Palestine Journal of Mathematics

Vol 14(1)(2025) , 293–297

On the solution of the Diophantine Equation $M_n^x + 8^y = z^2$ for a Mersenne number M_n

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11D61; Secondary 11D72.

Keywords and phrases: Mersenne numbers, Catalan's Conjecture, Exponential Diophantine equations.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions, which have improved the quality of our paper.

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Abstract In this paper, we establish that for any Mersenne number M_n, the exponential Diophantine equation $1+M_n^x = z^2$ has a solution if and only if n is even, and that the solution is unique. Further, using Catalan conjectures, we prove that the exponential Diophantine equation $M_1^x + 8^y = z^2$ has infinitely many solutions, and find them all. For non-unit $n \in \mathbb{N}$, we show that the exponential Diophantine equation $M_n^x + 8^y = z^2$ will have a solution if and only if $n \equiv 1($ mod $3)$. When $n = 3l + 1$, l being a non-negative integer, there are exactly two solutions, viz. $(x, y, z) = (0, 1, 3)$ and $(2, l+1, 2^n+1)$. Finally, we provide certain examples and non-examples alike!

1 Introduction

The word "Diophantine" comes from "Diophantus," a mathematician from Alexandria around 250 AD. Diophantine equations are useful across various fields. The realm of Diophantine equations is ancient and extensive, lacking a universal method to determine whether a given equation has solutions. In most cases, our focus narrows to examining a single equation rather than various equation types. The literature is replete with numerous articles delving into individual nonlinear equations involving diverse prime numbers and powers[2, 8].

If the Diophantine equation involves one or more additional variables or if a variable appears as an exponent, it is classified as a Diophantine exponent equation, akin to the equations found in the Fermat-Catalan conjecture and Beal's conjecture, $a^m + b^n = c^k$ with inequality restrictions on the exponents. The general theory of this type of equation is not available; however, special cases such as Catalan's conjecture have been resolved[15].

In 1844, the great Mathematician, Eugene Charles Catalan formulated a conjecture that the Diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$ has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$[1]. Authors J. H. E. Cohn[10], N. Terai, J. W. S. Cassels, S. A. Arif[11], F. S. Abu Muriefah etc. have did their extensive research works on the Diophantine equations such as $x^2 + c = y^n, x^4 - Dy^2 = 1, a^x + b^y = c^z, x^2 + 2^k = y^n$ etc. in the period of 1993-1997. Famous Mathematicians F. Luca [7], Z. Cao[20], F. Beukers[21] and many others have did considerable works (approx $1995 - 2001$) on various aspects of the Diophantine equations $x^2 + 3^m = y^n, a^x + b^y = c^z, Ax^p + By^q = Cz^r$ etc. Catalan conjecture was eventually proved by the Mathematician Preda Mihailescu in 2002 [4]. In the same year, S. A. Arif and F. S. A. Muriefah have worked on the Diophantine equation $x^2 + q^{2k+1} = y^n$ [11].

In 2007, Acu proved that the Diophantine equation $2^x + 5^y = z^2, x, y, z \in \mathbb{Z}^+$ has only two solutions i.e. $(3, 0, 3)$ and $(2, 1, 3)$ [5]. In the period $2010-2016$, the researchers Suvarnamani, et al.[6], B. Sroysang, et al.[3], J. J. Bravo, et al.[12] etc. have conducted thorough research on the different types of Diophantine equations $4^x+7^y=z^2, 4^x+11^y=z^2, 4^x+13^y=z^2, 4^x+17^y=z^2, A^x+B^y=C^z, 3^x+5^y=z^2, 8^x+19^y=z^2, 31^x+32^y=z^2, 7^x+8^y=z^2, F_n+F_m=2^a$ etc. In 2019, On Diophantine Equations Of Nathanson was studied at [9]. In 2020, Researcher M. Somanath, et al. [13] have worked on the Diophantine equations $x^2=29y^2-7^t, t\in\mathbb{N}, x^2=9y^2+11z^2, \alpha^2-90\beta^2-10\alpha-1260\beta=4401$. Recently, N. Burshtein [8], S. Aggarwal, S. Kumar, M. Buosi, et al. [16] etc. have did works in the field of Diophantine equation . In 2022, authors M. Karama, et al. [14] and M. Dutta et al. [17, 18, 19] did related works.

