Palestine Journal of Mathematics

Vol 14(1)(2025) , 262–268

On Matsumoto-Randers change on m-th root Finsler metric

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Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53B40; Secondary 53C60.

Keywords and phrases: Locally dually flat metric, projectively flat metric, Matsumoto change, Randers change, m-th root metric.

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Abstract In this article, we consider the Matsumoto-Randers change on m-th root Finsler metric. The necessary and sufficient conditions for the Matsumoto-Randers change on m-th root Finsler metric to be locally dually flat is obtained. Finally, we discuss the condition for which the Finsler metric is projectively flat.

1 Introduction

The class of Finsler metrics is a natural generalization of the class of Riemannian metrics. Compared to Riemannian metrics, Finsler metrics do not always exhibit reversibility condition. If one considers the reversibility of certain Finsler metrics, it can be simplified to a Riemannian metric, namely Randers-type metric [8]. This demonstrates the complexity of the Finsler metric in comparison to the Riemannian metric. On the other hand, an investigation into the geometrical structure of a family of probability distributions led to the development of information geometry. Amari and Nagaoka [2] initially established the concept of dually flat metrics while investigating the information geometry on Riemannian spaces. Later, Z. Shen [9] expanded the concept of dually flatness to Finsler metrics. In Finsler information geometry, dually flat Finsler metrics constitute a distinct and valuable class of Finsler metrics that are crucial for the study of flat Finsler information structure [9].

A Finsler metric $F = F(x, y)$ on a manifold is considered locally dually flat if at every point, there exists a coordinate system (x^i) in which the spray coefficients G^i can be expressed as

$$G^i = \frac{-1}{2} g^{ij} H_{y^j}, \tag{1.1}$$

where $H = H(x, y)$ is a local scalar function [4]. In this case, $H = H(x, y)$ in (1.1) is given by $H = \frac{-1}{6}[F^2]_{x^m} y^m$. Such a coordinate system refers to an adapted coordinate system. In other words, a Riemannian metric $F = \sqrt{g_{ij}(x) y^i y^j}$ is locally dually flat if and only if in an adapted coordinate system,

$$g_{ij}(x) = \frac{\partial^2 \phi}{\partial x^i \partial x^j}(x)$$

where $\phi = \phi(x)$ is a C^∞ function [1, 2]. There are many non Riemannian metrics which are dually flat, for example [5]

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - < x, y >^2)}}{1 - |x|^2} \pm \frac{< x, y >}{1 - |x|^2}$$

On the unit ball $B^n \subset \mathbb{R}^n$, the above metric is the Funk metric.

H. Shimada [10] developed the notion of m-th root Finsler metrics, which has been utilized in the field of Biology as an ecological metric. For a Finsler manifold (M, F) of dimension n, let TM denote its tangent bundle, and (x^i, y^i) represent the coordinates in a local chart on TM. The Finsler metric $F : TM \to \mathbb{R}$ be defined as $F = A^{\frac{1}{m}}$, where $A := a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ and $a_{i_1 i_2 \ldots i_m}$ is symmetric in all indices. Then F is called an m-th root Finsler metric on M. The second root metric ($m = 2$) is simply the Riemannian metric. The third root metric ($m = 3$) is referred to as the cubic metric, whereas the fourth root metric ($m = 4$) is known as the quartic metric. Recent researches indicate that the theory of m-th root Finsler metrics is highly significant in the fields of physics, space-time structure theory, seismic ray theory, general relativity and gravitation.

A Finsler metric is considered locally projectively flat if, for any given point, there exists a local coordinate system where the geodesics are represented as straight lines. The condition for the projective flatness is the existence of a local coordinate system on the base manifold in which the Finsler function satisfies the system of partial differential equations [3]

$$\frac{\partial^2 F}{\partial^2 x^k y^l} y^k - \frac{\partial F}{\partial x^l} = 0 \tag{1.2}$$

