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REAL COMPACTNESS VIA REAL MAXIMAL IDEALS OF $B_1(X)$

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Abstract *In this paper, we construct a class of ideals of $B_1(X)$ from proper ideals of $C(X)$, establishing a one-to-one correspondence between the class of real maximal ideals of $C(X)$ and those of $B_1(X)$. The collection of all real maximal ideals of $B_1(X)$, with the hull-kernel topology, is shown to be homeomorphic to the space of real maximal ideals of $C(X)$, endowed with a topology finer than the subspace topology induced from its structure space. It is also proven that a Tychonoff space is real compact if and only if every real maximal ideal of $B_1(X)$ is fixed. As a consequence, within the class of real compact T_4 spaces whose points are G_δ, $B_1(X) = {B_1}^*(X)$ if and only if X is finite.*

1 Introduction

Baire one functions are named in honor of René-Louis Baire (1874-1932), a French mathematician. Let X be a topological space. A function $f : X \to \mathbb{R}$ is called a $Baire\ one\ function$ if f is the pointwise limit of a sequence of continuous functions, that is, if there is a sequence $\{f_n\}$ of real-valued continuous functions on X such that for every $x \in X, f(x) = \lim\limits_{n\to\infty} f_n(x)$. Various characterizations of Baire one functions defined on metric spaces have been established by different authors [3], [5]. Inspired by the research on rings of continuous functions, in [1], we introduced two rings $B_1(X)$ and ${B_1}^*(X)$ consisting respectively of real valued Baire one functions and bounded Baire one functions on a topological space X. It has been observed in [1] that $B_1(X)$ is a commutative lattice ordered ring with unity with respect to the usual addition and multiplication of functions and it is an over-ring of the ring $C(X)$ of all real valued continuous functions on X. The main purpose of this paper is to establish that the class of all real maximal ideals of $B_1(X)$ is in one-one correspondence with the class of all real maximal ideals of $C(X)$. In fact, defining a sort of 'extension' of an ideal I of $C(X)$ in $B_1(X)$ (denoted by I_B), we show that the contraction $I_B \cap C(X)$ coincides with I if and only if I is a real maximal ideal in $C(X)$. That I_B is a real maximal ideal in $B_1(X)$, for any real maximal ideal I in $C(X)$ determines the said one-to-one correspondence. Moreover, it is proved that the collection $\mathcal{RM}(B_1(X))$ of all real maximal ideals of $B_1(X)$ with the subspace topology of the structure space of $B_1(X)$ (i.e., the hull-kernel topology) is homeomorphic to the collection $\mathcal{RM}(C(X))$ of all real maximal ideals of $C(X)$ equipped with a topology finer than the subspace topology of the structure space of $C(X)$.

The class of topological spaces on which every Baire one function is bounded, is yet to be determined completely. In Section 3, we prove a necessary and sufficient condition for $B_1(X)$ to coincide with ${B_1}^*(X)$ within the class of all T_4, real compact spaces whose points are G_δ.