2 Preliminaries

Definition 2.1. Mersenne numbers : Numbers of the form $M_n := 2^n - 1$ are called Mersenne numbers, where $n\in\mathbb{N}$. The first few Mersenne number are $1, 3, 7, 15, 31, 63, 127$ (corresponding to $n = 1, 2, ..., 7$). Prime Mersenne numbers are called Mersenne primes. There are only 51 known Mersenne primes [2].

Conjecture 2.2. Catalan's Conjecture : The unique solution for the Diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$ is $(3, 2, 2, 3)$ [4].

Lemma 2.3. *The only solution of the exponential Diophantine equation* $1 + 8^y = z^2$ *is* $(y, z) = (1, 3)$ *.*

Proof. The given equation is

$$1 + 8^y = z^2, \tag{2.1}$$

$z^2 = 1 + 2^{3y} \geq 1 + 2^0 = 2$ implies $z \geq 2$ (because z is non-negative integer), Since equation (2.1) is same as $z^2 - 2^{3y} = 1$, If $3y > 1$, i.e. $y > 0$, then Catalan conjecture implies that $(y, z) = (1, 3)$ is the only solution. Otherwise $y = 0$. In this case equation (2.1) implies $1 + 1 = z^2 = 2$ which has no solution in integers. Thus, the only solution of $1 + 8^y = z^2$ is $(y, z) = (1, 3)$. □

Lemma 2.4. *The exponential Diophantine equation* $1 + M_n^x = z^2$ *has a solution if and only if* n *is even. If* $n = 2l, l \in \mathbb{N}$*, then the unique solution is* $(x, z) = (1, 2^l)$*.*

Proof. $z^2 = M_n^x + 1 \geq 1 + 1 = 2$ implies $z \geq 2$. For $M_1 = 1, z^2 = 2$, and the given equation has no integral solution. For $n \geq 2, M_n = 2^n - 1 \geq 2^2 - 1 = 3$. Therefore for $z^2 - M_n^x = 1$ to have a solution, we must have by Catalan conjecture either $x = 0$ or 1. For $x = 0$ implies $z^2 = 1+1 = 2$ implies no solution exists, and $x = 1$ implies $z^2 = 1+M_n = 1+(2^n-1)$ implies $z^2 = 2^n$. This has a solution if and only if n is even. Let $n = 2l, l \in \mathbb{N}$, then $z^2 = 2^{2l}$ implies $z = 2^l$. Therefore $(x, z) = (1, 2^l)$ is the unique solution of $1 + M_n^x = z^2$ where $n = 2l, l \in \mathbb{N}$. For odd n, no solution exists. □

Example 1. For $1 + M_2^x = z^2, n = 2.1$ is even implies $z^2 = 1 + M_2^x$ has the unique solution $(x, z) = (1, 2^1)$ i.e. $z^2 = 1 + 3^x$, has the unique solution $(x, z) = (1, 2)$.

Example 2. For $1 + M_3^x = z^2, n = 3$ is odd implies $z^2 = 1 + M_3^x$ has no solution, i.e. $z^2 = 1 + 7^x$, has no solution.

Example 3. For $1 + M_6^x = z^2, n = 6 = 2.3$ is even implies $z^2 = 1 + M_6^x$ has the unique solution $(x, z) = (1, 2^3)$ i.e. $z^2 = 1 + 63^x$, has the unique solution $(x, z) = (1, 2^3) = (1, 8)$. etc.

In the following, we will use the Catalan's conjecture and the preceding lemmas in solving the exponential Diophantine equation $M_n^x + 8^y = z^2$.

3 Main Results

Theorem 3.1. *The exponential Diophantine equation* $M_1^x + 8^y = z^2$ *has the solution* $(x, y, z) = (l, 1, 3), l \in \mathbb{Z}$*.*

Proof. $M_1 = 2^1 - 1 = 1$, Therefore, the equation reduces to $1^x + 8^y = z^2$, By lemma 2.3 above, this has solution $(y, z) = (1, 3), \forall x \in \mathbb{Z}$. Thus, the original equation has $(x, y, z) = (l, 1, 3)$ as solution, $\forall l \in \mathbb{Z}$. □

