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PEBBLING ON SHADOW GRAPH OF CYCLE

S. Kither iammal, I. Dhivviyanandam and A.Lourdusamy

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Corresponding Author: I. Dhivviyanandam

Abstract *Consider a distribution of pebbles on the vertices of graph G. A pebbling move means to remove two pebbles from a vertex and place one pebble on the adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G, v)$ such that for every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of G is the smallest number, $f(G)$, such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ is the maximum value of $f(G, v)$ overall vertices v. Consider the property that from every placement of certain number of pebbles we can move t pebbles to any vertex of the graph through a sequence of pebbling moves. The least positive integer $f_t(G)$ which satisfies the above property is known as the t-pebbling number of G. This paper discusses $f_t(G)$ for shadow graph of cycles.*

1 Introduction

One recent development in graph theory suggested by, Lagarias and Saks on pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1] and has been developed by many others including Hulbert, who published a survey of graph pebbling [2]. There have been many developments since Hulbert's survey appeared in graph pebbling. For the past 30 *years, graph pebbling is an essential tool for the transportation of consumable resources. Lourdusamy. et.al introduced the some new variants in graph pebbling and labeling. (for example [7, 10, 11, 12]*

Throughout the paper, G stands for a simple connected graph. Let us now explain the pebbling number of a vertex v in a graph G. It is the least positive integer $f(G, v)$ with the following property: with every possible configuration of $f(G, v)$ pebbles there is a possibility to move a pebble to v where pebbling move is defined as removal of two pebbles from a vertex throwing one pebble away and placing another on the adjacent vertex.

This paper is organized as follows. In section 2, we give some preliminaries which we need for the subsequent sections. In section 3, we find the pebbling number of shadow graph of a cycle. In section 4, we find the t-pebbling number of shadow graph of a cycle.

2 Preliminaries

Definition 2.1. [4] Shadow graph of a graph G, denoted by $D_2(G)$, is obtained by having 2 copies of a graph G and adding edges between the vertices of copies if their corresponding vertices are adjacent in G. The Shadow graph of a cycle with n vertices is denoted by $D_2(C_n)$. Let us denote the vertices of the first copy of C_n by $u_1, u_2, \cdots, u_n$ and the second copy of C_n by $v_1, v_2, \cdots, v_n$.

Definition 2.2. [5] Given a pebbling of G, a transmitting subgraph of G is a path $x_0, x_1, x_2, \cdots,$ x_k such that there are at least two pebbles on x_0 and at least one pebble on each of the other vertices in the path, except possibly x_k. In this case, we can transmit a pebble from x_0 to x_k.

Theorem 2.3. *[3] Let G be a graph with $diam(G) = 2$, then $f(G) = n$ or $n + 1$.*
Fact 1. *For any vertex v of a graph G, $f(v, G) \geq n$ where $n = |V(G)|$.*

Lemma 2.4. *[9] Let G be a simple connected graph $|V(G)| \geq 2$. Then $f(G[\bar{K_n}]) \leq nf(G)$*

Definition 2.5. [6] The t-pebbling number, $f_t(G)$, of a graph G, is the least n such that, for any configuration of n pebbles on the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the pebbling number of G.

Theorem 2.6. *[3] For $k \geq 1$, $f(C_{2k}) = 2^k$ and $f(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$.*

Remark 2.7. Consider the graph G with n vertices and $f(G)$ pebbles on it. We select u as the target vertex from G. If $p(u) = 1$ or $p(v) \geq 2$ where $uv \in E(G)$, then we aer done. As a result, when u is the target vertex, we always assume that $p(u) = 0$ and $p(v) \leq 1$ for all $uv \in E(G)$.

Remark 2.8. Let $p(u)$ represents the number of pebbles on the vertex u. Let $p(V_1)$ represents the total number of pebbles placed on all the vertices of the set $V_1 \subseteq V$ where V is the vertex set of the graph.