To ensure this paper is self-contained, we now recall some known terminologies and facts. A zero set of $f \in B_1(X)$ is defined as usual by a set of the form $Z(f) = \{x \in X : f(x) = 0\}$. As an analogue of z-filter (or, z-ultrafilter) on X, we introduced in [2] the Z_B-filter (or respectively, Z_B-ultrafilter) on X, thereby investigating the duality between ideals (maximal ideals) of $B_1(X)$ and Z_B-filters (respectively, Z_B-ultrafilters) on X. The above mentioned duality existing between ideals in $B_1(X)$ and Z_B-filters on X is manifested by the fact that if I is an ideal in $B_1(X)$ then $Z_B[I] = \{Z(f) : f \in B_1(X)\}$ is a Z_B-filter on X and dually for a Z_B-filter $\mathscr{F}$ on X, $Z_B^{-1}[\mathscr{F}] = \{f \in B_1(X) : Z(f) \in \mathscr{F}\}$ is a proper ideal in $B_1(X)$. The assignment $M \mapsto Z_B[M]$ is a bijection from the set of all maximal ideals in $B_1(X)$ and to the family of all Z_B-ultrafilters on X. In the same paper [2], defining suitably the residue class fields $B_1(X)/M$, the concept of real and hyper-real maximal ideals of $B_1(X)$ was introduced. A maximal ideal M of $B_1(X)$ is called real if $B_1(X)/M \cong \mathbb{R}$ and in such case $B_1(X)/M$ is called real residue class field [2]. If M is not real then it is called hyper-real and $B_1(X)/M$ is called hyper-real residue class field [2]. Considering the structure space $\mathcal{M}(B_1(X))$ of $B_1(X)$, i.e., the collection $\mathcal{M}(B_1(X))$ of all maximal ideals of $B_1(X)$ with respect to the hull-kernel topology, we get the subspace topology on the collection $\mathcal{RM}(B_1(X))$ of all real maximal ideals of $B_1(X)$. We show that the subspace topology via the aforesaid bijection induces a topology on $\mathcal{RM}(C(X))$ of all real maximal ideals of $C(X)$ which is finer than the hull-kernel topology on $\mathcal{RM}(C(X))$.

Throughout this paper, unless stated otherwise, we consider X to be any Hausdorff topological space. We use the notation $f_n \xrightarrow{p.w.} f$ to denote $\{f_n\}$ pointwise converges to f.

Theorem 1.1 ([4, Theorem 5.14]). *For a maximal ideal M of $C(X)$ the following statements are equivalent:*

(i) M *is a real maximal ideal.*

(ii) $Z[M]$ *is closed under countable intersection.*

(iii) $Z[M]$ *has countable intersection property.*

An analogue of this theorem in the context of Baire one functions is the following:

Theorem 1.2 ([2, Theorem 4.26]). *For a maximal ideal M of $B_1(X)$ the following statements are equivalent:*

(i) M *is a real maximal ideal.*

(ii) $Z_B[M]$ *is closed under countable intersection.*

(iii) $Z_B[M]$ *has countable intersection property.*

2 A one-one correspondence between the real maximal ideals of $C(X)$ and the real maximal ideals of $B_1(X)$

It is easy to observe that for each proper ideal I of $C(X)$, $I_B = \{f \in B_1(X) : \exists\{f_n\} \subseteq I$ such that $f_n \xrightarrow{p.w.} f\}$ is an ideal of $B_1(X)$ such that $I \subseteq I_B \cap C(X)$. As a natural example, we obtain that $(M_p)_B$ is a fixed maximal ideal of $B_1(X)$ [2], where $M_p = \{f \in C(X) : f(p) = 0\}$ is a fixed maximal ideal of $C(X)$ [4] and this example prompts us to prove a more general result later in this section.

Example 2.1. For each $p \in X$, $(M_p)_B = \widehat{M_p} \equiv \{f \in B_1(X) : f(p) = 0\}$.
For each $f \in (M_p)_B$, there exists $\{f_n\} \subseteq M_p$ such that $f_n \xrightarrow{p.w.} f$. This implies $f_n(p) = 0$, for all $n \in \mathbb{N}$ and therefore, $f(p) = 0$, i.e., $(M_p)_B \subseteq \widehat{M_p}$. On the other hand, $f \in \widehat{M_p}$ implies $f(p) = 0$. Since $f \in B_1(X)$, there exists a sequence $\{g_n\} \subseteq C(X)$ such that $g_n \xrightarrow{p.w.} f$. Define $f_n = g_n - g_n(p)$, for all $n \in \mathbb{N}$. Clearly, $f_n(p) = 0$, for all $n \in \mathbb{N}$. Also, $f_n \xrightarrow{p.w.} f$. Therefore, $f \in (M_p)_B$ and $\widehat{M_p} \subseteq (M_p)_B$. Hence, $(M_p)_B = \widehat{M_p}$.

