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A Study On Newton-type Inequalities For Fractional Operator and Application

A. Munir*, H. Budak, I. Faiz, L. Rathour, H. Kara, A. Kashuri

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Abstract: *The main goal of this article is to establish the Newton-type inequalities for the Caputo-Fabrizio fractional operator. We use the Peano kernel to obtain the Newton-type inequalities for the s-φ-convex function. We present several new error bounds and estimations are appraised by employing some well-known inequalities Hölder and power-mean. Additionally, we have given an application to special means.*

1 Introduction

Inequalities are essential tools in mathematics that help us understand and describe relationships between quantities, and they have numerous practical applications in various fields of science and engineering. In all these fields, inequalities help in formulating mathematical models that provide insights and guide decision-making processes. By expressing real-world constraints and relationships in mathematical terms, inequalities allow for the analysis and optimization of complex systems, leading to better-informed decisions and solutions to practical problems. In many real-world applications, the quantities of interest are indeed subject to constraints, and inequalities are a powerful mathematical tool for modeling and analyzing these constraints. Inequalities are versatile tools in mathematical analysis that play a crucial role in comparing functions, setting bounds on integrals, and modeling real-world constraints. They provide a rigorous framework for understanding mathematical relationships, making approximations, and solving complex problems in both pure and applied mathematics. Integral inequalities encompass a broad spectrum of mathematical tools.

Many types of integral inequalities exist in the literature. A well-known one of these is the Simpson type inequality. Simpson type inequality, which gives an upper bound on the inaccuracy when utilizing Simpson type to approximate a function's integral. Simpson type inequality has applications in various fields of science and engineering. Simpson type inequality is an important result in numerical analysis. In numerical analysis, error bounds and inequalities associated with various numerical integration methods, including Simpson's Rule, are crucial. In all of this years, Thomas Simpson established fundamental methods for numerical integration and estimate of definite integrals that are now known as Simpson's law. (1710-1761). But J. Kepler utilized an identical approximation over a century before, which is because it is often referred to as Kepler's law. Estimates based just on a three-step quadratic kernel are often referred to as Newton-type results as Simpson's method utilizes the three-point Newton-Cotes quadrature rule. The following inequality which is known as Simpson type inequality has been studied by several authors (see the papers [1–5]).

Theorem 1.1. *Let $\delta : [\psi, \chi] \to \mathbb{R}$ be a four times continuously differentiable mapping on (ψ, χ) and $\left\|\delta^{(4)}\right\|_\infty = \sup_{x\in(\psi,\chi)} \left|\delta^{(4)}\right| < \infty$, then the following inequality holds:*

$$\left|\left[\frac{\delta(\psi)+\delta(\chi)}{6}+\frac{2}{3}\delta\left(\frac{\psi+\chi}{2}\right)\right]-\frac{1}{\chi-\psi}\int_\psi^\chi \delta(x)\,dx\right| \le \frac{1}{2880}\left\|\delta^{(4)}\right\|_\infty (\chi-\psi)^4 .$$

Simpson quadrature formula (Simpson's $1/3$) is followed as:

$$\left|\frac{\delta(\psi)}{6}+\frac{4}{6}\delta\left(\frac{\psi+\chi}{2}\right)+\frac{\delta(\chi)}{6}-\frac{1}{\chi-\psi}\int_\psi^\chi \delta(u)\,du\right| \le \frac{(\chi-\psi)^4}{2880}\left\|\delta^{(4)}\right\|_\infty .$$

Simpson second formula or Newton-Cotes quadrature formula (Simpson's $3/8$) is followed as:

$$\left|\frac{\delta(\psi)}{8}+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{2}\right)+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{2}\right)+\frac{\delta(\chi)}{8}-\frac{1}{\chi-\psi}\int_\psi^\chi \delta(u)\,du\right| \le \frac{(\chi-\psi)^4}{6480}\left\|\delta^{(4)}\right\|_\infty .$$

Fractional calculus, including fractional integral inequalities, is a growing and interdisciplinary field with a wide range of applications. It allows for a more nuanced understanding of systems with memory or long-range dependence and provides a powerful mathematical framework for modeling and solving problems in various scientific and engineering disciplines. Numerous authors have used different fractional operators and generalizations of convexity to obtain new error bounds. For more details different fractional operator to see these articles [6–19]. In [20] Dragomir et al. obtain the Simpson type inequalities for differentiable mapping of bounded variation and give some application to special means. In [21] Park et al establish the Simpson type inequality for differentiable functions whose absolute values are pre-invex. In [22] Hezenci et al. prove the Simpson type inequalities for twice differentiable functions whose absolute values are convex. In [23] Arslan et al. improved the error bounds of Simpson type inequality for four times differentiable functions whose absolute values *at a certain power are s-convex. The Simpson type inequality is also called Newton-type or (simpson second formula) if it consists of four points. In [24] Meftah et al. prove the Newton-type inequalities for differentiable functions whose absolute values are s-convex. In [25] Chasreechai et al. derive the Newton-type inequalities using the multiplicative functions. For more information on Newton-type inequalities we see these papers [26–29]. In [30] Iftikhar et al. establish the Simpson's 3/8 for p-harmonic convex function via a differentiable function as follows:*

