{g}({}^VX,{}^VY){}^Vu \\ &+ \frac{\delta^2}{\alpha}(g(X,\varphi Y)-\delta^2 g(u,\varphi u)\tilde{g}({}^VX,{}^VY)){}^V(\varphi u), \end{aligned}$$

for all vector fields X,Y on M, where ∇ denotes the Levi-Civita connection, R represents its Riemannian curvature tensor of (M,φ,g).

The hypersurface that corresponds to the φ-unit tangent (sphere) bundle over an anti-paraKähler manifold (M,φ,g) can be expressed as follows:

$$T_1^{\varphi}M=\big\{(x,u)\in TM,\ g(u,\varphi u)=1\big\}.$$

The unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N}=\sqrt{\frac{\delta^2}{\alpha-1}}\,{}^V(\varphi u),$$

where $\alpha=1+\delta^2g(u,u)$.

The tangential lift ${}^TX$ with respect to $\tilde{g}$ of a vector $X\in T_xM$ to $(x,u)\in T_1^{\varphi}M$ is the tangential projection of the vertical lift of X to (x,u) with respect to $\mathcal{N}$, that is

$${}^TX={}^VX-\tilde{g}_{(x,u)}({}^VX,\mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)}={}^VX-\frac{\delta^2}{\alpha-1}g_x(X,\varphi u){}^V(\varphi u)_{(x,u)}.$$

The tangent space $T_{(x,u)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at $(x,u)\in T_1^{\varphi}M$ is given by

$$T_{(x,u)}T_1^{\varphi}M=\{{}^HX+{}^TY\ /\ X\in T_xM, Y\in\{\varphi u\}^{\perp}\subset T_xM\},$$

where $\{\varphi u\}^{\perp}=\big\{Y\in T_xM,\ g(Y,\varphi u)=0\big\}$.

Given a vector field X on M, the tangential lift ${}^TX$ of X is given by

$${}^TX_{(x,u)}=\big({}^VX-\tilde{g}({}^VX,\mathcal{N})\mathcal{N}\big)_{(x,u)}={}^VX_{(x,u)}-\frac{\delta^2}{\alpha-1}g_x(X_x,\varphi u){}^V(\varphi u)_{(x,u)}.$$

For the sake of notational clarity, we will use $\bar{X}=X-\dfrac{\delta^2}{\alpha-1}g(X,\varphi u)\varphi u$, then ${}^TX={}^V\bar{X}$.

The Levi-Civita connection $\widehat{\nabla}$ on $T_1^{\varphi}M$ with respect to the Berger-type Cheeger-Gromoll metric is characterized by the Gauss formula:

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y}=\widetilde{\nabla}_{\widehat{X}}\widehat{Y}-\tilde{g}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y},\mathcal{N})\mathcal{N}, \tag{3.2}$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on $T_1^{\varphi}M$. Using Theorem 3.3, Gauss formula (3.2), and usual direct calculations, we find the following result:

Theorem 3.4. *Let (M,φ,g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. Then $\widehat{\nabla}$ satisfies the following:*

$$\begin{aligned}
\widehat{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_XY)-\frac{1}{2}{}^T(R(X,Y)u),\\
\widehat{\nabla}_{{}^HX}{}^TY &= \frac{1}{2\alpha}{}^H\big(R(u,Y)X\big)+{}^T(\nabla_XY),\\
\widehat{\nabla}_{{}^TX}{}^HY &= \frac{1}{2\alpha}{}^H\big(R(u,X)Y\big),\\
\widehat{\nabla}_{{}^TX}{}^TY &= -\frac{\delta^2}{\alpha}\big(g(Y,u)-\frac{\delta^2}{\alpha-1}g(Y,\varphi u)\big){}^TX-\frac{\delta^2}{\alpha}\big(g(X,u)-\frac{\delta^2}{\alpha-1}g(X,\varphi u)\big){}^TY\\
&\quad -\frac{\delta^2}{\alpha-1}g(Y,\varphi u){}^T(\varphi X)+\frac{\delta^2}{\alpha}\big(g(X,Y)+\frac{\delta^2}{(\alpha-1)^2}g(X,\varphi u)g(Y,\varphi u)\big){}^Tu,
\end{aligned}$$

for all vector fields X,Y on M, where ∇ represents the Levi-Civita connection and R denotes its Riemannian curvature tensor of (M,φ,g).

4 Geodesics of the Berger-type Cheeger-Gromoll metric

4.1 Geodesics of the Berger-type Cheeger-Gromoll metric on tangent bundle

Let $\Gamma=(\gamma(t),\xi(t))$ be a naturally parameterized curve on the tangent bundle TM (i.e., t is an arc length parameter on Γ), where γ is a curve on M and ξ is a vector field along this curve. Denote $\gamma'=\frac{d\gamma}{dt}$, $\gamma''=\nabla_{\gamma'}\gamma'$, $\xi'=\nabla_{\gamma'}\xi$, $\xi''=\nabla_{\gamma'}\xi'$ and $\Gamma'=\frac{d\Gamma}{dt}$. Then

$$\Gamma'={}^H\gamma'+{}^V\xi'. \tag{4.1}$$

Theorem 4.1. *Let (M,φ,g) be an anti-paraKähler manifold, $(TM,\tilde{g})$ its tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma=(\gamma(t),\xi(t))$ be a curve on TM, then Γ is a geodesic if and only if*

$$\left\{\begin{aligned}
\gamma'' &= \frac{1}{\alpha}R(\xi',\xi)\gamma'\\
\xi'' &= \frac{2\delta^2}{\alpha}g(\xi',\xi)\xi'-\frac{\delta^2}{\alpha}(|\xi'|^2+\delta^2g(\xi',\varphi\xi)^2)\xi\\
&\quad -(\frac{\delta^2}{\alpha}g(\xi',\varphi\xi')+\frac{\delta^4}{\alpha^2}g(\xi,\varphi\xi)|\xi'|^2+\frac{\delta^6}{\alpha^2}g(\xi,\varphi\xi)g(\xi',\varphi\xi)^2)\varphi\xi
\end{aligned}\right. \tag{4.2}$$