Theorem 2.2. *If I is an absolutely convex ideal in $C(X)$ then I_B is an absolutely convex ideal in $B_1(X)$.*

Proof. We first prove that I_B is a convex ideal in $B_1(X)$. If so, then $f \in I_B$ implies that there is $\{f_n\} \subseteq I$ such that $f_n \xrightarrow{p.w.} f$ and hence, $|f_n| \xrightarrow{p.w.} |f|$. As I is absolutely convex, we have $\{|f_n|\} \subseteq I$, which ensures $|f| \in I_B$. In such a case, I_B becomes absolutely convex.
Let $f, g \in B_1(X)$ such that $0 \leq f \leq g$ and $g \in I_B$. Then there is a sequence $\{f_n\}$ in $C(X)$ and $\{g_n\} \subseteq I$ such that $f_n \xrightarrow{p.w.} f$ and $g_n \xrightarrow{p.w.} g$. Choosing $h_n = f_n \wedge g_n$, we observe the following:

(i) $h_n \xrightarrow{p.w.} f \wedge g = f$.

(ii) For each $n \in \mathbb{N}$, $0 \leq h_n \leq g_n$ and $g_n \in I$ implies that $h_n \in I$ (since I is absolutely convex).

Hence, $f \in I_B$ and this proves that I_B is a convex ideal in $B_1(X)$. □

For any proper ideal I of $C(X)$, it is clear that $I \subseteq I_B \cap C(X)$. In the following theorem we show that the equality holds precisely for the class of all real maximal ideals of $C(X)$.

Theorem 2.3. *$M \in \mathcal{RM}(C(X))$ if and only if $M = M_B \cap C(X)$.*

Proof. Let M be a real maximal ideal of $C(X)$. Clearly, $M \subseteq M_B \cap C(X)$. Now let $g \in M_B \cap C(X)$. There exists $\{g_n\} \subseteq M$ such that $g_n \xrightarrow{p.w.} g$. Since M is real and $g_n \in M$, for all $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} Z(g_n) \in Z[M]$. Also, $\bigcap_{n=1}^{\infty} Z(g_n) \subseteq Z(g)$. Hence, $Z(g) \in Z[M]$. By maximality of M it follows that $g \in M$. Therefore, $M_B \cap C(X) \subseteq M$ and it implies that $M = M_B \cap C(X)$.
Conversely, let M be a maximal ideal of $C(X)$ such that $M = M_B \cap C(X)$.
Consider any countable family $\{Z(g_n) : n \in \mathbb{N}\}$ of $Z[M]$. By maximality of M, $g_n \in M$, for all $n \in \mathbb{N}$.
We now construct a sequence $\{s_n\}$ as follows : $s_n = \sum_{j=1}^{n} \left(\frac{1}{2^j} \wedge |g_j|\right)$, for each $n \in \mathbb{N}$. Certainly, for each j, $Z(g_j) = Z\left(\frac{1}{2^j} \wedge |g_j|\right)$ implies that $\frac{1}{2^j} \wedge |g_j| \in M$. M being an ideal, finite sum of each such member will also lie within M. This means $s_n \in M$, for all $n \in \mathbb{N}$.
Now $s = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \wedge |g_n|\right)$ is the uniform limit of the sequence $\{s_n\}$ of continuous functions and therefore, $s \in C(X)$. Again, $\{s_n\} \subseteq M$ ensures that $s \in M_B \cap C(X) = M$. So, $Z(s) \neq \emptyset$. Following the arguments used in [4, 1.14 (a)] we obtain $\bigcap_{n=1}^{\infty} Z(g_n) = Z(s) \neq \emptyset$. Therefore, by Theorem 1.1 M is real. □

That M_B is not even a proper ideal of $B_1(X)$ when M is hyper-real in $C(X)$ is observed in the next theorem.

