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Uniform ergodic theorem for operator matrices

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Abstract *For* $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, *and* $C \in \mathcal{L}(Y,X)$ *we denote by* M_C *the operator matrix defined on* $X \oplus Y$ *by* $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, *where* X *and* Y *are Banach spaces. In this paper, we study the ergodicity of* M_C *with respect to those of its diagonal terms* A *and* B.

1 Introduction

Throughout this paper, X *and* Y *denote infinite dimensional complex Banach spaces, and* $\mathcal{L}(X,Y)$ *denotes the set of all bounded linear operators from* X *into* Y. *If* $X = Y$ *we write* $\mathcal{L}(X)$ *instead of* $\mathcal{L}(X,X)$. *For* $T \in \mathcal{L}(X)$, *we denote by* $N(T)$ *the kernel of* T, $R(T)$ *the range of* T, $\sigma(T)$ *the spectrum of* T, $\oplus$ *the external direct sum and* $\oplus_i$ *the internal direct sum. The reduced minimum modulus of* T *is defined by [1]*

$$\gamma(T) = \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, N(T)) = 1\} & \text{if } T \neq 0 \\ 0 & \text{if } T = 0 \end{cases}.$$

Let $n \in \mathbb{N}$ *and* $T \in \mathcal{L}(X)$. *We define* $\mathcal{M}_n(T)$, *the Cesàro averages of* T, *by:*

$$\mathcal{M}_n(T) = \frac{I + T + T^2 + ... + T^{n-1}}{n}.$$

An operator $T \in \mathcal{L}(X)$ *is called mean ergodic if the Cesàro averages* $\mathcal{M}_n(T)$ *converges in the strong operator topology. If* $\mathcal{M}_n(T)$ *converges in the uniform operator topology,* T *is called uniform ergodic operator.*

In ergodic theory and functional analysis, the term "ergodic" often refers to properties related to the long-term behavior of dynamical systems or linear operators. An operator is said to be "mean ergodic" if it possesses certain properties related to convergence and averaging. The mean ergodic property is significant in the study of the long-term behavior of dynamical systems and linear operators. It ensures that, on average, the orbits of the system get closer to a fixed point (or converge to some limit) over time. This property has applications in various fields such as probability theory, functional analysis, and statistical physics.

In [8], J.V. Neumann prove that if T *is an ergodic unitary operator, then the temporal averages of successive powers of* T *converge to an orthogonal projection operator of rank one. This is an important result, it is a one of the cornerstones of ergodic theory and has applications in various areas of mathematics and physics. K. Yosida [13] extended this concept to the setting of Banach spaces, this extension is important because it generalizes the concept of ergodic theorems from Hilbert spaces to more general spaces.*

Through their papers [12, 13], Kakutani and Yoshida developed the mean ergodic theorem. Firstly, by generalizing the result of J.V. Neumann and by working on (real or complex) Banach spaces. Secondly, the strong limit of $\mathcal{M}_n(T)$ *is the projection of* X *onto the subspace* $N(I-T)$, *corresponding to the ergodic decomposition* $X = N(I-T) \oplus_i \overline{R(I-T)}$ *and the conditions imposed on an operator* $T \in \mathcal{L}(X)$ *for it to be mean ergodic always hold for any bounded linear operator in* L^p *with* $p > 1$.

In 1943, Nelson Dunford [3] studied the necessary and sufficient conditions for the convergence of a sequence of linear operators towards a projection, he treated the convergence, in different topologies (Uniform, strong, weak and almost everywhere convergence), of a sequence of polynomial operators towards a projection.

Based on Dunford's results, M.Lin [6] developed an uniform ergodic theorem, through which an operator $T \in \mathcal{L}(X)$ *verifying* $\frac{\|T\|^n}{n} \to 0$ *is uniformly ergodic if and only if* $(I-T)X$ *is closed if and only if* 1 *is a simple pole of the resolvent of* T*. Mbekhta and Zemanek [7] generalized this result by replacing the closure of* $(I-T)X$ *by that of* $(I-T)^k X$ *for some integer* $k > 0$.

In his paper [14], M. Yahdi introduced a concept known as super-ergodicity, which occupies a position between mean ergodicity and uniform ergodicity in terms of strength. This concept will be elaborated upon in the third section.

Let's consider the upper triangular operator matrix M_C *defined, on* $X \oplus Y$*, by :*

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

with $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ *and* $C \in \mathcal{L}(Y, X)$.