Theorem 2.9. *[8] For the shadow graph of a path P_2, the t-pebbling number is $f_t(D_2(P_2)) = 2t + 2$.*

Theorem 2.10. *[8] For the shadow graph of a path P_n, $f_t(D_2(P_n)) = t2^{n-1} + 2$, $n \geq 3$.*

Theorem 2.11. *[3] Let P_n be a path on n vertices. Then $f(P_n) = 2^{n-1}$.*

3 Pebbling number for shadow graph of cycles

Theorem 3.1. *For $D_2(C_n)$ where $3 \leq n \leq 5$, $f(D_2(C_n)) = 2n$.*

Proof. Let $V(D_2(C_n))$ be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Since $D_2(C_n)$ contains $2n$ vertices. Therefore by 2.3, $f(D_2(C_n)) \geq 2n$ and by lemma 2.4, $f(D_2(C_n)) \leq 2n$. Hence, $f(D_2(C_n)) = 2n$. □

Theorem 3.2. *For $D_2(C_6)$, $f(D_2(C_6)) = 12$.*

Proof. Let $V(D_2(C_6)) = \{u_1, u_2, \cdots, u_6, v_1, v_2, \cdots, v_6\}$ and the edge set of $D_2(C_6)$ be $E(D_2(C_6)) = \{u_ju_{j+1}, u_1u_6, v_jv_{j+1}, v_1v_6, u_jv_{j+1}, u_6v_1, v_ju_{j+1}, v_6u_1\}$ where $j = 1, 2, \cdots, 5$. Without loss of generality, let the target be u_1. Consider the paths $P_A : u_2, u_3$ and $P_B : u_5, u_6$. In the given graph $D_2(C_6)$, there are two shadow graphs of paths P_A and P_B. Then the vertex set of $D_2(P_A)$ is $\{u_2, u_3, v_2, v_3\}$ and that of $D_2(P_B)$ is $\{u_5, u_6, v_5, v_6\}$. Let $S = \{v_4, u_4\}$, $N'[S] = \{u_3, v_3\}$ and $N''[S] = \{u_5, v_5\}$. Note that $d(u_1, w) \leq 3$ where $w \in V(D_2(C_6))$. By Theorem 2.3, $f(D_2(C_6)) \geq 12$. Now we will show that $f(D_2(C_6)) \leq 12$. If we move 2 pebbles to either one of the vertices in $\{u_2, v_2\}$ in $D_2(P_A)$ or one of the vertices in $\{v_6, u_6\}$ in $D_2(P_B)$, then we are done.

Case 1: Let $p(V(D_2(P_A))) = p(V(D_2(P_B))) = 0$ and $p(v_1) \leq 1$.

It is easy to see that at least 5 pebbles can be transfered to any one of the vertices of either $D_2(P_A)$ or $D_2(P_B)$. The distance from $D_2(P_A)$ or $D_2(P_B)$ to the target is ≤ 2. Thus by Theorem 2.11, we are able to pebble the target.

Case 2: Suppose $1 \leq p(v_1) \leq 3$.

If $p(V(D_2(P_A))) \geq 6$ or $p(V(D_2(P_B))) \geq 6$, then we are done by Theorem 2.10. Otherwise if any one of the following inequalities holds,

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + \left\lfloor \frac{p(v_4)}{2} \right\rfloor + \left\lfloor \frac{p(u_4)}{2} \right\rfloor \geq 4 \tag{3.1}$$

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_B))) + \left\lfloor \frac{p(v_4)}{2} \right\rfloor + \left\lfloor \frac{p(u_4)}{2} \right\rfloor \geq 4 \tag{3.2}$$

then we are done by Theorem 2.11.
Thus, the configurations for which we cannot reach the target satisfy

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + \left\lfloor \frac{p(v_4)}{2} \right\rfloor + \left\lfloor \frac{p(u_4)}{2} \right\rfloor \leq 3 \tag{3.3}$$

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_B))) + \left\lfloor \frac{p(v_4)}{2} \right\rfloor + \left\lfloor \frac{p(u_4)}{2} \right\rfloor \leq 3 \tag{3.4}$$

Adding (3.3) and (3.4), we get

$$2\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + p(V(D_2(P_B))) + 2\left\lfloor \frac{p(v_4)}{2} \right\rfloor + 2\left\lfloor \frac{p(u_4)}{2} \right\rfloor \leq 6 \tag{3.5}$$