$$\begin{aligned}
&\left|\frac{\delta(\psi)}{8}+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{2}\right)+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{2}\right)+\frac{\delta(\chi)}{8}-\frac{1}{\chi-\psi}\int_{\psi}^{\chi}\delta(u)\,du\right|\\
\leq\;&(\chi-\psi)\left[\frac{17}{756}\left(\frac{973\left|\delta'(\psi)\right|^{q}+251\left|\delta'(\chi)\right|^{q}}{1224}\right)^{\frac{1}{q}}+\frac{1}{36}\left(\frac{\left|\delta'(\psi)\right|^{q}+\left|\delta'(\chi)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.\\
&\left.+\frac{17}{756}\left(\frac{251\left|\delta'(\psi)\right|^{q}+973\left|\delta'(\chi)\right|^{q}}{1224}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\delta(\psi)}{8}+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{2}\right)+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{2}\right)+\frac{\delta(\chi)}{8}-\frac{1}{\chi-\psi}\int_{\psi}^{\chi}\delta(u)\,du\right|\\
\leq\;&(\chi-\psi)\left[\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5\left|\delta'(\psi)\right|^{q}+\left|\delta'(\chi)\right|^{q}}{18}\right)^{\frac{1}{q}}+\left(\frac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\delta'(\psi)\right|^{q}+\left|\delta'(\chi)\right|^{q}}{6}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\delta'(\psi)\right|^{q}+5\left|\delta'(\chi)\right|^{q}}{18}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

In [31], S. Erden obtained the error bounds of Simpson type inequality for differentiable function.

Theorem 1.2. *Let $\delta:[\psi,\chi]\to\mathbb{R}$ be a differentiable mapping whose derivative is continuously on (ψ,χ). Then for all $x\in[\psi,\chi]$, then following inequality holds:*

$$\left|\frac{\delta(\psi)}{8}+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{2}\right)+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{2}\right)+\frac{\delta(\chi)}{8}-\frac{1}{\chi-\psi}\int_{\psi}^{\chi}\delta(u)\,du\right|\leq\frac{25(\chi-\psi)}{288}\left\|\delta\right\|_{\infty}.$$

2 Preliminaries

Convexity theory plays a significant role in integral inequalities, particularly in the analysis of inequalities involving convex functions.

Definition 2.1. [32] If $\delta: I\to\mathbb{R}$ is called convex on I for all $\psi,\chi\in I$ and $\eta\in[0,1]$, then following inequality:

$$\delta(\eta\psi+(1-\eta)\chi)\leq\eta\delta(\psi)+(1-\eta)\delta(\chi),$$

for all $\psi,\chi\in I$ and $\eta\in[0,1]$.

Definition 2.2. [33] Suppose $0<s\leq 1$. A function $\delta: I\subset\mathbb{R}$ is said to be s-φ-convex if the inequality holds:

$$\delta(\eta\psi+(1-\eta)\chi)\leq\delta(\chi)+\eta^{s}\varphi(\delta(\psi),\delta(\chi)),$$

for all $\psi,\chi\in I$ and $\eta\in[0,1]$.

Definition 2.3. [34] Let $\delta: I\subset\mathbb{R}$ is said to be φ-quasi-convex if the inequality holds:

$$\delta(\eta\psi+(1-\eta)\chi)\leq\max\left[\delta(\chi),\delta(\chi)+\varphi(\delta(\psi),\delta(\chi))\right],$$

for all $\psi,\chi\in I$ and $\eta\in[0,1]$.

Remark 2.4. If we put $\varphi(\psi,\chi)=\psi-\chi$ and $s=1$ in above definition, are reduced to the definition of convex and quasi-convex, respectively.

Next we give some examples for above definitions.

Example 2.5. Suppose $\delta(x)=x^{2}$, then δ is convex and $\frac{1}{2}$-φ-convex with $\varphi(\psi,\chi)=2\psi+\chi$, indeed:

$$\begin{aligned}
\delta(\eta\psi+(1-\eta)\chi)&=(\eta\psi+(1-\eta)\chi)^{2}=\eta^{2}\psi^{2}+2\eta(1-\eta)\psi\chi+(1-\eta)^{2}\chi^{2}\\
&\leq\chi^{2}+\eta\psi^{2}+2\eta\psi\chi=\chi^{2}+\eta^{\frac{1}{2}}\left[\eta^{\frac{1}{2}}\psi^{2}+2\eta^{\frac{1}{2}}\psi\chi\right].
\end{aligned}$$

On the other hand, we have

$$0<\eta<1\Rightarrow 0<\eta^{\frac{1}{2}}<1\Rightarrow\eta^{\frac{1}{2}}\psi^{2}+2\eta^{\frac{1}{2}}\psi\chi\leq\psi^{2}+2\psi\chi\leq\psi^{2}+\psi^{2}+\chi^{2}.$$

Hence, we get

$$\delta(\eta\psi+(1-\eta)\chi)\leq\chi^{2}+\eta^{\frac{1}{2}}\left[2\psi^{2}+\chi^{2}\right]=\delta(\chi)+\eta^{\frac{1}{2}}\varphi(\delta(\psi),\delta(\chi)).$$