Proof. From (4.1) and Theorem 3.3, we obtain

$$\begin{aligned}
\widetilde{\nabla}_{\Gamma'}\Gamma' &= \widetilde{\nabla}_{({}^H\gamma'+{}^V\xi')}({}^H\gamma'+{}^V\xi')\\
&= \widetilde{\nabla}_{{}^H\gamma'}{}^H\gamma'+\widetilde{\nabla}_{{}^H\gamma'}{}^V\xi'+\widetilde{\nabla}_{{}^V\xi'}{}^H\gamma'+\widetilde{\nabla}_{{}^V\xi'}{}^V\xi'\\
&= {}^H\gamma''+\frac{1}{\alpha}{}^H(R(\xi,\xi')\gamma')+{}^V\xi''-\frac{2\delta^2}{\alpha}g(\xi',\xi){}^V\xi'+\delta^2\tilde{g}({}^V\xi',{}^V\xi'){}^V\xi\\
&+ \frac{\delta^2}{\alpha}(g(\xi',\varphi\xi')-\delta^2g(\xi,\varphi\xi)\tilde{g}({}^V\xi',{}^V\xi'){}^V(\varphi\xi)\\
&= {}^H\big(\gamma''+\frac{1}{\alpha}R(\xi,\xi')\gamma'\big)+{}^V\big(\xi''-\frac{2\delta^2}{\alpha}g(\xi',\xi)\xi'+\frac{\delta^2}{\alpha}(|\xi'|^2+\delta^2g(\xi',\varphi\xi)^2)\xi\\
&\quad +(\frac{\delta^2}{\alpha}g(\xi',\varphi\xi')+\frac{\delta^4}{\alpha^2}g(\xi,\varphi\xi)|\xi'|^2+\frac{\delta^6}{\alpha^2}g(\xi,\varphi\xi)g(\xi',\varphi\xi)^2)\varphi\xi\big).
\end{aligned}$$

If we put $\widetilde{\nabla}_{\Gamma'}\Gamma'$ equal to zero, we find (4.2). □

If γ is a curve on M, then the curve $\Gamma=(\gamma(t),\gamma'(t))$ is called a natural lift of the curve γ [29]. Therefore, we have

Corollary 4.2. *Let (M, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. The natural lift $\Gamma = (\gamma(t), \gamma'(t))$ of any geodesic γ is a geodesic on $(TM, \tilde{g})$.*

A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is said to be a horizontal lift of the curve $\gamma(t)$ on M if and only if $\xi' = 0$ [29]. Therefore, we have

Corollary 4.3. *Let (M, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. The horizontal lift $\Gamma = (\gamma(t), \xi(t))$ of any geodesic γ is a geodesic on $(TM, \tilde{g})$.*

Remark 4.4. As a reminder, note that locally we have

$$\gamma'' = \sum_{k=1}^{2m} \left(\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{2m} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k} \tag{4.3}$$

and

$$\xi' = \sum_{k=1}^{2m} \left(\frac{d\xi^k}{dt} + \sum_{i,j=1}^{2m} \frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}. \tag{4.4}$$

Example 4.5. Let $(\mathbb{R}^2, g, \varphi)$ be an anti-paraKähler manifold such that

$$g = e^{2x}dx^2 + e^{2y}dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1.$$

1) Let γ be a curve on $\mathbb{R}^2$, such that $\gamma(t) = (x(t), y(t))$, γ is a geodesic if and only if $\gamma'' = 0$, from (4.3), we have

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{2} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k = 0 \quad \Leftrightarrow \quad \begin{cases} x'' + (x')^2 = 0 \\ y'' + (y')^2 = 0 \end{cases} \quad \Leftrightarrow \begin{cases} x(t) = \ln(c_1 t + c_2) \\ y(t) = \ln(c_3 t + c_4) \end{cases}$$

where c_1, c_2, c_3, c_4 are real constants.
Hence $\gamma(t) = (\ln(c_1 t + c_2), \ln(c_3 t + c_4))$ and $\gamma'(t) = (\frac{c_1}{c_1 t + c_2}, \frac{c_3}{c_3 t + c_4})$.
From Corollary 4.2, the curve $\Gamma_1 = (\gamma(t), \gamma'(t))$ is a geodesic on $T\mathbb{R}^2$.
2) If $\Gamma_2 = (\gamma(t), \xi(t))$ is horizontal lift of γ, such that $\xi(t) = (u(t), v(t))$ if and only if $\xi' = 0$, from (4.4), we have

$$\frac{d\xi^k}{dt} + \sum_{i,j=1}^{2} \frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^k = 0 \quad \Leftrightarrow \quad \begin{cases} u' + x'u = 0 \\ v' + y'v = 0 \end{cases} \quad \Leftrightarrow \begin{cases} u(t) = \dfrac{c_5}{c_1 t + c_2} \\ v(t) = \dfrac{c_6}{c_3 t + c_4} \end{cases}$$

where c_5, c_6 are real constants. Hence $\xi(t) = (\frac{c_5}{c_1 t + c_2}, \frac{c_6}{c_3 t + c_4})$. From Corollary 4.3, the curve $\Gamma_2 = (\gamma(t), \xi(t))$ is a geodesic on $T\mathbb{R}^2$.

4.2 Geodesics of the Berger-type Cheeger-Gromoll metric on φ-unit tangent bundle

Lemma 4.6. *Let (M, φ, g) be an anti-paraKähler manifold, $T_1^\varphi M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^\varphi M$. Then we have*

$$\Gamma' = {}^H\gamma' + {}^T\xi'. \tag{4.5}$$

Proof. Using (4.1), we have

$$\Gamma' = {}^H\gamma' + {}^V\xi' = {}^H\gamma' + {}^T\xi' + \frac{\delta^2}{\alpha-1} g(\xi', \varphi\xi)^V(\varphi\xi).$$

Since $\Gamma = (\gamma(t), \xi(t)) \in T_1^\varphi M$ then $g(\xi, \varphi\xi) = 1$, on the other hand

$$0 = (g(\xi, \varphi\xi))' = 2g(\xi', \varphi\xi),$$

i.e.,

$$g(\xi', \varphi\xi) = 0. \tag{4.6}$$

Hence, the proof of the lemma is completed. □

Subsequently, let t be an arc length parameter on Γ, From (4.5), we have

$$1 = |\gamma'|^2 + \frac{1}{\alpha}|\xi'|^2. \tag{4.7}$$

Theorem 4.7. *Let (M, φ, g) be an anti-paraKähler manifold, $T_1^\varphi M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^\varphi M$. Then Γ is a geodesic on $T_1^\varphi M$ if and only if*

$$\begin{cases} \gamma'' = \frac{1}{\alpha} R(\xi', \xi)\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \delta^2\rho^2\xi \end{cases} \tag{4.8}$$

Moreover,

$$\begin{cases} |\gamma'| = \sqrt{1-\rho^2} \\ \frac{1}{\alpha}|\xi'|^2 = \rho^2 \end{cases} \tag{4.9}$$

where $0 \leq \rho \leq 1$ and $\rho = const$.