Theorem 2.4. *If M is a hyper-real maximal ideal in $C(X)$ then $M_B = B_1(X)$.*

Proof. If M is hyper-real then by Theorem 2.3 $M_B \cap C(X) \neq M$. But for any ideal I, $I \subseteq I_B \cap C(X)$ holds. Therefore, $M \subsetneq M_B \cap C(X)$. Since M is maximal, $M_B \cap C(X) = C(X)$. Hence, $C(X) \subseteq M_B$, i.e., $1 \in M_B$. This proves $M_B = B_1(X)$. □

Theorem 2.5. *If $M \in \mathcal{RM}(C(X))$ then $M_B \in \mathcal{RM}(B_1(X))$.*

Proof. Let $f \in B_1(X) \setminus M_B$. Consider the ideal J generated by $M_B \cup \{f\}$.
Now $f \in B_1(X)$ implies that there exists $\{f_n\} \subseteq C(X)$ such that $f_n \xrightarrow{p.w.} f$.
Since M is a real maximal ideal in $C(X)$, for each $f_n \in C(X)$, there exists some $r_n \in \mathbb{R}$ such that $M(f_n) = M(r_n)$ and so, $f_n = r_n$ on $Z_n = Z(f_n - r_n) \in Z[M]$. As $Z = \bigcap_{n=1}^{\infty} Z_n \in Z[M]$, for each $n \in \mathbb{N}$, $f_n = r_n$ on Z. As a consequence, f is constant (say, r) on $Z \in Z[M] \subseteq Z_B[M_B]$, where $r = \lim_{n\to\infty} r_n$.
Since $Z \subseteq Z(f_n - r_n)$ implies that $Z(f_n - r_n) \in Z[M]$ and M is a z-ideal in $C(X)$, we have $f_n - r_n \in M$. By definition of M_B, $f - r \in M_B$. Now r will be a non-zero real number as

$f \notin M_B$. But $r = f - (f - r) \in J$ and $r \neq 0$ implies that $J = B_1(X)$. So, M_B is a maximal ideal of $B_1(X)$ such that $f - r \in M_B$, i.e., $M_B(f) = M_B(r)$, for some $r \in \mathbb{R}$. If $f \in M_B$ then $M_B(f) = M_B(0)$ and this proves that $M_B \in \mathcal{RM}(B_1(X))$. □

Theorem 2.6. *If $\widehat{M} \in \mathcal{RM}(B_1(X))$ then $\widehat{M} \cap C(X) \in \mathcal{RM}(C(X))$.*

Proof. Let $\widehat{M} \in \mathcal{RM}(B_1(X))$. Then for each $f \in B_1(X)$, there exists $r_f \in \mathbb{R}$ such that $f - r_f \in \widehat{M}$. In particular, for any $f \in C(X)$, there is $r_f \in \mathbb{R}$ such that $f - r_f \in \widehat{M}$. So, $f - r_f \in \widehat{M} \cap C(X) = M$ (say). We now define a function $\phi : C(X)/M \to \mathbb{R}$ by $M(f) \mapsto r_f$, whenever $f - r_f \in M$. We claim that ϕ is an isomorphism.
$M(f) = M(g) \Leftrightarrow f - g \in M$. If $\phi(M(f)) = r_f$ and $\phi(M(g)) = r_g$ then $f - r_f, g - r_g \in M$, i.e., $(f - g) - (r_f - r_g) \in M$. Since, $f - g \in M$ and M is an ideal, it follows that $r_f - r_g \in M$ - a contradiction to the fact that M is proper, unless $r_f - r_g = 0$. Hence, ϕ is well defined.
Now $\phi(M(f)) = \phi(M(g))$ implies that $r_f = r_g$, where $f - r_f, g - r_g \in M$. Therefore, $f - g = (f - r_f) - (g - r_g) \in M$ which in turn gives $M(f) = M(g)$, proving ϕ to be one-one. The function ϕ is clearly onto, as $\phi(M(r)) = r$, for each $r \in \mathbb{R}$. By routine arguments we easily see that ϕ is indeed a ring homomorphism. Hence, ϕ is a ring isomorphism and therefore, $M \in \mathcal{RM}(C(X))$. □

Corollary 2.7. *If $\widehat{M} \in \mathcal{RM}(B_1(X))$ then $(\widehat{M} \cap C(X))_B = \widehat{M}$.*