Since the article of P. Jin and H.K. Du [4], several papers have studied the upper triangular operator matrices M_C*. Most of them have treated the perturbation of different type of spectra of the matrix* M_C *in relation to those of its diagonal terms* A *and* B*, see [2], [9].*

In this paper, we propose to study some dynamic aspects of the matrix M_C*. So, we investigate the uniform ergodicity (resp. mean ergodicity and super-ergodicity) of* M_C.

2 Uniform ergodicity of M_C

In the sequel, we consider X *and* Y *as complex infinite dimensional Banach spaces.*
The following results are valid for any l_p *norm on the direct sum* $X \oplus Y$*, where* $1 \leq p < \infty$.

Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$*, and* $C \in \mathcal{L}(Y, X)$*. For* $n \in \mathbb{N}$ *such that* $n \geq 2$*, we have:*

$$M_C^n = \begin{pmatrix} A^n & \sum_{k=0}^{n-1} A^{n-1-k} C B^k \\ 0 & B^n \end{pmatrix}.$$

Then

$$\mathcal{M}_n(M_C) = \begin{pmatrix} M_n(A) & S_n \\ 0 & M_n(B) \end{pmatrix}.$$

with

$$S_n = \sum_{k=0}^{n-2} \sum_{i=0}^{k} \frac{1}{n} (A^{k-i} C B^i).$$

Theorem 2.1. *Let* $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$*, and* $C \in \mathcal{L}(Y, X)$*.* A *and* B *must be uniformly ergodic if* M_C *is.*

Proof. Assume that M_C is uniformly ergodic, then there exists $P = \begin{pmatrix} P_1 & P_3 \\ P_4 & P_2 \end{pmatrix} \in \mathcal{L}(X \oplus Y)$ such that $\|\mathcal{M}_n(M_C) - P\| \to 0$. Hence

$$Sup_{x \oplus y \neq 0 \oplus 0} \frac{\|[(\mathcal{M}_n(A) - P_1)x + (S_n - P_3)y] \oplus [P_4 x - (\mathcal{M}_n(B) - P_2)y]\|}{\|x \oplus y\|} \to 0.$$

So,

$$Sup_{x\neq 0}\frac{\|(\mathcal{M}_n(A)-P_1)x\oplus P_4x\|}{\|x\|}\to 0.$$

Let $p\geq 1$ be an integer. Hence

$$Sup_{x\neq 0}\frac{1}{\|x\|}(\|(\mathcal{M}_n(A)-P_1)x\|^p+\|P_4x\|^p)^{\frac{1}{p}}\to 0.$$

If $P_4\neq 0$, then there exists $x_0\in X$ such that $\|P_4x_0\|\neq 0$. Hence

$$Sup_{x\neq 0}\frac{1}{\|x\|}(\|(\mathcal{M}_n(A)-P_1)x\|^p+\|P_4x\|^p)^{\frac{1}{p}}\geq\frac{1}{\|x_0\|}(\|(\mathcal{M}_n(A)-P_1)x_0\|^p+\|P_4x_0\|^p)^{\frac{1}{p}}\nrightarrow 0,$$

which is absurd. Consequently, $P_4=0$.
Let us show the uniform ergodicity of A and B. We have

$$\begin{aligned}\|\mathcal{M}_n(M_C)-P\| &= Sup_{\|x\oplus y\|\leq 1}\|(\mathcal{M}_n(M_C)-P)x\oplus y\|\\ &\geq Sup_{\|x\|\leq 1}\|(\mathcal{M}_n(M_C)-P)x\oplus 0\|\\ &= Sup_{\|x\|\leq 1}\|(\mathcal{M}_n(A)-P_1)x\|\\ &= \|\mathcal{M}_n(A)-P_1\|.\end{aligned}$$

Since $\|\mathcal{M}_n(M_C)-P\|\to 0$, $\|\mathcal{M}_n(A)-P_1\|\to 0$. As a result, A is uniformly ergodic. Moreover, we have

$$\begin{aligned}\|\mathcal{M}_n(M_C)-P\| &= Sup_{\|x\oplus y\|\leq 1}\|(\mathcal{M}_n(M_C)-P)x\oplus y\|\\ &\geq Sup_{\|y\|\leq 1}\|(\mathcal{M}_n(M_C)-P)0\oplus y\|\\ &= Sup_{\|y\|\leq 1}\|(S_n-P_3)y\oplus(\mathcal{M}_n(B)-P_2)y\|\\ &= Sup_{\|y\|\leq 1}(\|(S_n-P_3)y\|^p+\|(\mathcal{M}_n(B)-P_2)y\|^p)^{\frac{1}{p}}.\end{aligned}$$

Then $Sup_{\|y\|\leq 1}(\|(S_n-P_3)y\|^p+\|(\mathcal{M}_n(B)-P_2)y\|^p)\to 0$.