Tayebi *et al.* studied the Randers change [12] and Matsumoto change [11] of m-th root metric and found the conditions for different kinds of flatness of Finsler metric. Gupta *et al.* [6] obtained the results regarding projectively flatness and locally dually flatness of Finsler metric. They also studied the curvature properties of the conformal Matsumoto metric. Motivated by the work done in this field, we have considered the Matsumoto and Randers change simultaneously on a m-th root Finsler metric F, resulting in a new metric denoted as $\overline{F}$. We refer this metric as the Matsumoto-Randers change of the m-th root Finsler metric and defined as

$$\overline{F} = \frac{F^2}{F - \beta} + \beta \tag{1.3}$$

where β is one form and $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric, with A being irreducible. In this article, we will prove the following theorems

Theorem 1.1. *Let $F = A^{\frac{1}{m}} (m > 2)$ be m-th root Finsler metric and $\overline{F}(x, y)$ be the Matsumoto-Randers change of F given in (1.3). Then $\overline{F}$ is projectively flat if and only if the following conditions holds*

$$\beta_{0l} - \beta_{x^l} = 0, \quad mA(A_{0l} - A_{x^l}) = (m-1)A_0 A_l, \quad \text{and} \quad \beta_0 = \beta\,\theta.$$

Theorem 1.2. *Let $F = A^{\frac{1}{m}} (m > 2)$ be m-th root Finsler metric. Assume that $\overline{F}(x, y)$ be the Matsumoto-Randers change of F given in (1.3). Then the necessary and sufficient condition for $\overline{F}$ to be locally dually flat is $A_{x^j} = 0$ and b_j are constant.*

2 Locally Projectively flat

A Finsler metric $F(x, y)$ on an open domain $U \subset R^n$ is said to be locally projectively flat if and only if its geodesic coefficients G^i can be expressed as $G^i(x, y) = P(x, y)y^i$ where P is a positively homogeneous function of degree one *i.e.* $P(x, \lambda y) = \lambda P(x, y)$, for $\lambda > 0$. The term $P(x, y)$ refers to "projective factor" [7]. Alternatively stated, a Finsler metric $F = F(x, y)$ is considered locally projectively flat on a manifold M^n if and only if [3]

$$[\overline{F}]_{x^k y^l} y^k - [\overline{F}]_{x^l} = 0 \tag{2.1}$$

For an m-th root metric $F = A^{\frac{1}{m}}$, we have used the following notations

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_{0l} = A_{x^k y^l} y^k = \frac{\partial^2 A}{\partial x^k y^l} y^k, \quad \beta_{0l} = \beta_{x^k y^l} y^k, \quad \beta_l = \frac{\partial \beta}{\partial y^l}.$$

Proof of Theorem 1.1: Differentiating equation (1.3) with respect to x^k, we have

$$[\overline{F}]_{x^k} = \frac{1}{m(A^{\frac{1}{m}} - \beta)^2} \left\{ A^{\frac{3}{m}-1} A_{x^k} - 2\beta\, A^{\frac{2}{m}-1} A_{x^k} + m\, A^{\frac{2}{m}} \beta_{x^k} \right\} + \beta_{x^k}. \tag{2.2}$$

Again differentiating with respect to y^l and multiplying the result by y^k gives us

$$\begin{aligned}[\overline{F}]_{x^k y^l} y^k &= \frac{1}{m(A^{\frac{1}{m}}-\beta)^3}\Big\{ \left(\tfrac{1}{m}-1\right) A_0 A_l A^{\frac{4}{m}-2} + A_{0l} A^{\frac{4}{m}-1} + [A_l\beta_0 - \beta_l A_0 - \beta A_{0l}] A^{\frac{3}{m}-1} \\ &\quad + m\beta_{0l} A^{\frac{3}{m}} + 3\left(1-\tfrac{1}{m}\right)\beta A_0 A_l A^{\frac{3}{m}-2} - 2\beta A_l \beta_0 A^{\frac{2}{m}-1} \\ &\quad + 2\left(\tfrac{2}{m}-1\right)\beta^2 A_0 A_l A^{\frac{2}{m}-2} + m[\beta_0\beta_l] - \beta\beta_{0l} A^{\frac{2}{m}} + \beta_{0l}\Big\}.\end{aligned} \tag{2.3}$$