Proof. As $\widehat{M} \in \mathcal{RM}(B_1(X))$, $\widehat{M} \cap C(X) \in \mathcal{RM}(C(X))$ (by Theorem 2.6). Using Theorem 2.5, $(\widehat{M} \cap C(X))_B \in \mathcal{RM}(B_1(X))$. Since $(\widehat{M} \cap C(X))_B$ is a maximal ideal, it is enough to show that $(\widehat{M} \cap C(X))_B \subseteq \widehat{M}$.
Let $g \in (\widehat{M} \cap C(X))_B$. Then there exists $\{g_n\} \subseteq \widehat{M} \cap C(X)$ such that $g_n \xrightarrow{p.w.} g$. So, $Z(g) \supseteq \bigcap_{i=1}^{\infty} Z(g_n)$. As $Z_B[\widehat{M}]$ is a Z_B-ultrafilter and $\widehat{M}$ is real, it follows that $Z(g) \in Z_B[\widehat{M}]$. Hence, $g \in \widehat{M}$ and therefore $(\widehat{M} \cap C(X))_B \subseteq \widehat{M}$. □

In view of Corollary 2.7, Theorem 2.5 and Theorem 2.6, we get a one-one correspondence between $\mathcal{RM}(C(X))$ and $\mathcal{RM}(B_1(X))$

Theorem 2.8. *If $\psi : \mathcal{RM}(C(X)) \to \mathcal{RM}(B_1(X))$ is defined by $M \mapsto M_B$ then ψ is a bijection.*

Proof. Let $\widehat{M}$ be any member of $\mathcal{RM}(B_1(X))$. Therefore, by Corollary 2.7 we get $(\widehat{M} \cap C(X))_B = \widehat{M}$, where $\widehat{M} \cap C(X) \in \mathcal{RM}(C(X))$ (By Theorem 2.6). Hence, for $\widehat{M} \in \mathcal{RM}(B_1(X))$ we get $\widehat{M} \cap C(X) \in \mathcal{RM}(C(X))$ such that $\psi(\widehat{M} \cap C(X)) = \widehat{M}$. This proves that ψ is surjective. To show that ψ is injective we assume $\psi(\widehat{M}) = \psi(\widehat{N})$. This implies $(\widehat{M})_B = (\widehat{N})_B$. Now by applying Theorem 2.3, we get $\widehat{M} = (\widehat{M})_B \cap C(X) = (\widehat{N})_B \cap C(X) = \widehat{N}$. Therefore, ψ is injective and hence, it is a bijection. □

Corollary 2.9. $|\mathcal{RM}(C(X))| = |\mathcal{RM}(B_1(X))|$.

It is well known that $\{\widehat{\mathscr{M}_f} : f \in B_1(X)\}$, where each $\widehat{\mathscr{M}_f} = \{M \in \mathcal{M}(B_1(X)) : f \in M\}$, forms a base for closed sets for the hull-kernel topology on $\mathcal{M}(B_1(X))$ and certainly $\mathcal{RM}(B_1(X))$ is a subspace of $\mathcal{M}(B_1(X))$. In the following theorem we show that the bijection ψ obtained above becomes a homeomorphism if $\mathcal{RM}(C(X))$ is endowed with a finer topology than the subspace topology induced from the hull-kernel topology of $\mathcal{M}(C(X))$.

Theorem 2.10. *Let (X, τ) be a Tychonoff space. Then for each $f \in B_1(X)$, the collection ${\mathscr{M}_f}^* = \{M \in \mathcal{RM}(C(X)) : f \in M_B\}$ forms a base for closed sets for some topology σ on $\mathcal{RM}(C(X))$ which is finer than the subspace topology of the structure space of $\mathcal{M}(C(X))$. Moreover, $\psi : (\mathcal{RM}(C(X)), \sigma) \to \mathcal{RM}(B_1(X))$ given by $M \mapsto M_B$ is a homeomorphism.*