If $\|\mathcal{M}_n(B)-P_2\|\nrightarrow 0$, then

$$Sup_{\|y\|\leq 1}\|(\mathcal{M}_n(B)-P_2)y\|^p\nrightarrow 0.$$

Thus

$$Sup_{\|y\|\leq 1}(\|(S_n-P_3)y\|^p+\|(\mathcal{M}_n(B)-P_2)y\|^p)\nrightarrow 0,$$

which is absurd. Therefore $\|\mathcal{M}_n(B)-P_2\|\to 0$. As a result, B is uniformly ergodic. □

In the context of the study of the matrix M_C, an important question arises regarding the relationship between the projection associated to M_C and the projections associated to A and B. In light of the demonstration of the aforementioned theorem, it is established that both A and B are uniformly ergodic operators, where $\mathcal{M}_n(A)$ uniformly converges to P_1 and $\mathcal{M}_n(B)$ uniformly converges to P_2. In the subsequent corollary, we add a condition ensuring that the limit of $\mathcal{M}_n(M_C)$ is exclusively formulated by those of $\mathcal{M}_n(A)$ and $\mathcal{M}_n(B)$.

Corollary 2.2. *Let $A\in\mathcal{L}(X)$, $B\in\mathcal{L}(Y)$, and $C\in\mathcal{L}(Y,X)$ such that $N(I-B)\subseteq N(C)$. If M_C is uniformly ergodic, then $\mathcal{M}_n(M_C)$ converges uniformly to $\begin{pmatrix}P_1 & 0\\ 0 & P_2\end{pmatrix}$ where $\mathcal{M}_n(A)$ converges uniformly to P_1 and $\mathcal{M}_n(B)$ converges uniformly to P_2.*

Proof. Suppose that M_C is uniformly ergodic, then there exists $P=\begin{pmatrix}P_1 & P_3\\ 0 & P_2\end{pmatrix}\in\mathcal{L}(X\oplus Y)$ such that $\|\mathcal{M}_n(M_C)-P\|\to 0$ and P is the projection of $X\oplus Y$ on the subspace $N(I-M_C)$, corresponding to the ergodic decomposition $X\oplus Y=N(I-M_C)\oplus_i\overline{(I-M_C)(X\oplus Y)}$. So, for all $x\oplus y=(z\oplus t)+(s\oplus q)\in N(I-M_C)\oplus_i\overline{(I-M_C)(X\oplus Y)}=X\oplus Y$ we have $P(x\oplus y)=z\oplus t$.

Let $x \oplus y = (z \oplus t) + (s \oplus q) \in N(I - M_C) \oplus_i \overline{(I - M_C)(X \oplus Y)} = X \oplus Y$. Hence $P(x \oplus y) = (P_1 x + P_3 y) \oplus P_2 y = z \oplus t$. By Theorem 2.1, P_1 is the projection of X on the subspace $N(I - A)$ and P_2 is the projection of Y on the subspace $N(I - B)$. Since $N(I - B) \subseteq N(C)$, $N(I - A) \oplus N(I - B) = N(I - M_C)$. Hence $(z + P_3 y) \oplus t = z \oplus t$, which implies that $P_3 y = 0$, for all $y \in Y$. Therefore $P_3 = 0$. Consequently, $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$. □

Through the subsequent example, we illustrate that the uniform ergodicity of A and B does not necessarily entails the same property for M_C.

Example 2.3. Let $A = I_X$, $B = \frac{I_Y}{2}$, and $C(\neq 0) \in \mathcal{L}(Y, X)$. Evidently A and B are uniformly ergodic operators, then $\mathcal{M}_n(A)$ and $\mathcal{M}_n(B)$ converge. Hence, if M_C is uniformly ergodic, then (by Corollary 2.2) S_n converges to 0. Therefore to show that M_C is not uniformly ergodic, it suffices to show that $S_n \not\to 0$. For $n \geq 2$, we have