Substituting the value of equation (2.2) and (2.3) in equation (2.1), we obtain

$$\begin{aligned}&\left(\tfrac{1}{m}-1\right) A_0 A_l A^{\frac{4}{m}-2} + (A_{0l} - A_{x_l}) A^{\frac{4}{m}-1} + 3(1-\tfrac{1}{m})\beta A_0 A_l A^{\frac{3}{m}-2} + (3\beta A_{x_l} - 3\beta A_{0l}) A^{\frac{3}{m}-1} \\ &+ \left(2m\beta_{0l} - 2m\beta_{x^l}\right) A^{\frac{3}{m}} + \left(2\beta^2 A_{0l} - 2\beta\beta_0 A_l - 2\beta\beta_l A_0 - 2\beta^2 A_{x^l}\right) A^{\frac{2}{m}-1} \\ &+ 2\beta^2\left(\tfrac{2}{m}-1\right) A_0 A_l A^{\frac{2}{m}-2} + \left(2m\beta_0\beta_l - 4m\beta\beta_{0l} + 4m\beta\beta_{x^l}\right) A^{\frac{2}{m}} + 3m\beta^2(\beta_{0l} - \beta_{x^l}) A^{\frac{1}{m}} \\ &- 3m\beta^3(\beta_{0l} - \beta_{x^l}) = 0.\end{aligned}$$

Separating the rational and irrational terms, we get

$$m\beta^3(\beta_{0l} - \beta_{x^l}) = 0, \tag{2.4}$$

and

$$\begin{aligned}&\left(\tfrac{1}{m}-1\right) A_0 A_l A^{\frac{4}{m}-2} + (A_{0l} - A_{x_l}) A^{\frac{4}{m}-1} + 3(1-\tfrac{1}{m})\beta A_0 A_l A^{\frac{3}{m}-2} \\ &+ (3\beta A_{x_l} - 3\beta A_{0l}) A^{\frac{3}{m}-1} + \left(2m\beta_{0l} - 2m\beta_{x^l}\right) A^{\frac{3}{m}} + 2\beta^2\left(\tfrac{2}{m}-1\right) A_0 A_l A^{\frac{2}{m}-2} \\ &+ \left(2\beta^2 A_{0l} - 2\beta\beta_0 A_l - 2\beta\beta_l A_0 - 2\beta^2 A_{x^l}\right) A^{\frac{2}{m}-1} \\ &+ \left(2m\beta_0\beta_l - 4m\beta\beta_{0l} + 4m\beta\beta_{x^l}\right) A^{\frac{2}{m}} + 3m\beta^2(\beta_{0l} - \beta_{x^l}) A^{\frac{1}{m}} = 0.\end{aligned} \tag{2.5}$$

On rewritting equations (2.4), we get the first condition of the theorem as

$$\beta_{0l} - \beta_{x^l} = 0. \tag{2.6}$$

Substituting above relation in equation (2.5), and simplyfying we obtain

$$\begin{aligned}&\left(\tfrac{1}{m}-1\right) A_0 A_l A^{\frac{4}{m}-2} + (A_{0l} - A_{x_l}) A^{\frac{4}{m}-1} + 3(1-\tfrac{1}{m})\beta A_0 A_l A^{\frac{3}{m}-2} + (3\beta A_{x_l} \\ &- 3\beta A_{0l}) A^{\frac{3}{m}-1} + \left(2\beta^2 A_{0l} - 2\beta\beta_0 A_l - 2\beta\beta_l A_0 - 2\beta^2 A_{x^l}\right) A^{\frac{2}{m}-1} \\ &+ 2\beta^2\left(\tfrac{2}{m}-1\right) A_0 A_l A^{\frac{2}{m}-2} + 2m\beta_0\beta_l A^{\frac{2}{m}} = 0.\end{aligned} \tag{2.7}$$

Taking irrational and rational parts separately from above equation, we get

$$\left(\frac{1}{m}-1\right) A_0 A_l A^{\frac{2}{m}} + (A_{0l} - A_{x_l}) A^{\frac{2}{m}+1} + 3\left(1-\frac{1}{m}\right)\beta A_0 A_l A^{\frac{1}{m}} + (3\beta A_{x_l} - 3\beta A_{0l}) A^{\frac{1}{m}+1} = 0 \tag{2.8}$$

and

$$\left(2\beta^2 A_{0l} - 2\beta\beta_0 A_l - 2\beta\beta_l A_0 - 2\beta^2 A_{x^l}\right) A + 2\beta^2\left(\frac{2}{m}-1\right) A_0 A_l + 2m\beta_0\beta_l A^2 = 0 \tag{2.9}$$