Note that $p(v_1) + p(V(D_2(P_A))) + p(V(D_2(P_B))) + p(v_4) + p(u_4) = 12$. Thus, to minimize the LHS of (3.5) it is sufficient to assume $p(V(D_2(P_A))) + p(V(D_2(P_B))) = 0$ and $p(v_1) = 1$ *or* 3. So, the remaining pebbles will be distributed on $u_4,\ v_4$ and exactly one of $p(v_4),\ p(u_4)$ is even. Consider $p(v_1) = 3$. Now suppose that as many pebbles as possible are moved from u_4 and v_4 to any one of the vertices in $\{u_3,\ v_3\}$ in $D_2(P_A)$ or any one of the vertices in $\{u_5,\ v_5\}$ in $D_2(P_B)$ and from the vertex v_1 to any one of the vertices in W where $W = \{u_2,\ u_6,\ v_2,\ v_6\}$. After completing all these pebbling moves we still have a pebble left at u_4 or v_4 and a pebble at v_1. Thus, from (3.5), we have $1 + p(v_4) + p(u_4) - 1 \leq 6$ and this is a contradiction. If $p(v_1) = 1$, then from (3.5), we have $p(v_4) + p(u_4) - 1 \leq 6$ which is also a contradiction.

Case 3: $p(v_1) \geq 4$.
Then obviously the target will receive a pebble because the distance from v_1 to the target is 2.
Thus, $f(D_2(C_6)) \leq 12$.
Hence, $f(D_2(C_6)) = 12$. □

Theorem 3.3. *For* $D_2(C_{2k})$ *where* $k \geq 4$, $f(D_2(C_{2k})) = 2^k + 2$.

Proof. Let $V(D_2(C_{2k})) = \{u_1,\ u_2,\ \cdots,u_{2k},\ v_1,\ v_2,\ \cdots,\ v_{2k}\}$ and $E(D_2(C_{2k})) = \{u_ju_{j+1},\ u_{2k}u_1,\ v_jv_{j+1},\ v_{2k}v_1,\ u_jv_{j+1},\ u_{2k}v_1,\ v_ju_{j+1},\ v_{2k}u_1\}$ where $j = 1,\ 2,\ \cdots,\ 2k-1$. Without loss of generality, let the target vertex be u_1. Note that $d(u_1,\ w) \leq k$ where $w \in V(D_2(C_{2k}))$. Consider the paths $P_A = u_2,\ u_3,\ \cdots,\ u_k$ and $P_B = u_{k+2},\ u_{k+3},\ \ldots,\ u_{2k}$. In the given graph $D_2(C_{2k})$ there are two shadow graphs of paths P_A and P_B. Then the vertex set of $D_2(P_A)$ is $\{u_2,\ u_3,\ \ldots,\ u_k,\ v_2,\ v_3,\ \cdots,\ v_k\}$ and that of $D_2(P_B)$ is $\{u_{k+2},\ u_{k+3},\cdots,\ u_{2k},\ v_{k+2},\ v_{k+3},\ \cdots,\ v_{2k}\}$. Let $S = \{u_{k+1},\ v_{k+1}\}$, $N'[S] = \{u_k,\ v_k\}$ and $N''[S] = \{u_{k+2},\ v_{k+2}\}$. Note that the vertices of S are at equidistance from the target. First, we will show the necessity. Placing one pebble each on $v_1,\ v_{k+1}$ and $2^k - 1$ pebbles on u_{k+1}, we cannot reach the target u_1. Hence, $f(D_2(C_{2k})) \geq 2^k + 2$. Now we prove the sufficient part.

Case 1: Let $p(V(D_2(P_A))) = p(V(D_2(P_B))) = 0$ and $p(v_1) \leq 3$.
Subcase 1.1: Let $p(V(D_2(P_A))) = p(V(D_2(P_B))) = 0$ and $p(v_1) = 0$ or 2.
Then distribute all $2^k + 2 - p(v_1)$ pebbles on u_{k+1} and v_{k+1}. Note that the total number of pebbles is even. Thus, there will be two types of pebble distribution.