$$
\begin{aligned}
\|S_n\| &= \|\sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{A^{k-i}CB^i}{n}\| \\
&= \|\frac{C}{n}\sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{1}{2^i}\| \\
&= \|\frac{C}{n}\sum_{k=0}^{n-2} \frac{1}{2} \times \frac{1-(\frac{1}{2})^{k+1}}{1-\frac{1}{2}}\| \\
&= \|\frac{C}{n}\sum_{k=0}^{n-2}(1-(\frac{1}{2})^{k+1})\| \\
&= \|\frac{C}{n}[\sum_{k=0}^{n-2} 1 - \sum_{k=0}^{n-2}(\frac{1}{2})^{k+1}]\| \\
&= \|\frac{C}{n}(n-1) - \frac{C}{n}(\frac{1-(\frac{1}{2})^{n-1}}{1-\frac{1}{2}})\frac{1}{2}\| \\
&= \|\frac{C}{n}(n-1) - \frac{C}{n}(1-(\frac{1}{2})^{n-1})\| \\
&= \|\frac{C}{n}(n-1) - \frac{C}{n} + \frac{2C}{n}(\frac{1}{2})^{n-1}\| \\
&= |\frac{n-1}{n} - \frac{1}{n} + \frac{1}{n}(\frac{1}{2})^{n-1}|\|C\|.
\end{aligned}
$$

Therefore $S_n \to \|C\| \neq 0$. As a result, M_C is not uniformly ergodic.

Theorem 2.4. *Given $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y, X)$. If M_C is mean ergodic, this implies the mean ergodicity of both A and B.*

Proof. It is similar to the proof of Theorem 2.1. □

Also, we have the following corollary.

Corollary 2.5. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y, X)$. Assuming either $I - B$ is injective or $N(C) \subset N(I - B)$. If M_C is mean ergodic, then $\mathcal{M}_n(M_C)$ converges strongly to $\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ where $\mathcal{M}_n(A)$ converges strongly to P_1 and $\mathcal{M}_n(B)$ converges strongly to P_2.*

By revisiting Example 2.3 once more and following the same procedure, it becomes evident that the mean ergodicity of A and B does not implies that of M_C. In light of this, we formulated the following theorem.

Theorem 2.6. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y, X)$. If $\|A\| < 1$ and $\|B\| < 1$, then M_C is uniformly ergodic.*

Proof. Assume that $\|A\| < 1$ and $\|B\| < 1$, then they are uniformly ergodic. We have

$$\mathcal{M}_n(M_C) = \begin{pmatrix} \mathcal{M}_n(A) & \sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{1}{n}(A^{k-i}CB^i) \\ 0 & \mathcal{M}_n(B) \end{pmatrix},$$

Since A and B are uniformly ergodic, $\mathcal{M}_n(A)$ and $\mathcal{M}_n(B)$ converges. So, to show that M_C is uniformly ergodic, it suffices to show that S_n converges. We have

$$\|\sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{A^{k-i}CB^i}{n}\| \leq \sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{\|A^{k-i}\|\|C\|\|B^i\|}{n}.$$

Without loss of generality, we suppose that $\|B\| \leq \|A\|$. Then

$$\|\sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{A^{k-i}CB^i}{n}\| \leq \sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{\|A\|^k\|C\|}{n}.$$

For all $k \in \mathbb{N}$, we pose $U_k = \sum_{i=0}^{k} \frac{\|A\|^k\|C\|}{n}$ and $a_k = \|A\|$. Then we have

$$\lim_{k\to\infty} (a_k \frac{U_k}{U_{k+1}} - a_{k+1}) = 1 - \|A\| > 0.$$

Then Kummer's test ensures the convergence of the series $\sum_{n\geq 0} U_n$. Consequently, $\|\sum_{k=0}^{n-2}\sum_{i=0}^{k} \frac{A^{k-i}CB^i}{n}\|$ converges. Which implies that M_C is uniformly ergodic. □

Next, we present the subsequent proposition.

Proposition 2.7. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y, X)$. The following statements holds.*

(1) Suppose that $\|A\| \neq \|B\|$. If $\frac{\|A^n\|}{n} \to 0$ and $\frac{\|B^n\|}{n} \to 0$ when $n \to \infty$, and the sequences $\frac{\|A\|^n}{n}$ and $\frac{\|B\|^n}{n}$ tend to the same limit in $\mathbb{R}$, when $n \to \infty$. Then $\frac{\|M_C^n\|}{n} \to 0$ when $n \to \infty$.

(2) Suppose that $\|A\| = \|B\|$. If $\|A\|^n \to 0$ when $n \to \infty$, then $\frac{\|M_C^n\|}{n} \to 0$, when $n \to \infty$.