Simplifying equation (2.8), we obtain

$$\left(\frac{1}{m}-1\right) A_0 A_l + (A_{0l} - A_{x^l}) A = 0. \tag{2.10}$$

On rewritten above equation, we have second condition of the theorem as

$$mA(A_{0l} - A_{x^l}) = (m-1) A_0 A_l. \tag{2.11}$$

Since, A is irreducible and $deg\,(A_l) = m-1 < deg\,(A)$. Thus their exists a one-form $\theta = \theta_l y^l$ on U such that

$$A_0 = m\,\theta A. \tag{2.12}$$

Substituting above value of A_0 in equation (2.9), and using (2.11), we have

$$(\beta\,\theta - \beta_0)\,\{\beta\,A_l - m\beta_l A\} = 0. \tag{2.13}$$

Since $\beta\,A_l - m\beta_l A \neq 0$, we have

$$\beta_0 = \beta\,\theta, \tag{2.14}$$

which is the last condition of theorem. The converse part is direct computation. This completes the proof.

3 Locally Dually Flat

In this section, we will prove the necessary and sufficient condtion under which Finsler metric change into the Matsumoto-Randers m-th root metric to be locally dually flat. A Finsler metric is locally dually flat if and only if, the following condition holds

$$[\overline{F}^2]_{x^k y^l} y^k - 2[\overline{F}^2]_{x^l} = 0 \tag{3.1}$$

Proof of Theorem 1.2: On squaring the Matsumoto-Randers m-th root Finsler metric $\overline{F}$ (1.3), we have

$$[\overline{F}^2] = \left[\frac{A^{\frac{2}{m}}}{A^{\frac{1}{m}} - \beta} + \beta\right]^2$$

Differentiating above equation with respect to x^k, we obtain

$$\begin{aligned}[\overline{F}^2]_{x^k} \;=\; & \frac{2}{(A^{\frac{1}{m}}-\beta)^4}\Big\{A_{x^l} A^{\frac{6}{m}-1} - 2\beta A_{x^l} A^{\frac{5}{m}-1} + 2m\beta_{x^l} A^{\frac{5}{m}} - 2m\beta\beta_{x^l} A^{\frac{4}{m}} \\ & -2\beta^2 A_{x^l} A^{\frac{4}{m}-1} + 5\beta^3 A_{x^l} A^{\frac{3}{m}-1} - 3m\beta^2\beta_{x^l} A^{\frac{3}{m}} + 6m\beta^3\beta_{x^l} A^{\frac{2}{m}} \\ & -4m\beta^4\beta_{x^l} A^{\frac{1}{m}} + m\beta^5\beta_{x^l}\Big\}.\end{aligned} \tag{3.2}$$

Again differentiating above equation with respect to y^l, and multiplying the result by y^k, we get

$$\begin{aligned}[\overline{F}^2]_{x^k y^l} y^k = \; & \frac{2}{m(A^{\frac{1}{m}}-\beta)^4}\Big\{\left(\tfrac{2}{m}-1\right) A_0 A_l A^{\frac{6}{m}-2} + A_{0l} A^{\frac{6}{m}-1} + 2\left(1-\tfrac{3}{m}\right)\beta A_0 A_l A^{\frac{5}{m}-2} \\ & +2\left(\beta_l A_0 + A_l\beta_0 - \beta A_{0l}\right) A^{\frac{5}{m}-1} + 2m\beta_{0l} A^{\frac{5}{m}} + 2\beta^2 A_0 A_l A^{\frac{4}{m}-2} \\ & \left(-8\beta\beta_0 A_l - 8\beta\beta_l A_0 - 2\beta^2 A_{0l}\right) A^{\frac{4}{m}-1} + \left(6m\beta_0\beta_l - 2m\beta\beta_{0l}\right) A^{\frac{4}{m}} \\ & + \left(3\beta^2\beta_0 A_l + 3\beta^2 A_0\beta_l + 5\beta^3 A_{0l}\right) A^{\frac{3}{m}-1} + \left(-6m\beta\beta_0\beta_l - 3m\beta^2\beta_{0l}\right) A^{\frac{3}{m}} \\ & -3\beta^3\left(1-\tfrac{1}{m}\right) A_0 A_l A^{\frac{3}{m}-2} - 2\beta^4 A_{0l} A^{\frac{2}{m}-1} + \left(6m\beta^2\beta_0\beta_l + 6m\beta^3\beta_{0l}\right) A^{\frac{2}{m}} \\ & -4m\beta\left(\beta_0\beta_l + \beta\beta_{0l}\right) A^{\frac{1}{m}} + m\beta^4\beta_0\beta_l + m\beta^5\beta_{0l}\Big\}.\end{aligned} \tag{3.3}$$