Proof. To prove $\mathscr{B}^* = \{{\mathscr{M}_f}^* : f \in B_1(X)\}$ forms a base for closed sets for some topology σ on $\mathcal{RM}(C(X))$, it is enough to show that $\emptyset \in \mathscr{B}^*$ and $\mathscr{B}^*$ is closed under finite union. It is easy to observe that, $\emptyset = {\mathscr{M}_1}^* \in \mathscr{B}^*$. Now let ${\mathscr{M}_f}^*, {\mathscr{M}_g}^* \in \mathscr{B}^*$, for some $f, g \in B_1(X)$. Take any $M \in {\mathscr{M}_f}^* \cup {\mathscr{M}_g}^*$. Therefore, $gf \in M_B$ and $M \in {\mathscr{M}_{fg}}^*$. This implies ${\mathscr{M}_f}^* \cup {\mathscr{M}_g}^* \subseteq {\mathscr{M}_{fg}}^*$.

On the other hand, if we take any memeber $M \in \mathscr{M}_{fg}{}^*$ then we get $fg \in M_B$. Now by Theorem 2.5 M_B is a maximal ideal and hence $f \in M_B$ or $g \in M_B$, i.e., $M \in \mathscr{M}_f{}^* \cup \mathscr{M}_g{}^*$. So $\mathscr{M}_{fg}{}^* \subseteq \mathscr{M}_f{}^* \cup \mathscr{M}_g{}^*$. This proves that $\mathscr{M}_f{}^* \cup \mathscr{M}_g{}^* = \mathscr{M}_{fg}{}^*$ and hence $\mathscr{B}^*$ is closed under finite union.

Now to prove that ψ is a homeomorphism, we need to show ψ is bijective and exchanges the basic closed sets of $(\mathcal{RM}(C(X)), \sigma)$ and $\mathcal{RM}(B_1(X))$. The map ψ is bijective is already proved in Thoeorem 2.8. Now for any $f \in B_1(X)$, $\psi(\mathscr{M}_f{}^*) = \{\psi(M) : f \in M_B\} = \{M_B : f \in M_B\} = \{N \in \mathcal{RM}(B_1(X)) : f \in N\} = \widehat{\mathscr{M}_f} \bigcap \mathcal{RM}(B_1(X))$, which is a basic closed set of $\mathcal{RM}(B_1(X))$ for the subspace topology induced from the hull-kernel topology on $\mathcal{M}(B_1(X))$. As ψ exchanges the basic closed sets, it is a homeomorphism. □

Before we conclude this section, we show that an injective map exists from $\mathcal{H}(C(X))$ into $\mathcal{H}(B_1(X))$, where $\mathcal{H}(C(X))$ and $\mathcal{H}(B_1(X))$ represent the collections of all hyper-real maximal ideals in $C(X)$ and $B_1(X)$ respectively. In what follows, we use the notation I^* for the ideal of $B_1(X)$ generated by the subset I of $B_1(X)$ and $m(I^*)$ for its maximal extension. The next theorem ensures that the ideal of $B_1(X)$ generated by a proper ideal of $C(X)$ is indeed proper, so that it has a maximal extension, say $m(I^*)$.

Theorem 2.11. *For any proper ideal I of $C(X)$, I^* is a proper ideal of $B_1(X)$, where I^* denotes the ideal of $B_1(X)$ generated by I as a subset of $B_1(X)$.*

Proof. If possible let, I^* is not proper. Then $I^* = B_1(X)$ and hence $\mathbf{1}$ (the constant function with value 1) can be written as $\mathbf{1} = \sum_{i=1}^{n} \alpha_i f_i$, where $\alpha_i \in B_1(X)$ and $f_i \in I$, for all $i = 1, 2, .., n$. For each $x \in X$, $\exists\, k \in \{1, 2, ..., n\}$, such that $f_k(x) \neq 0$, otherwise it contradicts that $\mathbf{1} = \sum_{i=1}^{n} \alpha_i f_i$. We consider the map $g(x) = \sum_{i=1}^{n} f_i^2(x), \forall\, x \in X$. Clearly, $g \in I \subseteq C(x)$ and $g(x) \neq 0, \forall\, x \in X$. So g is a unit in I, i.e., $I = C(X)$ - a contradiction. Hence, I^* is a proper ideal of $B_1(X)$. □