Subcase 1.1. (A): Let both $p(u_{k+1})$ and $p(v_{k+1})$ be even.
It is easy to see that we can transfer at least 2^{k-1} pebbles to either one of the vertices in $\{u_k,\ v_k\}$ in $D_2(P_A)$ or one of the vertice in $\{u_{k+2},\ v_{k+2}\}$ in $D_2(P_B)$ which is at the distance $k-1$ and at most one pebble from v_1 to W where $W = \{u_2,\ u_{2k},\ v_2,\ v_{2k}\}$. By Theorem 2.11, we can move two pebbles to the vertex adjacent to the target, and hence we are done.

Subcase 1.1. (B): Let both $p(u_{k+1})$ and $p(v_{k+1})$ be odd.
Then after the pebbling moves at least $2^{k-1}-1$ pebbles will be moved to any one of the vertices of either in $\{u_k, v_k\}$ in $D_2(P_A)$ or in $\{u_{k+1}, v_{k+1}\}$ in $D_2(P_B)$ and one pebble from v_1 to W. The distance from $D_2(P_A)$ or $D_2(P_B)$ to the target is $\leq k-1$. By Theorem 2.11, we are able to pebble u_1.

Subcase 1.2: Let $p(V(D_2(P_A))) = p(V(D_2(P_B))) = 0$ and $p(v_1) = 1$ *or* 3.
Then S will receive the odd number of pebbles. So exactly one of v_{k+1} or u_{k+1} is even and the other is odd. After the sequence of pebbling moves, one of the vertices of either $D_2(P_A)$ or $D_2(P_B)$ will receive 2^{k-1} pebbles. Thus, by Theorem 2.11, we are able to pebble the target.

Thus, the configurations for which we cannot place a pebble on u_1 satisfy

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + \left\lfloor \frac{p(u_{k+1})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{2} \right\rfloor \leq 2^{k-1}-1 \tag{3.6}$$

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_B))) + \left\lfloor \frac{p(u_{k+1})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{2} \right\rfloor \leq 2^{k-1}-1 \tag{3.7}$$

Adding (3.6) and (3.7), we get

$$2\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + p(V(D_2(P_B))) + 2\left\lfloor \frac{p(u_{k+1})}{2} \right\rfloor + 2\left\lfloor \frac{p(v_{k+1})}{2} \right\rfloor \leq 2^k - 2 \tag{3.8}$$

Note that $p(v_1) + p(V(D_2(P_A))) + p(V(D_2(P_B))) + \left\lfloor \frac{p(u_{k+1})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{2} \right\rfloor = 2^k + 2$. To minimize the LHS of (3.8) it is sufficient to assume $p(v_1) = 3$ and $p(V(D_2(P_A))) = p(V(D_2(P_B))) = 0$. Now the remaining pebbles will be distributed on S and exactly one of $p(u_{k+1})$, $p(v_{k+1})$ is even. Now suppose that as many pebbles as possible are moved from the vertices u_{k+1} and v_{k+1} to either u_k or v_k in $D_2(P_A)$ or u_{k+2} or v_{k+2} in $D_2(P_B)$ and from the vertex v_1 to any one of the vertices in W. When we are done we could still have a pebble left at either u_{k+1} or v_{k+1} and a pebble in v_1. Then in LHS of (3.8), we have $p(v_1) - 1 + p(u_{k+1}) + p(v_{k+1}) - 1 = 2^k \leq 2^k - 2$ and this is a contradiction.

Case 2: Suppose $1 \leq p(V(D_2(P_A))) + p(V(D_2(P_B))) < 2^k$ and $p(v_1) \leq 1$.
If $p(V(D_2(P_A))) \geq 2^{k-1}+2$ or $p(V(D_2(P_B))) \geq 2^{k-1}+2$, then by Theorem 2.10, we are able to pebble the target. If we distribute one pebble each on the vertices in $N'[S]$ and $2^{k-1}-1$ pebbles in $V(D_2(P_A)$ or one pebble each on the vertices in $N''[S]$ and $2^{k-1}-1$ pebbles in $V(D_2(P_B)$ then by Theorem 2.11, we are done. (i.e. If $p(D_2(P_A)) = 2^{k-1}+1$ or $p(D_2(P_B)) = 2^{k-1}+1$, then by Theorem 2.11, we are able to pebble the target). If $p(v_1) \leq 1, 1 \leq p(V(D_2(P_A))) \leq 2^{k-1}$ and $1 \leq p(V(D_2(P_B))) \leq 2^{k-1}$ then $p(S) \geq 2$. Consider the pebbling moves towards $D_2(P_A)$. After the sequence of pebbling moves from $D_2(P_B)$ and S to $D_2(P_A)$ and together with the pebbles already on $D_2(P_A)$ there will be at least $2^{k-1}+1$ pebbles on $D_2(P_A)$ and so by Theorem 2.11, we are able to pebble the target. The case of moving pebbles towards $D_2(P_B)$ is symmetric to the case with that of $D_2(P_A)$.