Proof. Using formula (4.5) and Theorem 3.4, we compute the derivative $\widehat{\nabla}_{\gamma'}\gamma'$.

$$\begin{aligned} \widehat{\nabla}_{\Gamma'}\Gamma' &= \widehat{\nabla}_{({}^H\gamma' + {}^T\xi')}({}^H\gamma' + {}^T\xi') \\ &= \widehat{\nabla}_{{}^H\gamma'}{}^H\gamma' + \widehat{\nabla}_{{}^H\gamma'}{}^T\xi' + \widehat{\nabla}_{{}^T\xi'}{}^H\gamma' + \widehat{\nabla}_{{}^T\xi'}{}^T\xi' \\ &= {}^H\gamma'' + \frac{1}{\alpha}{}^H(R(\xi,\xi')\gamma') + {}^T\xi'' - \frac{2\delta^2}{\alpha}g(\xi',\xi){}^T\xi' + \frac{\delta^2}{\alpha}g(\xi',\xi')^T\xi \\ &= {}^H\big(\gamma'' - \frac{1}{\alpha}R(\xi',\xi)\gamma'\big) + {}^T\big(\xi'' - \frac{2\delta^2}{\alpha}g(\xi',\xi)\xi' + \frac{\delta^2}{\alpha}|\xi'|^2\xi\big). \end{aligned} \tag{4.10}$$

On the one hand, we have

$$\alpha = 1 + \delta^2 g(\xi,\xi) \Rightarrow \alpha' = 2\delta^2 g(\xi',\xi) \Rightarrow (\ln\alpha)' = \frac{2\delta^2}{\alpha}g(\xi',\xi).$$

Substituting it into (4.10), we find

$$\widehat{\nabla}_{\Gamma'}\Gamma' = {}^H\big(\gamma'' - \frac{1}{\alpha}R(\xi',\xi)\gamma'\big) + {}^T\big(\xi'' - (\ln\alpha)'\xi' + \frac{\delta^2}{\alpha}|\xi'|^2\xi\big). \tag{4.11}$$

If we put $\widehat{\nabla}_{\Gamma'}\Gamma'$ equal to zero, we find

$$\begin{cases} \gamma'' = \frac{1}{\alpha} R(\xi', \xi)\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \frac{\delta^2}{\alpha}|\xi'|^2\xi \end{cases} \tag{4.12}$$

On the other hand, we have

$$(\frac{1}{\alpha}|\xi'|^2)' = (\frac{1}{\alpha})'|\xi'|^2 + \frac{1}{\alpha}(|\xi'|^2)' = -\frac{\alpha'}{\alpha^2}|\xi'|^2 + \frac{2}{\alpha}g(\xi'', \xi'),$$

using the second equation of (4.12), we find

$$\gamma'(\frac{1}{\alpha}|\xi'|^2) = -\frac{1}{\alpha}(\ln\alpha)'|\xi'|^2 + \frac{2}{\alpha}(\ln\alpha)'|\xi'|^2 - \frac{1}{\alpha}(\ln\alpha)'|\xi'|^2 = 0,$$

i.e., $\frac{1}{\alpha}|\xi'|^2 = \rho^2$ and $\rho = const$. Substituting it into (4.12), we find (4.8), from (4.7), we find $|\gamma'| = \sqrt{1-\rho^2}$ and $0 \leq \rho \leq 1$. □

Remark 4.8. According to (4.9), the geodesics $\Gamma = (\gamma(t), \xi(t))$ of $T_1^{\varphi}M$ can be splitted naturally into 3 classes, namely,
(1) horizontal geodesics, if $\rho = 0$, from (4.9), $|\gamma'| = 1$, then from (4.7), we have $\xi' = 0$ i.e., Γ is generated by parallel vector fields ξ along the geodesics γ on the base manifold,
(2) vertical geodesics, if $\rho = 1$, from (4.9), $|\gamma'| = 0$, then $\gamma(t)$ is a constant i.e., Γ is geodesic in Euclidean space, (on a fixed fiber),
(3) umbilical (oblique) geodesics corresponding to $0 < \rho < 1$, In this case, Γ can be regarded as a vector field $\xi \neq 0$ along the curve γ. see [27].

Proposition 4.9. *Let (M, φ, g) be an anti-paraKähler manifold manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. If $\Gamma = (\gamma(t), \xi(t))$ is a curve on $T_1^{\varphi}M$. Then, we have*
(1) $\Upsilon = (\gamma(t), \varphi\xi(t))$ is a curve on $T_1^{\varphi}M$.
(2) Υ is a geodesic on $T_1^{\varphi}M$ if and only if Γ is a geodesic on $T_1^{\varphi}M$.

Proof.
(1) We put $\zeta(t) = \varphi\xi(t)$, since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$, then $g(\xi, \varphi\xi) = 1$.
On the other hand, $g(\zeta, \varphi\zeta) = g(\varphi\xi, \varphi(\varphi\xi)) = g(\varphi\xi, \xi) = 1$ i.e.,

$$\Upsilon = (\gamma(t), \zeta(t)) \in T_1^{\varphi}M.$$

(2) In a similar way proof of (4.10), and using $\zeta' = \varphi\xi'$ and $\zeta'' = \varphi\xi''$, we have

$$\begin{aligned}\widehat{\nabla}_{\Upsilon'}\Upsilon' &= {}^{H}\big(\gamma'' - \frac{1}{\alpha}R(\zeta', \zeta)\gamma'\big) + {}^{T}\big(\zeta'' - \frac{2\delta^2}{\alpha}g(\zeta', \zeta)\zeta' + \frac{\delta^2}{\alpha}|\zeta'|^2\zeta\big) \\ &= {}^{H}\big(\gamma'' - \frac{1}{\alpha}R(\varphi\xi', \varphi\xi)\gamma'\big) + {}^{T}\big(\varphi\xi'' - \frac{2\delta^2}{\alpha}g(\varphi\xi', \varphi\xi)\varphi\xi' + \frac{\delta^2}{\alpha}|\varphi\xi'|^2\varphi\xi\big).\end{aligned}$$

Since the Riemannian curvature tensor is pure, we get

$$\widehat{\nabla}_{\Upsilon'}\Upsilon' = {}^{H}\big(\gamma'' - \frac{1}{\alpha}R(\xi', \xi)\gamma'\big) + {}^{T}\big(\varphi(\xi'' - (\ln\alpha)'\xi' + \frac{\delta^2}{\alpha}|\varphi\xi'|^2\xi)\big),$$

hence,

$$\begin{aligned}\widehat{\nabla}_{\Upsilon'}\Upsilon' = 0 \quad &\Leftrightarrow \quad \begin{cases} \gamma'' - \frac{1}{\alpha}R(\xi', \xi)\gamma' = 0 \\ \varphi(\xi'' - (\ln\alpha)'\xi' + \frac{\delta^2}{\alpha}|\varphi\xi'|^2\xi) = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} \gamma'' = \frac{1}{\alpha}R(\xi', \xi)\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \frac{\delta^2}{\alpha}|\varphi\xi'|^2\xi \end{cases} \\ &\Leftrightarrow \quad \widehat{\nabla}_{\Gamma'}\Gamma' = 0.\end{aligned}$$