Example 2.6. Suppose $\delta(x) = x^3$, then δ is not convex but is φ-convex with $\varphi(\psi,\chi) = 3\chi^2(\psi-\chi) + 3\chi(\psi-\chi)^2 + (\psi-\chi)^3$, indeed:

$$\begin{aligned}
\delta(\eta\psi + (1-\eta)\chi) &= (\eta\psi + (1-\eta)\chi)^3 = (\eta(\psi-\chi)+\chi)^3 \\
&= \chi^3 + 3\chi^2\eta(\psi-\chi) + 3\chi\eta^2(\psi-\chi)^2 + \eta^3(\psi-\chi)^3 \\
&= \delta(\chi) + \eta\left[3\chi^2(\psi-\chi) + 3\chi\eta(\psi-\chi)^2 + \eta^2(\psi-\chi)^3\right] \\
&\leq \delta(\chi) + \eta\left[3\chi^2(\psi-\chi) + 3\chi(\psi-\chi)^2 + (\psi-\chi)^3\right] \\
&= \delta(\chi) + \eta\varphi(\delta(\psi), \delta(\chi)).
\end{aligned}$$

Example 2.7. Suppose $\delta : [\psi,\chi] \to \mathbb{R}$, $0 < \psi < \chi$, with $\delta(x) = 2$. We obviously see that δ is a φ-quasi-convex with $\varphi(\psi,\chi) = \psi - \chi$.

Let's give some basic definitions in the literature without giving the definitions of fractional integral.

Definition 2.8. The Euler Gamma function and Euler Beta function are defined

$$\Gamma(x) := \int_0^\infty \eta^{x-1} e^{-\eta} d\eta,$$

$$B(x,y) := \int_0^1 \eta^{x-1}(1-\eta)^{y-1} d\eta.$$

Here, $0 < x, y < \infty$.

We shall need the following integral representations [35]:

$${}_2F_1(\alpha,\beta;\gamma;-z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)} \int_0^1 \frac{\eta^{\alpha-1}(1-\eta)^{\gamma-\alpha-1}}{(1+z)^\beta} d\eta,$$

where $\text{Re}(\gamma) > \text{Re}(\alpha)$, $|\arg(1+z)| < \pi$.

Definition 2.9. [36] Let $H^1(\psi,\chi)$ be the Sobolev space of order one defined as

$$H^1(\psi,\chi) = \left\{\delta \in L^2(\psi,\chi) : \delta' \in L^2(\psi,\chi)\right\},$$

where

$$L^2(\psi,\chi) = \left\{\delta(z) : \left(\int_\psi^\chi \delta^2(z)\, dz\right)^{\frac{1}{2}} < \infty\right\}.$$

Let $\delta \in H^1(\psi,\chi)$, $\psi < \chi$ and $\alpha \in [0,1]$, then the notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}^{CFD}_{\psi}D^\alpha\delta\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_\psi^x \delta'(\eta)\, e^{\frac{-\alpha(x-\eta)^\alpha}{1-\alpha}} d\eta,$$

where $x > \alpha$ and the associated integral operator is

$$\left({}^{CF}_{\psi}I^\alpha\delta\right)(x) = \frac{1-\alpha}{\beta(\alpha)}\delta(x) + \frac{\alpha}{\beta(\alpha)}\int_\psi^x \delta(\eta)\, d\eta,$$

where $\beta(\alpha) > 0$ is the normalization function satisfying $\beta(0) = \beta(1) = 1$. For $\alpha = 0, \alpha = 1$, the left derivative is defined as follows:

$$\begin{aligned}
\left({}^{CFD}_{\psi}D^0\delta\right)(x) &= \delta'(x) \\
\left({}^{CFD}_{\psi}D^1\delta\right)(x) &= \delta(x) - \delta(\psi).
\end{aligned}$$

For the right derivative operator, we have

$$\left({}^{CFD}_{\chi}D^\alpha\delta\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_x^\chi \delta'(\eta)\, e^{\frac{-\alpha(\eta-x)^\alpha}{1-\alpha}} d\eta,$$

where $x < \chi$ and the associated integral operator is given by

$$\left({}^{CF}I^\alpha_\chi\delta\right)(x) = \frac{1-\alpha}{\beta(\alpha)}\delta(x) + \frac{\alpha}{\beta(\alpha)}\int_x^\chi \delta(\eta)\, d\eta.$$

Motivated by the ongoing research, the main goal of this paper is to establish a new identity using the Caputo-Fabrizio fractional integral operator. By using the new identity to give the fractional bounds of Newton-type inequalities. Based on this identity, we have developed the Newton-type inequality for the s-φ-convex function. We also include an application regarding special means.

3 An Essential Identity

New Newton-type Lemma for a well-known Caputo-Fabrizio fractional operator is introduced below.