Proof. (1) We have

$$\|\sum_{k=0}^{n-1} \frac{A^{n-1-k}CB^k}{n}\| \leq \frac{\|C\|}{n}\sum_{k=0}^{n-1} \|A^{n-1-k}\|\|B^k\|.$$

Hence

$$\|\sum_{k=0}^{n-1} \frac{A^{n-1-k}CB^k}{n}\| \leq \frac{\|C\|}{n}\frac{\|A\|^n - \|B\|^n}{\|A\| - \|B\|}.$$

We have

$$\frac{\|M_C^n\|}{n} \leq \frac{1}{n}(\|A^n\| + \|\sum_{k=0}^{n-1} A^{n-1-k}CB^k\|) + \frac{\|B^n\|}{n}.$$

Indeed, we have

$$\begin{aligned}
\|M_C^n\| &= Sup_{\|x\oplus y\|\leq 1}\|M_C^n(x \oplus y)\| \\
&= Sup_{\|x\oplus y\|\leq 1}\|A^n x + \sum_{k=0}^{n-1} A^{n-1-k}CB^k \oplus B^n y\| \\
&= Sup_{\|x\oplus y\|\leq 1}(\|A^n x + \sum_{k=0}^{n-1} A^{n-1-k}CB^k\|^p + \|B^n y\|^p)^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$\|M_C^n\|^p = Sup_{\|x\oplus y\|\leq 1}(\|A^n x + \sum_{k=0}^{n-1} A^{n-1-k}CB^k\|^p + \|B^n y\|^p).$$

Moreover we have

$$\begin{aligned} Sup_{\|x\oplus y\|\leq 1}\|A^n x + \sum_{k=0}^{n-1} A^{n-1-k}CB^k y\|^p &\leq Sup_{\|x\oplus y\|\leq 1}(\|A^n x\| + \|\sum_{k=0}^{n-1} A^{n-1-k}CB^k y\|)^p \\ &= [Sup_{\|x\oplus y\|\leq 1}(\|A^n x\| + \|\sum_{k=0}^{n-1} A^{n-1-k}CB^k y\|)]^p \\ &\leq (Sup_{\|x\|\leq 1}\|A^n x\| + Sup_{\|y\|\leq 1}\|\sum_{k=0}^{n-1} A^{n-1-k}CB^k y\|)^p \\ &= (\|A^n\| + \|\sum_{k=0}^{n-1} A^{n-1-k}CB^k\|)^p. \end{aligned}$$

Assume that $\frac{\|A\|^n}{n}$ and $\frac{\|B\|^n}{n}$ tend to the same limit in $\mathbb{R}$, when $n \to \infty$. Hence $\|\sum_{k=0}^{n-1} \frac{A^{n-1-k}CB^k}{n}\| \to 0$, when $n \to \infty$. As a consequence, we have $\frac{\|M_C^n\|}{n} \to 0$.

(2) Assume that $\|A\|^n \to 0$, when $n \to \infty$ and $\|A\| = \|B\|$. We have

$$\|\sum_{k=0}^{n-1} \frac{A^{n-1-k}CB^k}{n}\| \leq \frac{\|C\|}{n}\sum_{k=0}^{n-1}\|A\|^{n-1-k}\|B\|^k.$$

Since $\|A\| = \|B\|$,

$$\|\sum_{k=0}^{n-1} \frac{A^{n-1-k}CB^k}{n}\| \leq \frac{\|C\|}{n}\sum_{k=0}^{n-1}\|A\|^{n-1-k}\|B\|^k = \|C\|\|A\|^{n-1}.$$

Moreover $\|A\|^n \to 0$, when $n \to \infty$, then $\|\sum_{k=0}^{n-1} \frac{A^{n-1-k}CB^k}{n}\| \to 0$, when $n \to \infty$. Consequently, $\frac{\|M_C^n\|}{n} \to 0$, when $n \to \infty$.

□

By Proposition 2.7, we can subsequently replace the condition $\frac{\|M_C^n\|}{n} \to 0$ *when* $n \to \infty$, *with specific requirements that exclusively involve the operators* A *and* B.