□

Lemma 4.10. *Let (M, φ, g) be a locally symmetric anti-paraKähler manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t), \xi(t))$ be a geodesic on $T_1^{\varphi}M$, then we have*

$$\begin{cases} \gamma^{(p+1)} = \dfrac{1}{\alpha} R(\xi', \xi)\gamma^{(p)} \\ |\gamma^{(p)}| = const \end{cases} \qquad p \geq 1 \tag{4.13}$$

Proof. Using the first equation of (4.8), we have $\gamma'' = \dfrac{1}{\alpha} R(\xi', \xi)\gamma'$.
It is easy to see that

$$(g(\gamma', \gamma'))' = 2g(\gamma'', \gamma') = \frac{2}{\alpha} g(R(\xi', \xi)\gamma', \gamma') = 0,$$

hence, $|\gamma'| = const$.
Calculate the third derivative, we get

$$\begin{aligned} \gamma''' &= (\frac{1}{\alpha} R(\xi', \xi)\gamma')' \\ &= (\frac{1}{\alpha})' R(\xi', \xi)\gamma' + \frac{1}{\alpha}(R(\xi', \xi)\gamma')' \\ &= -\frac{1}{\alpha}(\ln \alpha)' R(\xi', \xi)\gamma' + \frac{1}{\alpha} R(\xi'', \xi)\gamma' + \frac{1}{\alpha} R(\xi', \xi)\gamma''). \end{aligned}$$

Using the second equation of (4.8), we find

$$\frac{1}{\alpha} R(\xi'', \xi)\gamma' = \frac{1}{\alpha}(\ln \alpha)' R(\xi', \xi)\gamma',$$

then,

$$\gamma''' = \frac{1}{\alpha} R(\xi', \xi)\gamma'',$$

since

$$(g(\gamma'', \gamma''))' = 2g(\gamma''', \gamma'') = \frac{2}{\alpha} g(R(\xi', \varphi\xi)\gamma'', \gamma'') = 0,$$

hence, $|\gamma''| = const$.
Continuing the process by recurrence, we obtain

$$\gamma^{(p+1)} = \frac{1}{\alpha} R(\xi', \xi)\gamma^{(p)}, \quad p \geq 1$$

and

$$(g(\gamma^{(p)}, \gamma^{(p)}))' = 2g(\gamma^{(p+1)}, \gamma^{(p)}) = \frac{2}{\alpha} g(R(\xi', \xi)\gamma^{(p)}, \gamma^{(p)}) = 0.$$

Thus, we get

$$|\gamma^{(p)}| = const, \quad p \geq 1.$$

□

Let Γ be a curve on $T_1^{\varphi}M$, the cure $\pi \circ \Gamma$ is called the projection (projected curve) of the curve Γ on M, where $\pi : T_1^{\varphi}M \to M$ is a bundle projection.

Theorem 4.11. *Let (M, φ, g) be a locally symmetric anti-paraKähler manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t), \xi(t))$ be a non-vertical geodesic on $T_1^{\varphi}M$, then all Frenet curvatures of the projected curve $\gamma = \pi \circ \Gamma$ are constants.*

Proof. Denote by s an arc length parameter on γ, i.e., $(|\gamma'_s| = 1)$. Then $\dfrac{ds}{dt} = |\gamma'|$, and using (4.9), we get

$$\frac{ds}{dt} = \sqrt{1-\rho^2} = const. \tag{4.14}$$

Denote by $\nu_1 = \gamma'_s, \nu_2, \ldots, \nu_{2m-1}$ the Frenet frame along γ and by $\kappa_1, \ldots, \kappa_{2m-1}$ the Frenet curvatures of γ. Then the Frenet formulas hold

$$\begin{cases} (\nu_1)'_s &= \kappa_1\nu_2 \\ (\nu_i)'_s &= -\kappa_{i-1}\nu_{i-1} + \kappa_i\nu_{i+1} \qquad 2 \le i \le 2m-2 \\ (\nu_{2m-1})'_s &= -\kappa_{2m-2}\nu_{2m-2} \end{cases}$$

From (4.14), we have

$$\gamma' = \gamma'_s \frac{d\,s}{dt} = \sqrt{1-\rho^2}\,\nu_1.$$

From the Frenet formulas, we obtain

$$\begin{aligned} \gamma'' &= \sqrt{1-\rho^2}\,(\nu_1)' = \sqrt{1-\rho^2}\,(\nu_1)'_s \frac{d\,s}{dt} \\ &= (1-\rho^2)\kappa_1\nu_2. \end{aligned} \tag{4.15}$$

Now (4.13) implies $\kappa_1 = const$. Next, in a similar way, we have

$$\begin{aligned} \gamma''' &= (1-\rho^2)\kappa_1(\nu_2)' = (1-\rho^2)\kappa_1(\nu_2)'_s \frac{d\,s}{dt} \\ &= (1-\rho^2)^{\frac{3}{2}}\kappa_1(-\kappa_1\nu_1 + \kappa_2\nu_3), \end{aligned} \tag{4.16}$$

and again (4.13) implies $\kappa_2 = const$. By continuing the process, we finish the proof. □

From Theorem 4.11 and Proposition 4.9, we have the following theorem:

Theorem 4.12. *Let (M, φ, g) be a locally symmetric anti-paraKähler manifold, $T_1^\varphi M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t), \xi(t))$ be a non-vertical geodesic on $T_1^\varphi M$. Then all Frenet curvatures of the projected curve $\gamma = \pi \circ \Upsilon$ are constants, where $\Upsilon = (\gamma(t), \varphi\xi(t))$.*

Now, we study the geodesics on the φ-unit tangent bundle with the Berger-type Cheeger-Gromoll metric over anti-paraKähler manifold of constant sectional curvature. We recall that every manifold of constant sectional curvature is locally symmetric. From Theorem 4.7, we have the following:

Corollary 4.13. *Let (M, φ, g) be an anti-paraKähler manifold of constant sectional curvature $\varepsilon \neq 0$, $T_1^\varphi M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^\varphi M$. Then Γ is a geodesic on $T_1^\varphi M$ if and only if*

$$\begin{cases} \gamma'' &= \varepsilon(g(\xi, \gamma')\xi' - g(\xi', \gamma')\xi) \\ \xi'' &= (\ln\alpha)'\xi' - \delta^2\rho^2\xi \end{cases} \tag{4.17}$$

We recall that the power of the curvature operator $R^p(X, Y)$ is defined by recurrence as follows:

$$R^p(X, Y)Z = R^{p-1}(X, Y)R(X, Y)Z,$$

for any vector fields X and Y on M, where $p \ge 2$.