Substituting the values of equation (3.2) and (3.3), in equation (3.1), we have

$$\begin{aligned}& \left(\tfrac{2}{m}-1\right) A_0 A_l A^{\frac{6}{m}-2} + \left(A_{0l} - 2A_{x^l}\right) A^{\frac{6}{m}-1} + 2\left(1-\tfrac{3}{m}\right)\beta A_0 A_l A^{\frac{5}{m}-2} \\ & +2m(A_l\beta_0 - \beta_0 A_l - \beta A_{0l} + 2\beta A_{x^l})A^{\frac{5}{m}-1} + 2m(\beta_{0l} - 2\beta_{x^l})A^{\frac{5}{m}} + 2\beta^2 A_0 A_l A^{\frac{4}{m}-2} \\ & + \left(4\beta^2 A_{x^l} - 8\beta\beta_0 A_l - 8\beta\beta_l A_0 - 2\beta^3 A_{0l}\right) A^{\frac{4}{m}-1} + \left(6m\beta_0\beta_l - 2m\beta\beta_{0l} + 4m\beta\beta_{x^l}\right) A^{\frac{4}{m}} \\ & + \left(3\beta^2 A_0\beta_l + 3\beta^2\beta_0 A_l + 5\beta^3 A_{0l} - 10\beta^3 A_{x^l}\right) A^{\frac{3}{m}-1} - 3\beta^3\left(1-\tfrac{1}{m}\right) A_0 A_l A^{\frac{3}{m}-2} \\ & -3\beta^4 A_{0l} A^{\frac{2}{m}-1} + \left(6m\beta^2\beta_0\beta_l + 6m\beta^3\beta_{0l} - 12m\beta^3\beta_{x^l}\right) A^{\frac{2}{m}} \\ & + \left(6m\beta^2\beta_{x^l} - 6m\beta\beta_0\beta_l - 3m\beta^2\beta_{0l}\right) A^{\frac{3}{m}} + \left(8m\beta^4\beta_{x^l} - 4m\beta\beta_0\beta_l - 4m\beta^2\beta_{0l}\right) A^{\frac{1}{m}} \\ & +m\beta^4\beta_0\beta_l + m\beta^5\beta_{0l} - 2m\beta^5\beta_{x^l} = 0.\end{aligned}$$

On separating rational and irrational terms, we get

$$m\beta^4\beta_0\beta_l + m\beta^5\beta_{0l} - 2m\beta^5\beta_{x^l} = 0,$$

and

$$\begin{gathered}\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{6}{m}-2} + \left(A_{0l}-2A_{x^l}\right)A^{\frac{6}{m}-1} + 2\left(1-\tfrac{3}{m}\right)\beta A_0A_lA^{\frac{5}{m}-2} \\ +2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A^{\frac{5}{m}-1} + 2m(\beta_{0l}-2\beta_{x^l})A^{\frac{5}{m}} + 2\beta^2A_0A_lA^{\frac{4}{m}-2} \\ +\left(4\beta^2A_{x^l}-8\beta\beta_0A_l-8\beta\beta_lA_0-2\beta^3A_{0l}\right)A^{\frac{4}{m}-1} + \left(6m\beta_0\beta_l-2m\beta\beta_{0l}+4m\beta\beta_{x^l}\right)A^{\frac{4}{m}} \\ +\left(3\beta^2A_0\beta_l+3\beta^2\beta_0A_l+5\beta^3A_{0l}-10\beta^3A_{x^l}\right)A^{\frac{3}{m}-1} - 3\beta^3\left(1-\tfrac{1}{m}\right)A_0A_lA^{\frac{3}{m}-2} \\ -3\beta^4A_{0l}A^{\frac{2}{m}-1} + \left(6m\beta^2\beta_0\beta_l+6m\beta^3\beta_{0l}-12m\beta^3\beta_{x^l}\right)A^{\frac{2}{m}} \\ +\left(6m\beta^2\beta_{x^l}-6m\beta\beta_0\beta_l-3m\beta^2\beta_{0l}\right)A^{\frac{3}{m}} + \left(8m\beta^4\beta_{x^l}-4m\beta\beta_0\beta_l-4m\beta^2\beta_{0l}\right)A^{\frac{1}{m}} = 0.\end{gathered}$$