Case 3: If $p(v_1) \geq 4$.
By Theorem 2.11, we are able to pebble the target by using the pebbling moves since the distance from v_1 to the target is 2. Thus, $f(D_2(C_{2k})) \leq 2^k + 2$.
Therefore, the pebbling number of the shadow graph of the cycle C_{2K} is $f(D_2(C_{2K})) = 2^k + 2$. □

Theorem 3.4. *For $D_2(C_{2k+1})$ where $k \geq 3$, $f(D_2(C_{2k+1})) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$.*

Proof. Let $V(D_2(C_{2k+1})) = \{u_1, u_2, \cdots, u_{2k+1}, v_1, v_2, \ldots, v_{2k+1}\}$ and $E(D_2(C_{2K+1})) = \{u_j u_{j+1}, u_{2k+1}u_1, v_j v_{j+1}, v_{2k+1}v_1, u_j v_{j+1}, u_{2k+1}v_1, v_j u_{j+1}, v_{2k+1}u_1\}$ where $j = 1, 2, \cdots, 2k$. Without loss of generality, let the target vertex be u_1. Note that $d(u_1, w) \leq k$ where $w \in V(D_2(C_{2k+1}))$. Consider the paths $P_A = u_2, u_3, \ldots, u_k$ and $P_B = u_{k+3}, u_{k+4}, \ldots, u_{k+2}$. In the given graph $D_2(C_{2k+1})$ there are two shadow graphs of paths P_A and P_B. Let $V(D_2(P_A)) =$

$\{u_2,\ u_3,\cdots,\ u_k,\ v_2,\ v_3,\ldots,\ v_k\}, V(D_2(P_B)) = \{u_{k+3},\ u_{k+4},\ldots,\ u_{2k+1},$ $v_{k+3},\ v_{k+4},\ \cdots,\ v_{2k+1}\}$ and $S = \{u_{k+1},\ u_{k+2},\ v_{k+1},\ v_{k+2}\},\ N'[S] = \{u_k, v_k\}$ and $N''[S] = \{u_{k+3}, v_{k+3}\}$.

Case 1: k is odd.
Place $\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$ pebbles at u_{k+1} and $\left\lfloor \frac{2^{k+1}}{3} \right\rfloor$ pebbles at u_{k+2} and 1 pebble each on $v_{k+1},\ v_{k+2}$ and v_1. It is easy to see that at most $2^{k-1}-1$ pebbles can be moved to u_k or v_k (respectively u_{k+3} or v_{k+3}) which is at the distance $k-1$ from the target and so we cannot reach the target u_1. Hence, $f(D_2(C_{2k+1})) \geq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$. Now we show that $f(D_2(C_{2k+1})) \leq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$.

Subcase 1.1: Let $p(V(D_2(P_A))) = p(V(D_2(P_B))) = 0$ and $p(v_1) \leq 3$.
Then $p(S) \geq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$. Then distribute all $2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4 - p(v_1)$ pebbles on S. Note that $p(S)$ is even when $p(v_1) = 0$ or 2 and $p(S)$ is odd when $p(v_1) = 1$ or 3.
If one of the following inequalities holds,

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + \left\lfloor \frac{p(u_{k+2})}{4} \right\rfloor + \left\lfloor \frac{p(v_{k+2})}{4} \right\rfloor + \left\lfloor \frac{p(u_{k+1})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{2} \right\rfloor \geq 2^{k-1} \tag{3.9}$$