Proposition 2.8. *Let* $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, *and* $C \in \mathcal{L}(K,H)$, *where* H *and* K *are two complex infinite dimensional separable Hilbert spaces. If either* $I - B$ *or* $(I - A)^*$ *is injective and* $\frac{\|M_C^n\|}{n} \to 0$ *when* $n \to \infty$. *Then the following statements are equivalent:*

(1) M_C *is uniformly ergodic.*

(2) A *and* B *are uniformly ergodic.*

(3) The subspace $N(I-A)+R(I-A)$ *is closed in* X, *and the subspace* $N(I-B)+R(I-B)$ *is closed in* Y.

(4) The point 1 *is a pole of order at most equal to* 1 *of resolvant of* A *and resolvant of* B.

(5) The range $R((I-A)^m)$ *is closed for certain* $m = 1, 2, ...$, *and the range* $R((I-B)^m)$ *is closed for certain* $m = 1, 2, ...$

Proof. (1) $\Longrightarrow$ (2) from Theorem 2.1. (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) from [7, Theorem 1].

(5) $\Longrightarrow$ (1): We have $\frac{\|M_C^n\|}{n} \to 0$, then it is easy to see that $\frac{\|A^n\|}{n} \to 0$ and $\frac{\|B^n\|}{n} \to 0$. Assume that there exist $n, m = 1, 2, ...$ such that $R((I-A)^m)$ and $R((I-B)^m)$ are closed. So, according to [7, Theorem 1], A and B are uniformly ergodic. Then [6, Theorem] $R(I-A)$ and $R(I-B)$ are closed, which is equivalent to $\gamma(I-A)$ and $\gamma(I-B)$ are strictly positive. Since $I - B$ or $(I-A)^*$ is injective, [5, Theorem 3] ensures that $\gamma(I - M_C)$ is stricly positive. As consequence, $R(I - M_C)$ is closed. Thus M_C is uniformly ergodic. □

If we take the example 2.3, the sequences $(A^n)_{n\in\mathbb{N}}$ and $(B^n)_{n\in\mathbb{N}}$ converge uniformly but $(M_C^n)_{n\in\mathbb{N}}$ does not. By the following Theorem we give some conditions that ensure the equivalence between the uniform convergence of $(M_C^n)_{n\in\mathbb{N}}$ and that of $(A^n)_{n\in\mathbb{N}}$ and $(B^n)_{n\in\mathbb{N}}$.

Let us denote the unit circle by $\mathbb{T}$.

Theorem 2.9. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, and $C \in \mathcal{L}(K,H)$, where H and K are two complex infinite dimensional separable Hilbert spaces. If either $I - B$ or $(I-A)^*$ is injective and $\frac{\|M_C^n\|}{n} \to 0$, then the following statements are equivalent:*

(i) $(M_C^n)_{n\in\mathbb{N}}$ *converges uniformly;*

(ii) $(A^n)_{n\in\mathbb{N}}$ *and* $(B^n)_{n\in\mathbb{N}}$ *converge uniformly.*

Proof. (1) $\Longrightarrow$ (2): Assume that $(M_C^n)_{n\in\mathbb{N}}$ converges uniformly. Hence there exists $Q = \begin{pmatrix} Q_1 & Q_3 \\ Q_4 & Q_2 \end{pmatrix} \in \mathcal{L}(H \oplus K)$ such that $\|M_C^n - Q\| \to 0$. Hence

$$Sup_{x\oplus y\neq 0\oplus 0}\frac{\|[(A^n - Q_1)x + (\sum_{k=0}^{n} A^{n-1-k}CB^k - Q_3)y] \oplus [Q_4x - (B^n - Q_2)y]\|}{\|x \oplus y\|} \to 0.$$

Hence

$$Sup_{x\neq 0}\frac{\|(A^n - Q_1)x \oplus Q_4x\|}{\|x\|} \to 0.$$

So, it is easy to see that $Q_4 = 0$.
Let us show that $(A^n)_{n\in\mathbb{N}}$ and $(B^n)_{n\in\mathbb{N}}$ converges uniformly. We have

$$\begin{aligned}\|M_C^n - Q\| &= Sup_{\|x\oplus y\|\leq 1}\|(M_C^n - Q)x \oplus y\| \\ &\geq Sup_{\|x\|\leq 1}\|(M_C^n - Q)x \oplus 0\| \\ &= Sup_{\|x\|\leq 1}\|(A^n - Q_1)x\| \\ &= \|A^n - Q_1\|.\end{aligned}$$