On simplifying the above equations, we obtain

$$\beta_0\beta_l + \beta\beta_{0l} - 2\beta\beta_{x^l} = 0 \tag{3.4}$$

and

$$\begin{gathered}\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{4}{m}-1} + \left(A_{0l}-2A_{x^l}\right)A^{\frac{4}{m}} + 2\left(1-\tfrac{3}{m}\right)\beta A_0A_lA^{\frac{3}{m}-1} + 2m(\beta_{0l}-2\beta_{x^l})A^{\frac{3}{m}+1} \\ +2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A^{\frac{3}{m}} + 2\beta^2A_0A_lA^{\frac{2}{m}-1} + 4m\beta_0\beta_lA^{\frac{2}{m}+1} \\ +\left(4\beta^2A_{x^l}-8\beta\beta_0A_l-8\beta\beta_lA_0-2\beta^3A_{0l}\right)A^{\frac{2}{m}} - 3\beta^3\left(1-\tfrac{1}{m}\right)A_0A_lA^{\frac{1}{m}-1} \\ +\left(3\beta^2A_0\beta_l+3\beta^2\beta_0A_l+5\beta^3A_{0l}-10\beta^3A_{x^l}\right)A^{\frac{1}{m}} - 3m\beta\beta_0\beta_lA^{\frac{1}{m}+1} - 3\beta^4A_{0l} = 0\end{gathered} \tag{3.5}$$

Taking rational and irrational terms separately from above equation, and simplyfing it, gives us

$$A_{0l} = 0 \tag{3.6}$$

and

$$\begin{gathered}\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{3}{m}} + \left(A_{0l}-2A_{x^l}\right)A^{\frac{3}{m}+1} + 2\left(1-\tfrac{3}{m}\right)\beta A_0A_lA^{\frac{2}{m}} \\ +2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A^{\frac{2}{m}+1} + 2m(\beta_{0l}-2\beta_{x^l})A^{\frac{2}{m}+2} + 2\beta^2A_0A_lA^{\frac{1}{m}} \\ +\left(4\beta^2A_{x^l}-8\beta\beta_0A_l-8\beta\beta_lA_0-2\beta^3A_{0l}\right)A^{\frac{1}{m}+1} - 3\beta^3\left(1-\tfrac{1}{m}\right)A_0A_l \\ +\left(3\beta^2A_0\beta_l+3\beta^2\beta_0A_l+5\beta^3A_{0l}-10\beta^3A_{x^l}\right)A - 3m\beta\beta_0\beta_lA^2 + 4m\beta_0\beta_lA^{\frac{1}{m}+2} = 0\end{gathered} \tag{3.7}$$

Equation (3.7) again splits into two equations

$$\begin{gathered}\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{3}{m}} + \left(A_{0l}-2A_{x^l}\right)A^{\frac{3}{m}+1} + 2\left(1-\tfrac{3}{m}\right)\beta A_0A_lA^{\frac{2}{m}} \\ +2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A^{\frac{2}{m}+1} + 2m(\beta_{0l}-2\beta_{x^l})A^{\frac{2}{m}+2} + 2\beta^2A_0A_lA^{\frac{1}{m}} \\ +\left(4\beta^2A_{x^l}-8\beta\beta_0A_l-8\beta\beta_lA_0-2\beta^3A_{0l}\right)A^{\frac{1}{m}+1} + 4m\beta_0\beta_lA^{\frac{1}{m}+2} = 0,\end{gathered} \tag{3.8}$$

and

$$-3\beta^3\left(1-\frac{1}{m}\right)A_0A_l + \left(3\beta^2A_0\beta_l+3\beta^2\beta_0A_l+5\beta^3A_{0l}-10\beta^3A_{x^l}\right)A - 3m\beta\beta_0\beta_lA^2 = 0. \tag{3.9}$$