Lemma 4.14. *[27] Let (M, g) be a Riemannian manifold of constant sectional curvature ε. Then, we have*

$$R^p(X, Y) = \begin{cases} (-b^2\varepsilon^2)^{i-1}R(X, Y), & for \;\; p = 2i-1 \\ (-b^2\varepsilon^2)^{i-1}R^2(X, Y), & for \;\; p = 2i \end{cases} \qquad i \ge 1$$

for any vector fields X and Y on M, where $b^2 = |X|^2|Y|^2 - g(X, Y)^2$.

Lemma 4.15. *Let (M,φ,g) be an anti-paraKähler manifold of constant sectional curvature $\varepsilon \neq 0$, $T_1^\varphi M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t),\xi(t))$ be a non-vertical geodesic on $T_1^\varphi M$. Then, we have*

$$\gamma^{(p+1)} = \begin{cases} (\frac{-b^2\varepsilon^2}{\alpha^2})^{i-1}(1-\rho^2)\kappa_1\nu_2, & \text{for } p = 2i-1 \\ (\frac{-b^2\varepsilon^2}{\alpha^2})^{i-1}(1-\rho^2)^{\frac{3}{2}}\kappa_1(-\kappa_1\nu_1+\kappa_2\nu_3), & \text{for } p = 2i \end{cases} \tag{4.18}$$

where, $b^2 = |\xi'|^2|\xi|^2 - g(\xi',\xi)^2$, and ν_1,ν_2,ν_3 are the three first vectors of the Frenet frame along γ, and κ_1 and κ_2 are the two first Frenet curvatures.

Proof. Using the first equation of (4.8), we have

$$\gamma'' = \frac{1}{\alpha}R(\xi',\xi)\gamma'.$$

From the proof of Lemma 4.10, we have, by recurrence, the following:

$$\gamma''' = \frac{1}{\alpha}R(\xi',\xi)\gamma'' = \frac{1}{\alpha^2}R^2(\xi',\xi)\gamma'.$$

Continuing the process, we find

$$\gamma^{(p+1)} = \frac{1}{\alpha}R(\xi',\xi)\gamma^{(p)} = \frac{1}{\alpha^p}R^p(\xi',\xi)\gamma', \quad p \geq 1 \tag{4.19}$$

From the Lemma 4.14 and (4.19), we find

$$\begin{aligned} \gamma^{(p+1)} &= \begin{cases} \frac{1}{\alpha^p}(-b^2\varepsilon^2)^{i-1}R(\xi',\xi)\gamma', & \text{for } p = 2i-1 \\ \frac{1}{\alpha^p}(-b^2\varepsilon^2)^{i-1}R^2(\xi',\xi)\gamma', & \text{for } p = 2i \end{cases} \\ &= \begin{cases} (\frac{-b^2\varepsilon^2}{\alpha^2})^{i-1}\gamma'', & \text{for } p = 2i-1 \\ (\frac{-b^2\varepsilon^2}{\alpha^2})^{i-1}\gamma''', & \text{for } p = 2i \end{cases} \end{aligned}$$

Using (4.15) and (4.16), we get the result. □

Remark 4.16. Note that, from (4.13) and (4.18), we find $\frac{b^2}{\alpha^2}$ =const.

In what follows, we study the non-vertical geodesics $\Gamma = (\gamma(t),\xi(t))$ on $T_1^\varphi M$ whose projected curves $\gamma = \pi \circ \Gamma$ have vanishing the first three Frenet curvatures κ_1, κ_2, and κ_3.

Case: $\kappa_1 \equiv 0$

Comparing the first equation of (4.8) and (4.15), we see that

$$\gamma'' = \frac{1}{\alpha}R(\xi',\xi)\gamma' = (1-\rho^2)\kappa_1\nu_2.$$

We take $\kappa_1 = 0$. Hence, we have the following result:

Theorem 4.17. *Let (M,φ,g) be an anti-paraKähler manifold, $T_1^\varphi M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and $\Gamma = (\gamma(t),\xi(t))$ be a non-vertical geodesic on $T_1^\varphi M$. Then the following conditions are equivalent:*
(i) $\kappa_1 = 0$,
(ii) M is flat,
(iii) The projected curve $\gamma = \pi \circ \Gamma$ is a geodesic on M.

Theorem 4.18. *Let (M,φ,g) be an anti-paraKähler manifold of constant sectional curvature $\varepsilon\neq 0$, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric, and $\Gamma=(\gamma(t),\xi(t))$ be a non-vertical geodesic on $T_1^{\varphi}M$, such that the projected curve $\gamma=\pi\circ\Gamma$ has a $\kappa_1=0$. If along γ, choose the orthonormal frame $\{u_1=\frac{\gamma'}{|\gamma'|},u_2,\dots,u_{2m-1}\}$, consisting of parallel vector fields along γ. Then the coordinates ξ_i of ξ with respect to this frame, verify the following:*

$$\begin{cases}\xi_1 &= c_1e^{\delta\rho t}+c_2e^{-\delta\rho t}\\ \xi_i &= \lambda_i\xi_1, \qquad\qquad i=2,\dots,2m-1\\ c_1c_2 &= \dfrac{-1}{4\delta^2\lambda^2}\end{cases}$$

where c_1,c_2, λ_i and $\lambda^2=1+\lambda_2^2+\ldots+\lambda_{2m-1}^2$, are real constants.

Proof. Using the first equation of (4.9), we have

$$\gamma'=\sqrt{1-\rho^2}u_1.$$

Then, from the first equation of (4.17), we calculate

$$\begin{aligned} g(\xi,\gamma') &= g(\xi_1u_1+\ldots+\xi_{2m-1}u_{2m-1},\sqrt{1-\rho^2}u_1)=\xi_1\sqrt{1-\rho^2},\\ g(\xi',\gamma') &= g(\xi_1'u_1+\ldots+\xi_{2m-1}'u_{2m-1},\sqrt{1-\rho^2}u_1)=\xi_1'\sqrt{1-\rho^2}.\end{aligned}$$

We also have $\kappa_1=0\Leftrightarrow\gamma''=0$, then

$$\begin{aligned}\varepsilon(g(\xi,\gamma')\xi'-g(\xi',\gamma')\xi)=0 \quad\Leftrightarrow\quad & \varepsilon\xi_1\sqrt{1-\rho^2}(\xi_1'u_1+\ldots+\xi_{2m-1}'u_{2m-1})\\ & -\varepsilon\xi_1'\sqrt{1-\rho^2}(\xi_1u_1+\ldots+\xi_{2m-1}u_{2m-1})=0\\ \Leftrightarrow\quad & \xi_1\xi_i'-\xi_1'\xi_i=0,\quad i=2,\dots,2m-1.\end{aligned}$$