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_B))) + \left\lfloor \frac{p(u_{k+2})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+2})}{2} \right\rfloor + \left\lfloor \frac{p(u_{k+1})}{4} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{4} \right\rfloor \geq 2^{k-1} \tag{3.10}$$

Then by Theorem 2.11, we are done.
Thus, the configurations for which we cannot reach the target satisfy

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + \left\lfloor \frac{p(u_{k+2})}{4} \right\rfloor + \left\lfloor \frac{p(v_{k+2})}{4} \right\rfloor + \left\lfloor \frac{p(u_{k+1})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{2} \right\rfloor \leq 2^{k-1}-1 \tag{3.11}$$

$$\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_B))) + \left\lfloor \frac{p(u_{k+2})}{2} \right\rfloor + \left\lfloor \frac{p(v_{k+2})}{2} \right\rfloor + \left\lfloor \frac{p(u_{k+1})}{4} \right\rfloor + \left\lfloor \frac{p(v_{k+1})}{4} \right\rfloor \leq 2^{k-1}-1 \tag{3.12}$$

Adding (3.11) and (3.12), we get

$$2\left\lfloor \frac{p(v_1)}{2} \right\rfloor + p(V(D_2(P_A))) + p(V(D_2(P_B))) + \frac{3}{4}\lfloor p(u_{k+1}) + p(v_{k+1}) + p(u_{k+2}) + p(v_{k+2}) \rfloor \leq 2^k - 2. \tag{3.13}$$

Note that $p(v_1) + p(V(D_2(P_A))) + p(V(D_2(P_B))) + p(S) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$. To minimize the LHS of (3.13) it is sufficient to assume $p(v_1) = 3$. Now the remaining pebbles will be distributed in S and so at most 3 vertices in S will be distributed with an odd number of pebbles and the remaining vertices will receive an even number of pebbles.

Now suppose that as many pebbles as possible are moved from the vertices in S to either u_k or v_k in $D_2(P_A)$ or u_{k+2} or v_{k+2} in $D_2(P_B)$ and from the vertex v_1 to any one of the vertices in W where $W = \{u_2,\ u_{2k},\ v_2,\ v_{2k}\}$. When we are done we could still have a pebble left in S and a pebble at v_1. Without loss of generality, assume $p(v_{k+2})$ is even. Then we move as many pebbles as possible from S to $D_2(P_A)$. Thus, $p(u_{k+2}) \equiv 1\ (mod\ 4),\ p(u_{k+1}) \equiv 1\ (mod\ 2)$ and $p(v_{k+1}) \equiv 1\ (mod\ 2)$ and if we move as many pebbles as possible from S to $D_2(P_B)$, $p(u_{k+2}) \equiv 1\ (mod\ 2),\ p(u_{k+1}) \equiv 1\ (mod\ 4)$, and $p(v_{k+1}) \equiv 3\ (mod\ 4)$ also $p(v_1) \equiv 1\ (mod\ 2)$. Thus, from (13) we have $\frac{3}{4}(p(u_{k+1}) + p(v_{k+1}) + p(u_{k+2}) + p(v_{k+2})) + p(v_1) - \frac{13}{4} \leq 2^k - 2$. (The - $\frac{13}{4}$ comes from the possible pebbles left behind at v_1 and S). But $p(v_1) + p(S) \geq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4 \geq 2\left(\frac{2^{k+1}-2}{3}\right) + 4 = \frac{4}{3}(2^k - 1) + 4$. So $\frac{3}{4}(p(u_{k+1}) + p(v_{k+1}) + p(u_{k+2}) + p(v_{k+2})) +$

$p(v_1) - \frac{13}{4} = \frac{3}{4}(p(u_{k+1}) + p(v_{k+1}) + p(u_{k+2}) + p(v_{k+2}) + p(v_1)) + \frac{1}{4}p(v_1) - \frac{13}{4} \geq 2^k - 1 + 3 + \frac{3}{4} - \frac{13}{4} = 2^k - \frac{1}{2}$ and this is a contradiction. Thus, we are able to pebble the target.