Equation (3.8) can be rewritten as

$$\begin{gathered}\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{2}{m}} + \left(A_{0l}-2A_{x^l}\right)A^{\frac{2}{m}+1} + 2\left(1-\tfrac{3}{m}\right)\beta A_0A_lA^{\frac{1}{m}} \\ +2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A^{\frac{1}{m}+1} + 2m(\beta_{0l}-2\beta_{x^l})A^{\frac{1}{m}+2} + 2\beta^2A_0A_l \\ +\left(4\beta^2A_{x^l}-8\beta\beta_0A_l-8\beta\beta_lA_0-2\beta^3A_{0l}\right)A + 4m\beta_0\beta_lA^2 = 0,\end{gathered} \tag{3.10}$$

which on separating rational and irrational terms, we get

$$\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{2}{m}}+\left(A_{0l}-2A_{x^l}\right)A^{\frac{2}{m}+1}+2\left(1-\tfrac{3}{m}\right)\beta A_0A_lA^{\frac{1}{m}}$$
$$+2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A^{\frac{1}{m}+1}+2m(\beta_{0l}-2\beta_{x^l})A^{\frac{1}{m}+2}=0, \tag{3.11}$$

and

$$4m\beta_0\beta_lA^2+\left(4\beta^2A_{x^l}-8\beta\beta_0A_l-8\beta\beta_lA_0-2\beta^3A_{0l}\right)A+2\beta^2A_0A_l=0. \tag{3.12}$$

Equation (3.11) can be rewritten as

$$\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{1}{m}}+\left(A_{0l}-2A_{x^l}\right)A^{\frac{1}{m}+1}+2\left(1-\tfrac{3}{m}\right)\beta A_0A_l$$
$$+2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A+2m(\beta_{0l}-2\beta_{x^l})A^2=0. \tag{3.13}$$

Separating rational and irrational terms from the above equation, we get

$$\left(\tfrac{2}{m}-1\right)A_0A_lA^{\frac{1}{m}}+\left(A_{0l}-2A_{x^l}\right)A^{\frac{1}{m}+1}=0, \tag{3.14}$$

and

$$2\left(1-\frac{3}{m}\right)\beta A_0A_l+2m(A_l\beta_0-\beta_0A_l-\beta A_{0l}+2\beta A_{x^l})A+2m(\beta_{0l}-2\beta_{x^l})A^2=0. \tag{3.15}$$

From equation (3.14), we have the following relation

$$\left(\frac{2}{m}-1\right)A_0A_l=-\left(A_{0l}-2A_{x^l}\right)A \tag{3.16}$$

In view of equations (3.4), (3.9), and (3.15), we have the relation

$$A_0A_l=0 \tag{3.17}$$

and then equation (3.16) gives

$$A_{0l}-2A_{x^l}=0 \tag{3.18}$$

Solving equation (3.9) and (3.12), with the help of equation (3.17) and (3.18), gives us

$$\beta_lA_0+A_l\beta_0=0, \quad \textit{and} \quad \beta_0\beta_l=0 \tag{3.19}$$

Using above equations in (3.15), we have

$$\beta_{0l}-2\beta_{x^l}=0 \tag{3.20}$$

Since, $A_l\neq 0$, equation (3.17) gives $A_0=0$. In view of equation (3.6) and (3.18), we have $A_{x^l}=0$. Using the fact $A_0=0$ in equation (3.19), it follows that $\beta_0=0$. Taking vertical derivation of of the expression yields the result $\beta_{x^l}+\beta_{0l}=0$. Thus $\beta_{x^l}=0$, by equation (3.20), indicating that b_i are constants. The converse part is direct computation. This completes the proof of the theorem.

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Received: 2024-01-25
Accepted: 2024-06-27