Theorem 2.12. *If M is a hyper-real maximal ideal of $C(X)$ then $m(M^*)$ is a hyper-real maximal ideal of $B_1(X)$.*

Proof. If $m(M^*)$ is a real maximal ideal of $B_1(X)$ then by Theorem 2.6, $m(M^*) \cap C(X)$ is a real maximal ideal of $C(X)$. Since $M \subseteq m(M^*) \cap C(X)$ and M is maximal it follows that $M = m(M^*) \cap C(X)$ - a contradiction to the fact that M is hyper-real. □

Theorem 2.13. *The function $\zeta : \mathcal{H}(C(X)) \to \mathcal{H}(B_1(X))$ given by $\zeta(M) = m(M^*)$ is an injective function.*

Proof. Let $M, N \in \mathcal{H}(C(X))$ be such that $m(M^*) = m(N^*)$. Then by maximality of M and N it follows that $M = m(M^*) \cap C(X) = m(N^*) \cap C(X) = N$. □

Corollary 2.14. $|\mathcal{M}(C(X))| \leq |\mathcal{M}(B_1(X))|$.

Proof. This is immediate from Theorem 2.8 and Theorem 2.13. □

3 Characterization of Real compact spaces

From the discussion of the last section it follows that there is a one-one correspondence between the collections $\mathcal{RM}(C(X))$ and $\mathcal{RM}(B_1(X))$ given by $M \mapsto M_B$. It is well known in [4] that a Tychonoff space X is real compact if and only if every real maximal ideal of $C(X)$ is fixed. Utilizing the one-to-one correspondence as mentioned above, we get a characterization of real compact spaces via real maximal ideals of $B_1(X)$.

Theorem 3.1. *A Tychonoff space X is real compact if and only if every real maximal ideal of $B_1(X)$ is fixed.*

Proof. Let X be a real compact space and $\widehat{M} \in \mathcal{RM}(B_1(X))$. By Theorem 2.8, there exists $M \in \mathcal{RM}(C(X))$ such that $\widehat{M} = M_B$. Since X is real compact, M is fixed; i.e., $M = M_p$, for some $p \in X$. Hence, $\widehat{M} = M_B = (M_p)_B = \widehat{M_p}$ (by Example 2.1).
Conversely, let M be any real maximal ideal of $C(X)$. Then $M_B \in \mathcal{RM}(B_1(X))$ and so, M_B is fixed. Therefore, M ($\subseteq M_B$) is a fixed ideal. Hence, X is real compact. □

In [2, Theorem 3.9], we proved a result for perfectly normal T_1-spaces, though the same proof applies to a larger class of spaces. The following lemma states the result for a broader class of spaces without providing the proof.

Lemma 3.2. *If X is a T_4-space in which every point is a G_δ point then the following statements are equivalent:*

(i) X is finite.

(ii) Every maximal ideal in $B_1(X)$ is fixed.

(iii) Every ideal in $B_1(X)$ is fixed.

Theorem 3.3. *Let X be a T_4 real compact space in which every point is a G_δ-point. Then $B_1(X) = {B_1}^*(X)$ if and only if X is finite.*

Proof. If X is finite then certainly, $B_1(X) = B_1^*(X)$.
Conversely, let $\widehat{M}$ be any maximal ideal of $B_1(X)(= B_1^*(X))$. By [2, Theorem 4.21] $\widehat{M}$ is a real maximal ideal. Since X is real compact, by Theorem 3.1 $\widehat{M}$ is fixed. Finally, using Lemma 3.2 we can conclude that X is finite. □

Corollary 3.4. *For a perfectly normal T_1 real compact space X, $B_1(X) = B_1^*(X)$ if and only if X is finite.*

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