Subcase 1.2: Let $1 \leq p(V(D_2(P_A))) + p(V(D_2(P_B))) \leq 2k+1$ and $p(v_1) \leq 1$.
If $p(V(D_2(P_A))) \geq 2^{k-1} + 2$ or $p(V(D_2(P_B))) \geq 2^{k-1} + 2$ then we are able to pebble the target by Theorem 2.11. If we distribute one pebble each on the vertices in $N'[S]$ and $2^{k-1} - 1$ pebbles in $V(D_2(P_A)$ or one pebble each on the vertices in $N''[S]$ and $2^{k-1} - 1$ pebbles in $V(D_2(P_B)$ then by Theorem 2.11, we are done. (i.e. If $p(V(D_2(P_A))) = 2^{k-1} + 1$ or $p(V(D_2(P_B))) = 2^{k-1} + 1$ then we move a pebble to the target by Theorem 2.11). If $p(v_1) \leq 1, 1 \leq p(V(D_2(P_A))) \leq 2^{k-1}$ and $1 \leq p(V(D_2(P_B))) \leq 2^{k-1}$ then $p(S) \geq 2$. Consider the pebbling moves towards $D_2(P_A)$. After the sequence of pebbling moves from $D_2(P_B)$ and S to $D_2(P_A)$ and together with the pebbles already on $D_2(P_A)$ there will be at least $2^{k-1} + 1$ pebbles on $D_2(P_A)$ and so by Theorem 2.11, we are able to pebble the target. The case of moving pebbles to the vertices in $D_2(P_B)$ is symmetric to the case with that of $D_2(P_A)$.

Subcase 1.3: Let $p(v_1) \geq 4$.
Then obviously the target will receive a pebble. Since the distance from v_1 to the target is 2.
Hence, the pebbling number of the shadow graph of the odd cycle is $f(D_2(C_{2k+1})) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$ when k is odd.

case 2: When k is even.
Place $\left\lfloor \frac{2^{k+1}}{3} \right\rfloor - 1$ pebbles at u_{k+1} and $\left\lfloor \frac{2^{k+1}}{3} \right\rfloor - 1$ pebbles at u_{k+2}, 3 pebbles at v_{k+1} and 1 pebble each on v_1, v_{k+2}. It is easy to see that at most $2^{k-1} - 1$ pebbles can be moved to $u_k or v_k$ (respectively u_{k+3} or v_{k+3}) which is at the distance $k - 1$ from the target and we cannot reach the target. Hence, $(D_2(C_{2k+1})) \geq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$. Now we show that $f(D_2(C_{2k+1})) \leq 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4$. The proof follows from case 1. □

4 The t-pebbling number for shadow graph of cycles.

In this section, we find the t-pebbling number for $D_2(C_n)$ graphs.

Theorem 4.1. *For $D_2(C_{2k})$ where $k \geq 4$, the t-pebbling number is $f_t(D_2(C_{2k})) = t2^k + 2$.*

Proof. Suppose we have $t2^k + 1$ pebbles. Placing $t2^k - 1$ pebbles on u_{k+1} and one pebble each on the vertices v_1 and v_{k+1}, we cannot move t pebbles to the target vertex u_1. Hence, $f_t(D_2(C_{2k})) \geq t2^k + 2$. Now we prove $f_t(D_2(C_{2k})) \leq t2^k + 2$ by induction on t. Let us consider any distribution of $t2^k + 2$ pebbles on the vertices of $D_2(C_{2k})$. If $t = 1$, then the theorem is true by Theorem 3.3. Assume that the theorem is true for $2 \leq t' < t$. Let w be any target vertex.

Case 1: Let $p(w) = 0$.
As the graph $D_2(C_{2k})$ contains at least $2^{k+1} + 2$ pebbles we can move one pebble to any target vertex at a cost of at most 2^k pebbles. Hence, the remaining number of pebbles distributed on the vertices of the graph other than the vertex w is at least $f_t(D_2(C_{2k})) - 2^k = t2^k + 2 - 2^k = (t-1)2^k + 2 = f_{t-1}(D_2(C_{2k}))$. Therefore, by induction, we can move $t - 1$ additional pebbles to w.