Theorem 3.2. *Let $n \neq 1$, the exponential Diophantine equation $M_n^x + 8^y = z^2$ will have solution if and only if $n \equiv 1(\mod 3)$. When $n = 3l + 1$, $l \in \mathbb{N}$, the equation has exactly two solutions given by $(x, y, z) = (0, 1, 3)$ and $(2, l + 1, 2^n + 1)$.*

Proof. Given equation is

$$M_n^x + 8^y = z^2 \tag{3.1}$$

Here, $M_n = 2^n - 1, n \in \mathbb{N}, n \neq 1$ implies $M_n \equiv -1(\mod 4)$ implies $M_n^x \equiv (-1)^n(\mod 4)$ implies $M_n^x + 8^y \equiv (-1)^x + 0(\mod 4)$

$$\Rightarrow z^2 \equiv (-1)^x(\mod 4) \tag{3.2}$$

Now M_n is odd implies $M_n^x + 8^y = z^2$ is odd implies z is odd too implies $z^2 \equiv 1(\mod 4)$. Therefore equation (3.2) implies x is even. Let $x = 2k, k \geq 0$ is a non-negative integer. Then equation (3.1) implies $M_n^{2k} + 8^y = z^2$ implies $8^y = z^2 - (M_n^k)^2$ implies $2^{3y} = (z - M_n^k)(z + M_n^k), z \pm M_n^k$ are even implies $z - M_n^k = 2^u$ and $z + M_n^k = 2^{3y-u}, u \geq 1, 3y - u > u$. This in turn implies $2.M_n^k = 2^u(2^{3y-2u} - 1)$, because M_n is odd. Therefore $u = 1, M_n^k = 2^{3y-2} - 1$

$$\Rightarrow 2^{3y-2} - M_n^k = 1 \tag{3.3}$$

Now $k \geq 0$ implies $2^{3y-2} \geq 1 + M^0 = 2^1$ implies $3y - 2 \geq 1$

$$\Rightarrow y \geq 1 \tag{3.4}$$

Also, $n > 1$

$$\Rightarrow M_n = 2^n - 1 \geq 3 \tag{3.5}$$

From equation (3.3), By Catalan conjecture, we have $y = 1$ or $k = 0, 1$. We consider them one by one.

Case 1: $y = 1$, therefore equation (3.3) implies $2 - M_n^k = 1$ implies $M_n^k = 1$ implies $k = 0$, (since $M_n \neq 1$) implies $x = 2k = 0$. Therefore equation (3.1) implies $z^2 = M_n^0 + 8^1 = 1 + 8 = 9$ implies $z = 3$ (since $z \geq 0$). Therefore $(x, y, z) = (0, 1, 3)$.

Case 2: $k = 0$ implies $x = 2k = 0$ implies $2^{3y-2} = 1 + M_n^0$ implies $2^{3y-2} = 2^1$ implies $3y - 2 = 1$ implies $y = 1$ implies $z = 3$, by case 1. Therefore $(x, y, z) = (0, 1, 3)$.

Case 3: $k = 1$ implies $x = 2k = 2$, equation (3.3) implies $2^{3y-2} - M_n = 1$ implies $2^{3y-2} - 1 = M_n = 2^n - 1$ (By definition of Mersenne number) implies $2^{3y-2} = 2^n$. To have solution, we must have $n \equiv 1(\mod 3)$ implies $3y - 2 = n = 3l + 1, l \in \mathbb{N}$ implies $y = l + 1$. Therefore equation (3.1) implies $z^2 = M_n^2 + 8^{l+1} = (2^n - 1)^2 + 2^{3(l+1)} = (2^{2n} + 1 - 2^{n+1}) + 2^{3l+1+2} = 2^{2n} + 1 - 2^{n+1} + 2^{n+2} = 2^{2n} + 1 + 2^{n+1} = (2^n + 1)^2$ implies $z = 2^n + 1$. Therefore $(x, y, z) = (2, l + 1, 2^n + 1), l \in \mathbb{N}$. Converse is clearly true as can be seen from direct substitution. This completes the theorem.