Lemma 3.1. *Let $\delta : I^o \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^o where $\psi, \chi \in I^o$ with $\psi < \chi$. If $\delta'' \in L^1[\psi, \chi]$, then the following fractional equality holds:*

$$\begin{aligned} & \frac{1}{8}\left[\delta(\psi) + 3\delta\left(\frac{2\psi+\chi}{3}\right) + 3\delta\left(\frac{\psi+2\chi}{3}\right) + \delta(\chi)\right] \\ & -\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k) + \left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)}\delta(k) \\ = & (\chi-\psi)^2 \int_0^1 M(\eta)\,\delta''((1-\eta)\psi + \eta\chi)\,d\eta, \end{aligned}$$

where

$$M(\eta) := \begin{cases} \frac{\eta}{2}\left(\frac{1}{4}-\eta\right), & \eta \in \left[0, \frac{1}{3}\right) \\ \frac{\eta}{2} - \frac{\eta^2}{2} - \frac{1}{8}, & \eta \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ (1-\eta)\left(\frac{\eta}{2} - \frac{3}{8}\right), & \eta \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Proof. Let

$$\begin{aligned} & (\chi-\psi)^2 \int_0^1 M(\eta)\,\delta''((1-\eta)\psi+\eta\chi)\,d\eta \\ = & (\chi-\psi)^2 \left[\int_0^{\frac{1}{3}} \frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\delta''((1-\eta)\psi+\eta\chi)\,d\eta + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{\eta}{2} - \frac{\eta^2}{2} - \frac{1}{8}\right)\delta''((1-\eta)\psi+\eta\chi)\,d\eta \right. \\ & \left. \int_{\frac{2}{3}}^{1} (1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\delta''((1-\eta)\psi+\eta\chi)\,d\eta\right] \\ = & I_1 + I_2 + I_3. \end{aligned}$$

Integration by parts, we have

$$\begin{aligned} I_1 = & \int_0^{\frac{1}{3}} \frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\delta''((1-\eta)\psi+\eta\chi)\,d\eta \\ = & \left.\frac{\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)}{\chi-\psi}\delta'((1-\eta)\psi+\eta\chi)\right|_0^{\frac{1}{3}} - \frac{1}{\chi-\psi}\int_0^{\frac{1}{3}}\left(\frac{1}{8}-\eta\right)\delta'((1-\eta)\psi+\eta\chi)\,d\eta \\ = & \frac{-1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right) - \frac{1}{\chi-\psi}\int_0^{\frac{1}{3}}\left(\frac{1}{8}-\eta\right)\delta'((1-\eta)\psi+\eta\chi)\,d\eta \\ = & \frac{-1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right) - \frac{1}{\chi-\psi}\left[\left.\frac{\left(\frac{1}{8}-\eta\right)}{\chi-\psi}\delta((1-\eta)\psi+\eta\chi)\right|_0^{\frac{1}{3}} \right. \\ & \left. + \frac{1}{\chi-\psi}\int_0^{\frac{1}{3}}\delta((1-\eta)\psi+\eta\chi)\,d\eta\right] \\ = & \frac{-1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right) + \frac{5}{24(\chi-\psi)^2}\delta\left(\frac{2\psi+\chi}{3}\right) + \frac{1}{8(\chi-\psi)^2}\delta(\psi) \\ & - \frac{1}{(\chi-\psi)^2}\int_0^{\frac{1}{3}}\delta((1-\eta)\psi+\eta\chi)\,d\eta \\ = & \frac{-1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right) + \frac{5}{24(\chi-\psi)^2}\delta\left(\frac{2\psi+\chi}{3}\right) + \frac{1}{8(\chi-\psi)^2}\delta(\psi) \\ & - \frac{1}{(\chi-\psi)^3}\int_{\psi}^{\frac{2\psi+\chi}{3}}\delta(u)\,du. \end{aligned} \tag{3.1}$$

Similarly, we get

$$\begin{aligned} I_2 = & \int_{\frac{1}{3}}^{\frac{2}{3}}\left(\frac{\eta}{2} - \frac{\eta^2}{2} - \frac{1}{8}\right)\delta''((1-\eta)\psi+\eta\chi)\,d\eta \\ = & \frac{-1}{72(\chi-\psi)}\delta'\left(\frac{\psi+2\chi}{3}\right) + \frac{1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right) + \frac{1}{6(\chi-\psi)^2}\delta\left(\frac{\psi+2\chi}{3}\right) \end{aligned}$$

$$+\frac{1}{6(\chi-\psi)^2}\delta\left(\frac{2\psi+\chi}{3}\right)-\frac{1}{(\chi-\psi)^3}\int_{\frac{2\psi+\chi}{3}}^{\frac{\psi+2\chi}{3}}\delta(u)\,du, \tag{3.2}$$

and

$$\begin{aligned}I_3 &= \int_{\frac{2}{3}}^{1}(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\delta''((1-\eta)\psi+\eta\chi)\,d\eta\\ &= \frac{1}{72(\chi-\psi)}\delta'\left(\frac{\psi+2\chi}{3}\right)+\frac{5}{24(\chi-\psi)^2}\delta\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{8(\chi-\psi)^2}\delta(\chi)\\ &\quad-\frac{1}{(\chi-\psi)^3}\int_{\frac{\psi+2\chi}{3}}^{\chi}\delta(u)\,du. \end{aligned}\tag{3.3}$$