Case 2: Suppose $p(w) = y$, where $1 \leq y \leq t - 1$.
Then the total number of pebbles distributed on the vertices of the graph is $f_t(D_2(C_{2k})) - y = t2^k + 2 - y$. Since $1 \leq y \leq t - 1$. $t2^k + 2 - y \geq 2^k(t - y) + 2 = f_{t-y}(D_2(C_{2k}))$. Thus, we can move $t - y$ additional pebbles to the target. Hence, we are done. □

Theorem 4.2. *For $D_2(C_{2k+1})$ where $k \geq 3$, the t-pebbling number is $f_t(D_2(C_{2k+1})) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 4 + (t-1)2^k$.*

Proof. To prove this theorem let us consider the following 2 cases.
Case 1: When k is odd.

Suppose we have a distribution of $2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 3 + (t-1)\,2^k$ pebbles. Placing $\left\lfloor\frac{2^{k+1}}{3}\right\rfloor$ pebbles on u_{k+1}, $\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + (t-1)\,2^k$ on u_{k+2} and 1 pebble each on the vertices $v_{k+1},\ v_{k+2}, v_1$, we cannot move t pebbles to the target vertex u_1. Hence, $f_t\left(D_2\left(C_{2k+1}\right)\right) \geq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)2^k$. Now we prove $f_t\left(D_2\left(C_{2k+1}\right)\right) \leq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)2^k$ by induction on t. Let us consider any distribution $2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)2^k$ of pebbles on the vertices of $D_2\left(C_{2k+1}\right)$. If t = 1, then the theorem is true by Case 1 in Theorem 3.4. Assume that the theorem is true for $2 \leq t^{`} < t$. Let w be any target vertex.

Subcase 1.1: Let $p\,(w) = 0$.
As the graph $D_2\left(C_{2k+1}\right)$ contains at least $2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)2^k$ pebbles and we use 2^k pebbles to place a pebble at our target vertex w, the remaining number of pebbles is $2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)\,2^k - 2^k = 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-2)\,2^k = f_{t-1}\left(D_2\left(C_{2k+1}\right)\right)$. By induction we place $t-1$ pebbles at w.

Subcase 1.2: Suppose $p\,(w) = y$, where $1 \leq y \leq t-1$.
Then the total number of pebbles distributed on the vertices of the graph is $f_t\left(D_2\left(C_{2k+1}\right)\right) - y = 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)\,2^k - y$. Since $1 \leq y \leq t-1$. $2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)\,2^k - y \geq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-(y+1))2^k = f_{t-y}\left(D_2\left(C_{2k+1}\right)\right)$. Thus, we can move $t-y$ additional pebbles to the target. Hence, we are done.

Case 2: When k is even.
Placing $\left\lfloor\frac{2^{k+1}}{3}\right\rfloor - 1$ pebbles on u_{k+1} , $\left\lfloor\frac{2^{k+1}}{3}\right\rfloor - 1 + (t-1)2^k$ on u_{k+2}, 3 pebbles at v_{k+1} and 1 pebble each on v_1, v_{k+2}, we cannot move t pebbles to the target vertex u_1. Hence, $f_t\left(D_2\left(C_{2k+1}\right)\right) \geq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)2^k$. Now we prove $f_t\left(D_2\left(C_{2k}\right)\right) \leq 2\left\lfloor\frac{2^{k+1}}{3}\right\rfloor + 4 + (t-1)2^k$. Proof follows from Case 1. □

5 Conclusion

This paper aims is to obtain the pebbling and t-pebbling number of shadow graph of cycles. Using this results we can find the pebbling number monophonic pebbling number of different types of shadow graphs. Therefore, the results of this work is interesting and capable to develop its study in the future.

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Author information

S. Kither iammal, Department of Mathematics, Jayaraj Annapackiam College for women (Autonomous), Periyakulam Tamilnadu., India.
E-mail: cathsat86@gmail.com

I. Dhivviyanandam, Department of Mathematics, North Bengal st. Xavier's college, Rajganj, west Bengal., India.
E-mail: divyanasj@gmail.com

A.Lourdusamy, Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai-627002, Tamil Nadu., India.
E-mail: lourdusamy15@gmail.com

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