Since $\|M_C^n - Q\| \to 0$, $\|A^n - Q_1\| \to 0$. Hence $(A^n)_{n\in\mathbb{N}}$ converges uniformly to Q_1.
Moreover, we have

$$\begin{aligned}\|M_C^n - Q\| &= Sup_{\|x\oplus y\|\leq 1}\|(M_C^n - Q)x \oplus y\| \\ &\geq Sup_{\|y\|\leq 1}\|(M_C^n - Q)0 \oplus y\| \\ &= Sup_{\|y\|\leq 1}\|(S_n - P_3)y \oplus (B^n - Q_2)y\| \\ &= Sup_{\|y\|\leq 1}(\|(\sum_{k=0}^{n-1} A^{n-1-k}CB^k - D_3)y\|^p + \|B^n - D_2)y\|^p)^{\frac{1}{p}}.\end{aligned}$$

Hence

$$Sup_{\|y\|\leq 1}(\|(\sum_{k=0}^{n-1} A^{n-1-k}CB^k - Q_3)y\|^p + \|(B^n - Q_2)y\|^p) \to 0.$$

If $\|B^n - Q_2\|_U \not\to 0$, then

$$Sup_{\|y\|\leq 1}\|(B^n - Q_2)y\|^p \not\to 0.$$

Consequently,

$$Sup_{\|y\|\leq 1}(\|(\sum_{k=0}^{n-1} A^{n-1-k}CB^k - Q_3)y\|^p + \|B^n - Q_2)y\|^p) \not\to 0,$$

which is absurd. Hence $\|B^n - Q_2\| \to 0$. Therefore $(B^n)_{n\in\mathbb{N}}$ converges uniformly to Q_2.

(2) $\Longrightarrow$ (1): Assume that $(A^n)_n$ and $(B^n)_n$ converge uniformly. According to [7, Corollary 3], A and B are uniformly ergodic and $\begin{cases} \sigma(A) \cap \mathbb{T} \subset \{1\} \\ \sigma(B) \cap \mathbb{T} \subset \{1\} \end{cases}$. By [7, Corollary 3], we have

$\begin{cases} |\sigma(A)| \leq 1 \\ |\sigma(B)| \leq 1 \end{cases}$ with $\begin{cases} \sigma(A) \cap \mathbb{T} \subset \{1\} \\ \sigma(B) \cap \mathbb{T} \subset \{1\} \end{cases}$ and the point 1 is a pole of order at most equal to 1 of resolvant of A and resolvant of B. Since either $I - B$ or $(I - A)^*$ is injective and $\frac{\|M_C^n\|}{n} \to 0$, by Proposition 2.8 and [7, Theorem 1], the point 1 is a pole of order at most equal to 1 of resolvant of M_C. Furthermore, we have $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$. Hence $|\sigma(M_C)| \leq 1$ and $\sigma(M_C) \cap \mathbb{T} \subset \{1\}$. So, in virtue of [7, Corollary 3], $(M_C)_{n\geq 1}$ converges uniformly. □

3 Super-ergodicity of M_C

Let X be a Banach space and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. We denote by

(i) *$l^\infty(X)$ the space of all bounded sequences in X.*

(ii) *$\mathcal{C}^{\mathcal{U}}(X)$ the subspace of $l^\infty(X)$, of all sequences $(x_n)_n$ such that $\lim_{\mathcal{U}} \|x_n\| = 0$.*

(iii) *$X^{\mathcal{U}} = l^\infty(X)/C^{\mathcal{U}}(X)$ the quotient space called the $\mathcal{U}$-ultrapower of X, with the canonical norm*

$$\|\overline{(x_n)_n}\| = \lim_{\mathcal{U}} \|x_n\|.$$

(iv) *$T^{\mathcal{U}}$ the ultrapower operator of T defined on $X^{\mathcal{U}}$ by*

$$T^{\mathcal{U}}(\overline{(x_n)_n}) = \overline{(Tx_n)_n} = (Tx_n)_n + \mathcal{C}^{\mathcal{U}}(X).$$

Recall that [15, pp 451] for every ultrafilter $\mathcal{U}$ there is an isometry between $(X \oplus Y)^{\mathcal{U}}$ and $X^{\mathcal{U}} \oplus Y^{\mathcal{U}}$. Exactly, for all $\overline{(x_n)_n} \in X^{\mathcal{U}}$ and $\overline{(y_n)_n} \in X^{\mathcal{U}}$ we have

$$\|\overline{(x_n)_n} \oplus \overline{(y_n)_n}\|_{X^{\mathcal{U}} \oplus Y^{\mathcal{U}}} = \|\overline{(x_n)_n \oplus (y_n)_n}\|_{(X\oplus Y)^{\mathcal{U}}} = \lim_{\mathcal{U}} \|x_n \oplus y_n\|,$$

where $X \oplus Y$ is equipped by l_p-norm with $1 \leq p < \infty$.
Due to that we can identify each of $(X \oplus Y)^{\mathcal{U}}$ and $X^{\mathcal{U}} \oplus Y^{\mathcal{U}}$ by the other space.