Adding the equalities (3.1)-(3.3), we get

$$\begin{aligned}I_1+I_2+I_3 &= \frac{-1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right)+\frac{5}{24(\chi-\psi)^2}\delta\left(\frac{2\psi+\chi}{3}\right)+\frac{1}{8(\chi-\psi)^2}\delta(\psi)-\frac{1}{(\chi-\psi)^3}\int_{\psi}^{\frac{2\psi+\chi}{3}}\delta(u)\,du\\ &\quad+\frac{-1}{72(\chi-\psi)}\delta'\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{72(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right)+\frac{1}{6(\chi-\psi)^2}\delta\left(\frac{\psi+2\chi}{3}\right)\\ &\quad+\frac{1}{6(\chi-\psi)^2}\delta\left(\frac{2\psi+\chi}{3}\right)-\frac{1}{(\chi-\psi)^3}\int_{\frac{2\psi+\chi}{3}}^{\frac{\psi+2\chi}{3}}\delta(u)\,du+\frac{1}{72(\chi-\psi)}\delta'\left(\frac{\psi+2\chi}{3}\right)\\ &\quad+\frac{5}{24(\chi-\psi)^2}\delta\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{8(\chi-\psi)^2}\delta(\chi)-\frac{1}{(\chi-\psi)^3}\int_{\frac{\psi+2\chi}{3}}^{\chi}\delta(u)\,du\\ &= \frac{-1}{8(\chi-\psi)}\delta'\left(\frac{2\psi+\chi}{3}\right)+\frac{1}{8(\chi-\psi)}\delta'\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{8(\chi-\psi)^2}\delta(\psi)+\frac{1}{8(\chi-\psi)^2}\delta(\chi)\\ &\quad+\frac{3}{8(\chi-\psi)^2}\delta\left(\frac{2\psi+\chi}{3}\right)+\frac{3}{8(\chi-\psi)^2}\delta\left(\frac{\psi+2\chi}{3}\right)-\frac{1}{(\chi-\psi)^3}\int_{\psi}^{\chi}\delta(u)\,du. \end{aligned}\tag{3.4}$$

By multiplying equality (3.4) by $(\chi-\psi)^2$ and subtracting $\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)$, we obtain

$$\begin{aligned}&(I_1+I_2+I_3)(\chi-\psi)^2-\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\\ =\ & \frac{-(\chi-\psi)}{8}\delta'\left(\frac{2\psi+\chi}{3}\right)+\frac{(\chi-\psi)}{8}\delta'\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{8}(\delta(\psi)+\delta(\chi))+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{3}\right)\\ &+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{3}\right)-\frac{1}{(\chi-\psi)}\int_{\psi}^{\chi}\delta(u)\,du-\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\\ =\ & \frac{-(\chi-\psi)}{8}\delta'\left(\frac{2\psi+\chi}{3}\right)+\frac{(\chi-\psi)}{8}\delta'\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{8}(\delta(\psi)+\delta(\chi))+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{3}\right)+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{3}\right)\\ &-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left(\frac{\alpha}{\beta(\alpha)}\int_{\psi}^{k}\delta(u)\,du-\frac{(1-\alpha)}{\beta(\alpha)}\delta(k)+\frac{\alpha}{\beta(\alpha)}\int_{k}^{\chi}\delta(u)\,du-\frac{(1-\alpha)}{\beta(\alpha)}\delta(k)\right)\\ =\ & \frac{-(\chi-\psi)}{8}\delta'\left(\frac{2\psi+\chi}{3}\right)+\frac{(\chi-\psi)}{8}\delta'\left(\frac{\psi+2\chi}{3}\right)+\frac{1}{8}(\delta(\psi)+\delta(\chi))+\frac{3}{8}\delta\left(\frac{2\psi+\chi}{3}\right)+\frac{3}{8}\delta\left(\frac{\psi+2\chi}{3}\right)\\ &-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\\ &-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\\ =\ &(\chi-\psi)^2\int_0^1 M(\eta)\,\delta''((1-\eta)\psi+\eta\chi)\,d\eta. \end{aligned}$$

This completes the proof. □

4 Newton type inequalities for s-φ-convex functions

In this section, we acquire fractional Newton-type inequalities via s-φ-convex function.

Theorem 4.1. *Under the assumptions of Lemma 3.1, if $|\delta''|$ is s-φ-convex on $[\psi,\chi]$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right|\\
\leq\;&(\chi-\psi)^2\left[\left(\frac{19}{10368}\right)\left|\delta''(\psi)\right|+\left(\frac{2^{2s+3}s+3^{s+3}-2^{2s+3}}{2^{2s+6}3^{s+3}(5s+s^s+6)}\right)\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right.\\
&+\left(\frac{1}{648}\right)\left|\delta''(\psi)\right|+\left(\frac{2^{s+1}s^2+7\times 2^{2+s}-9s-3\times 2^{1+s}s-s^2-26}{2\times 3^{s+3}(s+2)(s+3)}\right)\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\\
&+\left(\frac{19}{10368}\right)\left|\delta''(\psi)\right|-\frac{1}{(s+1)(s+2)(s+3)}\left(3^{2s+5}s-15\times 2^{3s+4}s-2^{3s+4}s^2+3^{2s+7}+2^{2s+3}3^{s+3}s\right.\\
&\left.\left.+\frac{1}{2^{2s+6}3^{s+3}}\left(5\times 2^{2s+3}s+3^{s+3}+17\times 2^{3s+6}\right)\right)\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right].
\end{aligned}$$