From this, we find

$$\xi_i=\lambda_i\xi_1,\qquad \lambda_i=const,\ i=2,\dots,2m-1. \tag{4.20}$$

From the second equation of (4.17) and (4.20) we calculate

$$\xi'' \;=\; \xi_1''(u_1+\lambda_2u_2+\ldots+\lambda_{2m-1}u_{2m-1}). \tag{4.21}$$

$$\begin{aligned}(\ln\alpha)' &= \frac{2\delta^2}{\alpha}g(\xi',\xi)=\frac{2\delta^2}{\alpha}(\xi_1'\xi_1+\ldots+\xi_{2m-1}'\xi_{2m-1})\\ &= \frac{2\delta^2}{\alpha}\xi_1'\xi_1(1+\lambda_2^2+\ldots+\lambda_{2m-1}^2)=\frac{2\delta^2}{\alpha}\xi_1'\xi_1\lambda^2,\end{aligned}$$

where, we denote $\lambda^2=1+\lambda_2^2+\ldots+\lambda_{2m-1}^2$.

$$\begin{aligned}(\ln\alpha)'\xi'-\delta^2\rho^2\xi &= \frac{2\delta^2}{\alpha}\xi_1'\xi_1\lambda^2(\xi_1'u_1+\ldots+\xi_{2m-1}'u_{2m-1})\\ &\quad -\delta^2\rho^2(\xi_1u_1+\ldots+\xi_{2m-1}u_{2m-1})\\ &= (\frac{2\delta^2}{\alpha}(\xi_1')^2\lambda^2-\delta^2\rho^2)\xi_1(u_1+\lambda_2u_2+\ldots+\lambda_{2m-1}u_{2m-1}).\end{aligned} \tag{4.22}$$

From (4.21) and (4.22), we have

$$\xi_1''=(\frac{2\delta^2}{\alpha}(\xi_1')^2\lambda^2-\delta^2\rho^2)\xi_1. \tag{4.23}$$

Using the second equation of(4.9), we find

$$\begin{aligned}|\xi'|^2=\rho^2\alpha \quad\Leftrightarrow\quad & (\xi_1')^2+\ldots+(\xi_{2m-1}')^2=\rho^2\alpha\\ \Leftrightarrow\quad & (\xi_1')^2\lambda^2=\rho^2\alpha.\end{aligned} \tag{4.24}$$

From (4.23) and (4.24), we have

$$\xi_1'' = \delta^2\rho^2\xi_1.$$

then, we find $\xi_1 = c_1e^{\delta\rho t} + c_2e^{-\delta\rho t}$, where c_1, c_2 are real constants.
On the other hand, we have

$$\alpha = 1 + \delta^2|\xi|^2 = 1 + \delta^2\xi_1^2\lambda^2 = 1 + \delta^2(c_1e^{\delta\rho t} + c_2e^{-\delta\rho t})^2\lambda^2,$$

From (4.24), we find

$$(c_1\delta\rho e^{\delta\rho t} - c_2\delta\rho e^{-\delta\rho t})^2\lambda^2 = \rho^2(1 + \delta^2(c_1e^{\delta\rho t} + c_2e^{-\delta\rho t})^2\lambda^2)$$

After simplifying, we find

$$c_1c_2 = \frac{-1}{4\delta^2\lambda^2}.$$

□

Case: $\kappa_1 \neq 0$ and $\kappa_2 \equiv 0$

Theorem 4.19. *Let (M, φ, g) be an anti-paraKähler manifold of constant sectional curvature $\varepsilon \neq 0$, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. Then the projected curve $\gamma = \pi \circ \Gamma$ of any non-vertical geodesic Γ on $T_1^{\varphi}M$ has $\kappa_2 = 0$ if and only if*

$$\frac{b^2\varepsilon^2}{\alpha^2} = (1 - \rho^2)\kappa_1^2,$$

where $b^2 = |\xi'|^2|\xi|^2 - g(\xi', \xi)^2$.

Proof. By the Frenet formulas, we have

$$\nu_1' = (1 - \rho^2)^{\frac{1}{2}}\kappa_1\nu_2.$$

Using (4.16), we have

$$\gamma^{(4)} = -(1 - \rho^2)^2\kappa_1^3\nu_2,$$

from (4.18), we find

$$\gamma^{(4)} = \frac{-b^2\varepsilon^2}{\alpha^2}(1 - \rho^2)\kappa_1\nu_2,$$

by comparing the last two equations, we obtain

$$\frac{b^2\varepsilon^2}{\alpha^2} = (1 - \rho^2)\kappa_1^2.$$

□

Case: $\kappa_1 \neq 0$, $\kappa_2 \neq 0$ and $\kappa_3 \equiv 0$

Theorem 4.20. *Let (M, φ, g) be an anti-paraKähler manifold of constant sectional curvature $\varepsilon \neq 0$, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. Then the projected curve $\gamma = \pi \circ \Gamma$ of any non-vertical geodesic Γ on $T_1^{\varphi}M$ has $\kappa_3 \equiv 0$ if and only if*

$$\frac{b^2\varepsilon^2}{\alpha^2} = (1 - \rho^2)(\kappa_1^2 + \kappa_2^2).$$

Proof. Using (4.16), we have

$$\gamma^{(4)} = -(1-\rho^2)^2\kappa_1(\kappa_1^2+\kappa_2^2)\nu_2 + (1-\rho^2)^2\kappa_1\kappa_2\kappa_3\nu_4. \tag{4.25}$$

On the other hand, from Lemma 4.14, (4.15) and (4.19), we have

$$\begin{aligned}\gamma^{(4)} &= \frac{1}{\alpha^3}R^3(\xi',\xi)\gamma' = \frac{-b^2\varepsilon^2}{\alpha^3}R(\xi',\xi)\gamma' = \frac{-b^2\varepsilon^2}{\alpha^2}\gamma'' \\ &= \frac{-b^2\varepsilon^2}{\alpha^2}(1-\rho^2)\kappa_1\nu_2, \end{aligned}\tag{4.26}$$

where $b^2 = |\xi'|^2|\xi|^2 - g(\xi',\xi)^2$, from (4.25) and (4.26), we have

$$(1-\rho^2)\kappa_1\big((\frac{b^2\varepsilon^2}{\alpha^2} - (1-\rho^2)(\kappa_1^2+\kappa_2^2))\nu_2 + (1-\rho^2)\kappa_2\kappa_3\nu_4\big) = 0.$$

Since $\kappa_1 \neq 0$, then

$$(\frac{b^2\varepsilon^2}{\alpha^2} - (1-\rho^2)(\kappa_1^2+\kappa_2^2))\nu_2 + (1-\rho^2)k_2\kappa_3\nu_4 = 0.$$