Definition 3.1. [14] Let $T \in \mathcal{L}(X)$. We say that T is super-ergodic if the operator $T^{\mathcal{U}}$ is mean ergodic on $X^{\mathcal{U}}$.

Recall that we have

$$T \text{ is uniformly ergodic} \implies T \text{ is super-ergodic} \implies T \text{ is mean ergodic.}$$

But the inverse implications are not necessarily true (See [14]).

The key to this section lies in the following lemma.

Lemma 3.2. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y, X)$. We have*

$$M_C^{\mathcal{U}} = \begin{pmatrix} A^{\mathcal{U}} & C^{\mathcal{U}} \\ 0 & B^{\mathcal{U}} \end{pmatrix}.$$

Proof. Assume that $M_C^{\mathcal{U}} = \begin{pmatrix} T & D \\ Q & S \end{pmatrix}$.

We claim that $Q = 0$. In fact, by way of contradiction, we suppose that there exists $\overline{(x_n^0)_n} \in X^{\mathcal{U}}$ such that $Q\overline{(x_n^0)_n} \neq \overline{0}$. So, we have

$$M_C^{\mathcal{U}}(\overline{(x_n^0)_n} \oplus \overline{0}) = \overline{(M_C(x_n^0 \oplus 0))_n}.$$

Since $(X \oplus Y)^{\mathcal{U}} = X^{\mathcal{U}} \oplus Y^{\mathcal{U}}$, we deduce that

$$T\overline{(x_n^0)_n} \oplus Q\overline{(x_n^0)_n} = \overline{(Ax_n^0)_n} \oplus \overline{0},$$

which is absurd. Consequently, $Q = 0$.
Let $\overline{(y_n)_n} \in Y^{\mathcal{U}}$. We have

$$M_C^{\mathcal{U}}(\overline{0} \oplus \overline{(y_n)_n}) = \overline{(M_C(0 \oplus y_n))_n}.$$

Since $(X \oplus Y)^{\mathcal{U}} = X^{\mathcal{U}} \oplus Y^{\mathcal{U}}$, we have

$$D\overline{(y_n)_n} \oplus S\overline{(y_n)_n} = \overline{(Cy_n \oplus By_n)_n} = \overline{(Cy_n)_n} \oplus \overline{(By_n)_n}.$$

Therefore $S = B^{\mathcal{U}}$ and $D = C^{\mathcal{U}}$. In a similar manner, we have $T = A^{\mathcal{U}}$. □

Theorem 3.3. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $C \in \mathcal{L}(Y, X)$. If M_C is super-ergodic, then A and B are super-ergodic.*

Proof. Assume that M_C is super-ergodic. Hence $M_C^{\mathcal{U}}$ is mean ergodic. Moreover, by Lemma 3.2, we have $M_C^{\mathcal{U}} = \begin{pmatrix} A^{\mathcal{U}} & C^{\mathcal{U}} \\ 0 & B^{\mathcal{U}} \end{pmatrix}$. In accordance with Theorem 2.4, $A^{\mathcal{U}}$ and $B^{\mathcal{U}}$ are mean ergodic. Consequently, A and B are super-ergodic operators.

□

Remark 3.4. If $\|A\| < 1$ and $\|B\| < 1$, then M_C is super-ergodic. Indeed: Theorem 2.6 ensures that M_C is uniformly ergodic. As a result, M_C is super-ergodic.

Based on Corollary 2.5 and Lemma 3.2, we notice the following.

Remark 3.5. Assume that $N((I - B)^{\mathcal{U}}) \subseteq N(C^{\mathcal{U}})$. So, if M_C is super-ergodic, then $\mathcal{M}_n(M_C^{\mathcal{U}})$ converges strongly to $\begin{pmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{pmatrix}$ where $\mathcal{M}_n(A^{\mathcal{U}}) \xrightarrow{\text{Strongly}} \mathcal{P}_1$ and $\mathcal{M}_n(B^{\mathcal{U}}) \xrightarrow{\text{Strongly}} \mathcal{P}_2$.

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