Proof. By using the Lemma 3.1, since $|\delta''|$ is a s-φ-convex, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right|\\
\leq\;&(\chi-\psi)^2\int_0^1|M(\eta)|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta\\
\leq\;&(\chi-\psi)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta\right.\\
&\left.+\int_{\frac{2}{3}}^1\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta\right]\\
\leq\;&(\chi-\psi)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left(\left|\delta''(\psi)\right|+\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right)d\eta\right.\\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left(\left|\delta''(\psi)\right|+\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right)d\eta\\
&\left.+\int_{\frac{2}{3}}^1\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left(\left|\delta''(\psi)\right|+\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right)d\eta\right]\\
=\;&(\chi-\psi)^2\left(D_1+D_2+D_3\right), \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
D_1&=\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left(\left|\delta''(\psi)\right|+\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right)d\eta\\
&=\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left|\delta''(\psi)\right|d\eta+\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)d\eta\\
&=\left(\frac{19}{10368}\right)\left|\delta''(\psi)\right|+\left(\frac{2^{2s+3}s+3^{s+3}-2^{2s+3}}{2^{2s+6}3^{s+3}(5s+s^s+6)}\right)\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right), \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
D_2&=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left(\left|\delta''(\psi)\right|+\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right)d\eta\\
&=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left|\delta''(\psi)\right|d\eta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)d\eta\\
&=\frac{1}{648}\left|\delta''(\psi)\right|+\left(\frac{2^{s+1}s^2+7\times 2^{2+s}-9s-3\times 2^{1+s}s-s^2-26}{2\times 3^{s+3}(s+2)(s+3)}\right)\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right), \qquad (4.3)
\end{aligned}$$

and

$$D_3=\int_{\frac{2}{3}}^1\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left(\left|\delta''(\psi)\right|+\eta^s\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right)d\eta$$

$$
\begin{aligned}
&= \int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left|\delta''(\psi)\right|d\eta+\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\eta^{s}\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)d\eta \\
&= \left(\frac{19}{10368}\right)\left|\delta''(\psi)\right|-\frac{1}{(s+1)(s+2)(s+3)}\left(3^{2s+5}s-15\times 2^{3s+4}s-2^{3s+4}s^{2}+3^{2s+7}+2^{2s+3}3^{s+3}s\right. \\
&\quad \left.+\frac{1}{2^{2s+6}3^{s+3}}\left(5\times 2^{2s+3}s+3^{s+3}+17\times 2^{3s+6}\right)\right)\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right). \qquad (4.4)
\end{aligned}
$$

Using the equalities (4.2)-(4.4) in (4.1), we get the required result. □

Corollary 4.2. *If we choose $s=1$ in Theorem 4.1, then we get*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\
\leq\ & (\chi-\psi)^{2}\left(\frac{1}{192}\right)\left[\left|\delta''(\psi)\right|+\frac{1}{2}\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right].
\end{aligned}
$$

Corollary 4.3. *If we choose $\varphi(\psi,\chi)=\psi-\chi$ in Corollary 4.2, then we get*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\
\leq\ & (\chi-\psi)^{2}\left(\frac{1}{384}\right)\left(\left|\delta''(\psi)\right|+\left|\delta''(\chi)\right|\right).
\end{aligned}
$$

Remark 4.4. If we choose $\alpha=1$ and $\beta(0)=\beta(1)=1$ in Corollary 4.3, then we have the inequality obtained by Gao in [37, Theorem 2.1].

Theorem 4.5. *Under the assumption of Lemma 3.1, if $|\delta''|^{q}$ is s-φ-convex on $[\psi,\chi]$ and $q>1$, then the following fractional inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\
\leq\ & (\chi-\psi)^{2}\left[(M_1)^{\frac{1}{r}}\left(\frac{1}{3}\left|\delta''(\psi)\right|^{q}+\frac{\varphi\left(|\delta''(\psi)|^{q},|\delta''(\chi)|^{q}\right)}{3^{s+1}(s+1)}\right)^{\frac{1}{q}}\right. \\
&+(M_2)^{\frac{1}{r}}\left(\frac{1}{3}\left|\delta''(\psi)\right|^{q}+\frac{3^{-s-1}\left(-1+2^{1+s}\right)\left(\varphi\left(|\delta''(\psi)|^{q},|\delta''(\chi)|^{q}\right)\right)}{s+1}\right)^{\frac{1}{q}} \\
&\left.+(M_3)^{\frac{1}{r}}\left(\frac{1}{3}\left|\delta''(\psi)\right|^{q}+\frac{2^{s}\times 3^{-s-1}\left(-2+2^{-s}\times 3^{1+s}\right)\left(\varphi\left(|\delta''(\psi)|^{q},|\delta''(\chi)|^{q}\right)\right)}{s+1}\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $\frac{1}{r}+\frac{1}{q}=1$ and