□

Corollary 3.3. *The exponential Diophantine equation $M_n^x + 8^y = w^4$ has no solution for any natural number n.*

Proof. Suppose (x, y, w) is a solution of $M_n^x + 8^y = w^4, n \in \mathbb{N}$. This means that, (x, y, w^2) is a solution of $M_n^x + 8^y = z^2$. By theorem 3.1 and 3.2 above, we have $w^2 = 3$ or $2^n + 1$. This implies $w^2 = 2^n + 1$, (since $w^2 \neq 3$)

$$\Rightarrow w^2 - 2^n = 1 \tag{3.6}$$

Now $w^2 = 1 + 2^n \geq 3$ implies $w \geq 2$. We consider two cases,

Case 1:$n = 1$, therefore equation (3.6) implies $w^2 = 1 + 2 = 3$ implies not possible.

Case 2: $n > 1$, then $min\{w, 2, n\} \geq 2 > 1$, Therefore Catalan conjecture implies $w = 3$ and $n = 3$ implies $n \equiv 0(mod3)$. Therefore by theorem 3.2, we get a contradiction. □

Corollary 3.4. *The exponential Diophantine equation $M_n^x + 8^y = w^{2m}, m > 2$ has no solution in non-negative integers.*

Proof. Suppose (x, y, w) is a solution. Then (x, y, w^m) is a solution of (3.1). By theorem 3.1 and 3.2, $w^m = 3$ or $2^n + 1, w$ is non-negative and $m \geq 3$ implies $w^m \neq 3$ implies $w^m = 2^n + 1$

$$\Rightarrow w^m - 2^n = 1 \tag{3.7}$$

Now $w^m = 1 + 2^n > 1$ implies $w > 1$ implies $w \geq 2$ implies $w^m \geq 2^3 = 8$ as $m \geq 3$. Therefore from (3.7), by Catalan conjecture we get $n = 0, 1$. For $n = 0$ implies $w^m = 2$ implies contradiction as $w^m \geq 8$. For $n = 1$ implies $w^m = 3$ implies contradiction again. Therefore the given equation has no solution. □

Example 1. Because $2 \not\equiv 1(\mod 3)$. Therefore $M_2^x + 8^y = z^2$ has no solution except $(0, 1, 3)$, by theorem 3.2, i.e. $3^x + 8^y = z^2$ has unique solution $(0, 1, 3)$.

Example 2. Because $3 \equiv 0 \not\equiv 1(\mod 3)$. Therefore $M_3^x + 8^y = z^2$ has no solution except $(0, 1, 3)$, by theorem 3.2, i.e. $7^x + 8^y = z^2$ has unique solution $(0, 1, 3)$.

Example 3. Because $4 \equiv 1(\mod 3), 4 = 3.1 + 1$. Therefore $M_4^x + 8^y = z^2$ has exactly two solutions , by theorem 3.2. These are $(0, 1, 3)$ and $(2, 2, 17)$ i.e. $15^x + 8^y = z^2$ has exactly two solutions. The trivial solution is $(0, 1, 3)$ and the non-trivial solution is $(2, 2, 17)$.

Example 4. Because $5 \equiv 2 \not\equiv 1(\mod 3)$. Therefore $M_5^x + 8^y = z^2$ has no solution except $(0, 1, 3)$, by theorem 3.2 i.e. $31^x + 8^y = z^2$ has unique solution $(0, 1, 3)$.

Example 5. Because $7 \equiv 1(\mod 3), 7 = 3.2 + 1$. Therefore $M_7^x + 8^y = z^2$ has exactly two solutions , by theorem 3.2. These are $(0, 1, 3)$ and $(2, 3, 129)$ i.e. $127^x + 8^y = z^2$ has exactly two solutions. The trivial solution is $(0, 1, 3)$ and the non-trivial solution is $(2, 3, 129)$.

4 Conclusion remarks

In this article, we have showed that the exponential Diophantine equation $1 + M_n^x = z^2$ has a solution if and only if n is even. If $n = 2l, l \in \mathbb{N}$, then the unique solution is $(x, z) = (1, 2^l)$. We also proved that the exponential Diophantine equation $M_n^x + 8^y = z^2$ will have solution if and only if $n \equiv 1(\mod 3)$. When $n = 3l + 1, l \in \mathbb{N}$, the solutions are given by $(0, 1, 3)$ and $(x, y, z) = (2, l + 1, 2^n + 1)$. Finally, we concluded with certain examples and non-examples.

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Received: 2024-02-01
Accepted: 2024-05-17