Likewise, since $\kappa_2 \neq 0$, then, we find $\kappa_3 = 0$, and $\dfrac{b^2\varepsilon^2}{\alpha^2} = (1-\rho^2)(\kappa_1^2+\kappa_2^2)$. □

Theorem 4.21. *Let (M,φ,g) be an anti-paraKähler manifold of constant sectional curvature $\varepsilon \neq 0$, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. Then any non-vertical geodesic $\Gamma = (\gamma(t),\xi(t))$ on $T_1^{\varphi}M$ can be expressed as*

$$\begin{cases} \gamma'' &= \varepsilon(A\xi' - B\xi) \\ \xi &= \dfrac{\alpha A}{\sqrt{1-\rho^2}}\nu_1 + \dfrac{A'-B}{\kappa_1(1-\rho^2)}\nu_2 - \dfrac{\alpha\kappa_2 A}{\kappa_1\sqrt{1-\rho^2}}\nu_3 \end{cases}$$

where $A = g(\xi,\gamma')$ and $B = g(\xi',\gamma')$.

Proof. We put $A = g(\xi,\gamma')$, $B = g(\xi',\gamma')$ and $C = g(\xi',\xi)$, using the first equation of (4.17), we have

$$\gamma'' = \varepsilon(A\xi' - B\xi), \tag{4.27}$$

$$A' = g(\xi',\gamma') + g(\xi,\gamma'') = B + \varepsilon g(\xi, A\xi' - B\xi) = \varepsilon AC + (1-\varepsilon|\xi|^2)B,$$

$$\begin{aligned} B' &= g(\xi'',\gamma') + g(\xi',\gamma'') = g(\frac{2\delta^2}{\alpha}C\xi' - \delta^2\rho^2\xi,\gamma') + \varepsilon g(\xi, A\xi' - B\xi) \\ &= (\varepsilon\alpha - \delta^2)\rho^2 A + (\frac{2\delta^2 - \alpha C}{\alpha})BC. \end{aligned}$$

From (4.27), we have

$$\xi' = \frac{1}{\varepsilon A}\gamma'' + \frac{B}{A}\xi, \tag{4.28}$$

using the first equation of (4.13), we have

$$\gamma''' = \frac{1}{\alpha}R(\xi',\xi)\gamma'' = \frac{\varepsilon}{\alpha}(g(\xi,\gamma'')\xi' - g(\xi',\gamma'')\xi).$$

Using (4.28) and direct calculation, we get

$$\begin{aligned}\gamma''' &= \frac{1}{\alpha A}g(\xi,\gamma'')\gamma'' + \frac{\varepsilon B}{\alpha A}g(\xi,\gamma'')\xi - \frac{\varepsilon}{\alpha}g(\xi',\gamma'')\xi, \\ &= \frac{\varepsilon}{\alpha A}(AC - B|\xi|^2)\gamma'' + \frac{\varepsilon^2}{\alpha A}(2ABC - B^2|\xi|^2 - \alpha\rho^2 A^2)\xi \\ &= \frac{A'-B}{\alpha A}\gamma'' - \frac{|\gamma''|^2}{\alpha A}\xi, \end{aligned}$$

then,

$$\xi = \frac{1}{|\gamma''|^2}((A' - B)\gamma'' - \alpha A\gamma''').$$

From (4.15) and (4.16)

$$\begin{aligned}\xi &= \frac{1}{(1-\rho^2)^2\kappa_1^2}((A' - B)(1-\rho^2)\kappa_1\nu_2 - \alpha A(1-\rho^2)^{\frac{3}{2}}\kappa_1(-\kappa_1\nu_1 + \kappa_2\nu_3)) \\ &= \frac{\alpha A}{\sqrt{1-\rho^2}}\nu_1 + \frac{A' - B}{\kappa_1\sqrt{1-\rho^2}}\nu_2 - \frac{\alpha\kappa_2 A}{\kappa_1\sqrt{1-\rho^2}}\nu_3.\end{aligned}$$

□

Remark 4.22. Note that the Theorem 4.21 also holds for non-vertical geodesic $\Gamma = (\gamma(t), \xi(t))$ such that $\gamma = \pi \circ \Gamma$ has $\kappa_1 \neq 0$ and $\kappa_2 = 0$.

5 F-geodesics on φ-unit tangent bundle with the Berger-type Cheeger-Gromoll metric

Let (M^m, g) be a Riemannian manifold and F be a $(1,1)$-tensor field on (M^m, g). A curve γ on M is called F-planar if its speed remains, under parallel translation along the curve γ, in the distribution generated by the vector γ' and $F\gamma'$ along γ. This is equivalent to the fact that the tangent vector γ' satisfies:

$$\gamma'' = \varrho_1(t)\gamma' + \varrho_2 F\gamma',$$

where ϱ_1 and ϱ_2 are some functions of the parameter t, see [17, 12]. The F-planar curves generalize the magnetic curves and therefore, the geodesics.

We say that a curve γ on M is an F-geodesic if γ satisfies:

$$\gamma'' = F\gamma',$$

One can see that an F-geodesic is an F-planar curve, but generally, an F-planar curve is not always an F-geodesic; see [6].

Let $\widehat{\nabla}$ be the Levi-Civita connection of the Berger-type Cheeger-Gromoll metric on φ-unit tangent bundle $T_1^{\varphi}M$, given in the Theorem 3.4.

Theorem 5.1. *Let (M, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^HF$-planar with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma'' = \dfrac{1}{\alpha}R(\xi', \xi)\gamma' + \varrho_1\gamma' + \varrho_2 F\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \dfrac{\delta^2}{\alpha}|\xi'|^2\xi + \varrho_1\xi' + \varrho_2 F\xi' \end{cases} \tag{5.1}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

Proof. Γ be an ${}^HF$-planar with respect to $\widehat{\nabla}$ if and only if the Γ satisfies

$$\widehat{\nabla}_{\Gamma'}\Gamma' = \varrho_1\Gamma' + \varrho_2{}^HF\Gamma',$$

where ϱ_1 and ϱ_2 are some functions of the parameter t. By (4.5), we get

$$\widehat{\nabla}_{\Gamma'}\Gamma' = \varrho_1({}^H\gamma' + {}^T\xi') + \varrho_2{}^HF({}^H\gamma' + {}^T\xi').$$

From (4.6), we have ${}^T\xi' = {}^V\xi'$, then

$$\begin{aligned}\widehat{\nabla}_{\Gamma'}\Gamma' &= \varrho_1{}^H\gamma' + \varrho_2{}^HF{}^H\gamma' + \varrho_1{}^V\xi' + \varrho_2{}^HF{}^V\xi' \\ &= {}^H(\varrho_1\gamma' + \varrho_2F\gamma') + {}^V(\varrho_1\xi' + \varrho_2F\xi') \\ &= {}^H(\varrho_1\gamma' + \varrho_2F\gamma') + {}^T(\varrho_1\xi' + \varrho_2F\xi').\end{aligned} \tag{5.2}$$