$$
\begin{aligned}
M_1 &= \int_{0}^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|^{r}d\eta \qquad (4.5) \\
&= \frac{1}{8^{r}}\left[\frac{B(r+1,r+1)}{4^{r+1}}+\frac{-1^{r}}{3^{r+1}(r+1)}\times{}_2F_1\left(-r,r+1;r+2;\frac{4}{3}\right)\right. \\
&\quad \left.+\frac{(-1)^{r}\times B(r+1,r+1)}{4^{r+1}}\times{}_2F_1(-r,r+1;r+2;1)\right]. \\
M_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^{2}}{2}-\frac{1}{8}\right|^{r}d\eta \\
&= \frac{1}{3^{r+1}(r+1)}\left[2\times{}_2F_1\left(-r,r+1;r+2;\frac{2}{3}\right)-\frac{1}{2^{r}}\times{}_2F_1\left(-r,r+1;r+2;\frac{1}{3}\right)\right], \\
M_3 &= \int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|^{r}d\eta \\
&= \left(\frac{3}{8}\right)^{r}\left[\frac{1}{36^{r}\times 12}B(r+1,1)\times{}_2F_1\left(-r,r+1;r+2;\frac{-1}{3}\right)+\frac{1}{4^{r+1}3^{r}}B(r+1,r+1)\right].
\end{aligned}
$$

Proof. By using the Lemma 3.1, with the help of Hölder inequality and s-φ-convexity of $|\delta''|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\
\leq\ &(\chi-\psi)^2\int_0^1|M(\eta)|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta \\
\leq\ &(\chi-\psi)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|^r d\eta\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta\right)^{\frac{1}{q}}\right. \\
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|^r d\eta\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|^r d\eta\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Employing the s-φ-convexity of $|\delta''|^q$, we have

$$\begin{aligned}
\int_0^{\frac{1}{3}}\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta &\leq \int_0^{\frac{1}{3}}(1-\eta)\left|\delta''(\psi)\right|^q d\eta+\int_0^{\frac{1}{3}}\eta^s\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)d\eta \\
&= \frac{1}{3}\left|\delta''(\psi)\right|^q+\frac{\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)}{3^{s+1}(s+1)}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta \\
\leq\ &\frac{1}{3}\left|\delta''(\psi)\right|^q+\frac{3^{-s-1}\left(-1+2^{1+s}\right)\left(\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)}{s+1},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{2}{3}}^{1}\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta \\
\leq\ &\frac{1}{3}\left|\delta''(\psi)\right|^q+\frac{2^s\times 3^{-s-1}\left(-2+2^{-s}\times 3^{1+s}\right)\left(\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)}{s+1}.
\end{aligned}$$

We also have

$$\begin{aligned}
M_1 &= \int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|^r d\eta \\
&= \int_0^{\frac{1}{4}}\left(\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right)^r d\eta+\int_{\frac{1}{4}}^{\frac{1}{3}}\left(\frac{\eta}{2}\left(\eta-\frac{1}{4}\right)\right)^r d\eta \\
&= \frac{1}{8^r}\left[\int_0^{\frac{1}{4}}(\eta(1-4\eta))^r d\eta+\int_0^{\frac{1}{3}}(\eta(4\eta-1))^r d\eta-\int_0^{\frac{1}{4}}(\eta(4\eta-1))^r d\eta\right] \\
&= \frac{1}{8^r}\left[\frac{B(r+1,r+1)}{4^{r+1}}+\frac{-1^r}{3^{r+1}(r+1)}\times{}_2F_1\left(-r,r+1;r+2;\frac{4}{3}\right)\right. \\
&\quad\left.+\frac{(-1)^r\times B(r+1,r+1)}{4^{r+1}}\times{}_2F_1(-r,r+1;r+2;1)\right], \\
M_2 &= \int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|^r d\eta \\
&= \frac{1}{3^{r+1}(r+1)}\left[2\times{}_2F_1\left(-r,r+1;r+2;\frac{4}{3}\right)-\frac{1}{2^r}.{}_2F_1\left(-r,r+1;r+2;\frac{1}{3}\right)\right], \\
M_3 &= \int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|^r d\eta \\
&= \int_{\frac{2}{3}}^{\frac{3}{4}}\left[(1-\eta)\left(\frac{3}{8}-\frac{\eta}{2}\right)\right]^r d\eta+\int_{\frac{3}{4}}^{1}\left[(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right]^r d\eta \\
&= \left(\frac{3}{8}\right)^r\left[\frac{1}{36^r\times 12}B(r+1,1)\times{}_2F_1\left(-r,r+1;r+2;\frac{-1}{3}\right)+\frac{1}{4^{r+1}3^r}B(r+1,r+1)\right].
\end{aligned}$$

This completes the proof. □

Corollary 4.6. *If we choose $s = 1$ in Theorem 4.5, then we get*

$$\begin{aligned} & \left| \frac{1}{8} \left[\delta(\psi) + 3\delta\left(\frac{2\psi+\chi}{3}\right) + 3\delta\left(\frac{\psi+2\chi}{3}\right) + \delta(\chi) \right] \right. \\ & \left. - \frac{\beta(\alpha)}{\alpha(\chi-\psi)} \left[\left({}_{\psi}^{CF}I^{\alpha}\delta \right)(k) + \left({}^{CF}I_{\chi}^{\alpha}\delta \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)}\delta(k) \right| \\ \leq\; & (\chi-\psi)^2 \left[(M_1)^{\frac{1}{r}} \left(\frac{6|\delta''(\psi)|^q + \varphi(|\delta''(\psi)|^q, |\delta''(\chi)|^q)}{18} \right)^{\frac{1}{q}} \right. \\ & + (M_2)^{\frac{1}{r}} \left(\frac{2|\delta''(\psi)|^q + \varphi(|\delta''(\psi)|^q, |\delta''(\chi)|^q)}{6} \right)^{\frac{1}{q}} \\ & \left. + (M_3)^{\frac{1}{r}} \left(\frac{6|\delta''(\psi)|^q + 5\varphi(|\delta''(\psi)|^q, |\delta''(\chi)|^q)}{18} \right)^{\frac{1}{q}} \right], \end{aligned}$$

where M_i for $i = 1, 2, 3$ are denoted as in (4.5).