By comparing (4.11) and (5.2), the equations (5.1) are immediately obtained. □

Corollary 5.2. *Let (M, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-planar with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma'' = \dfrac{1}{\alpha} R(\xi', \xi)\gamma' + \varrho_1\gamma' + \varrho_2\varphi\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \dfrac{\delta^2}{\alpha}|\xi'|^2\xi + \varrho_1\xi' + \varrho_2\varphi\xi' \end{cases}$$

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 5.1, we obtain the following result:

Theorem 5.3. *Let (M, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}F$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma'' = \dfrac{1}{\alpha} R(\xi', \xi)\gamma' + F\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \dfrac{\delta^2}{\alpha}|\xi'|^2\xi + F\xi' \end{cases} \tag{5.3}$$

Corollary 5.4. *Let (M, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma'' = \dfrac{1}{\alpha} R(\xi', \xi)\gamma' + \varphi\gamma' \\ \xi'' = (\ln\alpha)'\xi' - \dfrac{\delta^2}{\alpha}|\xi'|^2\xi + \varphi\xi' \end{cases}$$

Theorem 5.5. *Let (M, φ, g) be an anti-paraKähler manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the Berger-type Cheeger-Gromoll metric and F be a $(1,1)$-tensor field on M. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of γ and $\Gamma \in T_1^{\varphi}M$, then Γ is an ${}^{H}F$-planar curve (resp., ${}^{H}F$-geodesic) if and only if γ is an F-planar curve (resp., F-geodesic).*

Proof. Since $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of a curve γ, then $\xi' = 0$. from the Theorem 5.3, we find

$$(5.1) \Leftrightarrow \gamma'' = \varrho_1\gamma' + \varrho_2 F\gamma'.$$

Hence, $\Gamma = (\gamma(t), \xi(t))$ is an ${}^{H}F$-planar if and only if γ is an F-planar curve.
In the case of $\varrho_1 = 0$ and $\varrho_2 = 1$, we get that Γ is an ${}^{H}F$-geodesic if and only γ is an F-geodesic. □

Example 5.6. Let $(\mathbb{R}^2, \varphi, g)$ be an anti-paraKähler manifold such that

$$g = e^{2x}dx^2 + e^{2y}dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \ a, b \in \mathbb{R}^*.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1.$$

Let $\Gamma = (\gamma(t), \xi(t))$ be a horizontal lift of a curve γ, such that $\gamma(t) = (x(t), y(t))$ and $\xi(t) = (u(t), v(t))$ then $\xi' = 0$, from (4.4), we have

$$\frac{d\xi^h}{dt} + \sum_{i,j=1}^{2} \frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^h = 0 \Leftrightarrow \begin{cases} u' + x'u = 0 \\ v' + y'v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = k_1 e^{-x(t)} \\ v(t) = k_2 e^{-y(t)} \end{cases}$$

where k_1, k_2 are real constants.
(i) γ is an F-geodesic if and only if $\gamma'' = F\gamma'$, from (4.3), we have

$$\begin{cases} x'' + (x')^2 = ax' \\ y'' + (y')^2 = by' \end{cases} \Leftrightarrow \begin{cases} x(t) = \ln(\frac{c_1}{a}e^{at} + c_2) \\ y(t) = \ln(\frac{c_3}{b}e^{bt} + c_4) \end{cases}$$

and

$$\begin{cases} u(t) = \dfrac{k_1}{\frac{c_1}{a}e^{at} + c_2} \\ v(t) = \dfrac{k_2}{\frac{c_3}{b}e^{bt} + c_4} \end{cases}$$

where c_i are real constants, hence

$$\gamma(t) = (\ln(\frac{c_1}{a}e^{at} + c_2), \ln(\frac{c_3}{b}e^{bt} + c_4))$$

and

$$\xi(t) = (\frac{k_1}{\frac{c_1}{a}e^{at} + c_2}, \frac{k_2}{\frac{c_3}{b}e^{bt} + c_4}).$$

But when

$$g(\xi, \varphi\xi) = 1 \Leftrightarrow k_1 = \pm\sqrt{1 + k_2^2},$$

become $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}\mathbb{R}^2$. Then, from Theorem 5.5, the horizontal lift $\Gamma = (\gamma(t), \xi(t))$ is an ${}^HF$-geodesic on $T\mathbb{R}^2$.

(ii) γ is an F-planar if and only if $\gamma'' = \varrho_1\gamma' + \varrho_2 F\gamma'$, where ϱ_1 and ϱ_2 are some functions of the parameter t, hence, we have

$$\begin{cases} x'' + (x')^2 = (\varrho_1 + a\varrho_2)x' \\ y'' + (y')^2 = (\varrho_1 + b\varrho_2)y' \end{cases} \Leftrightarrow \begin{cases} x(t) = \ln(\int(e^{\int(\varrho_1 + a\varrho_2)dt})dt) \\ y(t) = \ln(\int(e^{\int(\varrho_1 + b\varrho_2)dt})dt) \end{cases}$$

and

$$\begin{cases} u(t) = \dfrac{k_1}{\int(e^{\int(\varrho_1 + a\varrho_2)dt})dt} \\ v(t) = \dfrac{k_2}{\int(e^{\int(\varrho_1 + b\varrho_2)dt}} \end{cases}$$

For example: If $\varrho_1(t) = \dfrac{1}{t+1}$ and $\varrho_2(t) = \dfrac{1}{t}$, we find

$$\begin{cases} x(t) = \ln(\frac{c_1}{a+2}t^{a+2} + \frac{c_1}{a+1}t^{a+1} + c_2) \\ y(t) = \ln(\frac{c_3}{b+2}t^{b+2} + \frac{c_3}{b+1}t^{b+1} + c_4) \\ u(t) = \dfrac{k_1}{\frac{c_1}{a+2}t^{a+2} + \frac{c_1}{a+1}t^{a+1} + c_2} \\ v(t) = \dfrac{k_2}{\frac{c_3}{b+2}t^{b+2} + \frac{c_3}{b+1}t^{b+1} + c_4} \end{cases}$$

then $\Gamma = (x(t), y(t), u(t), v(t))$ is an ${}^HF$-planar on $T\mathbb{R}^2$, where c_i, k_i are real constants.

6 Conclusion remarks

Some results on the geodesic, F-geodesic, and F-planar curves on the tangent bundle and the φ-unit tangent bundle using the Berger-type Cheeger-Gromoll metric are studied in this paper. Therefore, this work's results are varied and significant, so it is interesting.

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