Corollary 4.7. *If we choose $\varphi(\psi, \chi) = \psi - \chi$ in Corollary 4.6, then we get*

$$\begin{aligned} & \left| \frac{1}{8} \left[\delta(\psi) + 3\delta\left(\frac{2\psi+\chi}{3}\right) + 3\delta\left(\frac{\psi+2\chi}{3}\right) + \delta(\chi) \right] \right. \\ & \left. - \frac{\beta(\alpha)}{\alpha(\chi-\psi)} \left[\left({}_{\psi}^{CF}I^{\alpha}\delta \right)(k) + \left({}^{CF}I_{\chi}^{\alpha}\delta \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)}\delta(k) \right| \\ \leq\; & (\chi-\psi)^2 \left[(M_1)^{\frac{1}{r}} \left(\frac{5|\delta''(\psi)|^q + |\delta''(\chi)|^q}{18} \right)^{\frac{1}{q}} + (M_2)^{\frac{1}{r}} \left(\frac{|\delta''(\psi)|^q + |\delta''(\chi)|^q}{6} \right)^{\frac{1}{q}} \right. \\ & \left. + (M_3)^{\frac{1}{r}} \left(\frac{|\delta''(\psi)|^q + 5|\delta''(\chi)|^q}{18} \right)^{\frac{1}{q}} \right], \end{aligned}$$

where M_i for $i = 1, 2, 3$ are denoted as in (4.5).

Theorem 4.8. *Under the assumptions of Lemma 3.1, if $|\delta''|^q$ is s-φ-convex on $[\psi, \chi]$ and $q \geq 1$, then the following fractional inequality holds:*

$$\begin{aligned} & \left| \frac{1}{8} \left[\delta(\psi) + 3\delta\left(\frac{2\psi+\chi}{3}\right) + 3\delta\left(\frac{\psi+2\chi}{3}\right) + \delta(\chi) \right] \right. \\ & \left. - \frac{\beta(\alpha)}{\alpha(\chi-\psi)} \left[\left({}_{\psi}^{CF}I^{\alpha}\delta \right)(k) + \left({}^{CF}I_{\chi}^{\alpha}\delta \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)}\delta(k) \right| \\ \leq\; & (\chi-\psi)^2 \left[\left(\frac{19}{10368} \right)^{1-\frac{1}{q}} \left(\frac{152}{82944} |\delta''(\psi)|^q + \frac{27\left(2^{2s+3}s + 3^{s+3} - 2^{2s+3}\right)}{2^{2s+6}3^{s+3}(5s + s^s + 6)} \varphi\left(|\delta''(\psi)|^q, |\delta''(\chi)|^q \right) \right)^{\frac{1}{q}} \right. \\ & + \left(\frac{1}{648} \right)^{1-\frac{1}{q}} \left(\frac{|\delta''(\psi)|^q}{648} + \left(\frac{2^{s+1}s^2 + 7 \times 2^{2+s} - 9s - 3 \times 2^{1+s}s - s^2 - 26}{2 \times 3^{s+3}(s+2)(s+3)} \right) \varphi\left(|\delta''(\psi)|^q, |\delta''(\chi)|^q \right) \right)^{\frac{1}{q}} \\ & + \left(\frac{19}{10368} \right)^{1-\frac{1}{q}} \left(\frac{152}{82944} |\delta''(\psi)|^q - \frac{1}{(s+1)(s+2)(s+3)} \left(3^{2s+5}s - 15 \times 2^{3s+4}s - 2^{3s+4}s^2 + 3^{2s+7} + 2^{2s+3}3^{s+3}s \right. \right. \\ & \left. \left. \left. + \frac{1}{2^{2s+6}3^{s+3}} \left(5 \times 2^{2s+3}s + 3^{s+3} + 17 \times 2^{3s+6} \right) \varphi\left| \delta''(\psi) \right|^q, \left| \delta''(\chi) \right|^q \right) \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. By using the Lemma 3.1, with the help of power-mean inequality s-φ-convexity of $|\delta''|^q$, we have

$$\begin{aligned} & \left| \frac{1}{8} \left[\delta(\psi) + 3\delta\left(\frac{2\psi+\chi}{3}\right) + 3\delta\left(\frac{\psi+2\chi}{3}\right) + \delta(\chi) \right] \right. \\ & \left. - \frac{\beta(\alpha)}{\alpha(\chi-\psi)} \left[\left({}_{\psi}^{CF}I^{\alpha}\delta \right)(k) + \left({}^{CF}I_{\chi}^{\alpha}\delta \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)}\delta(k) \right| \\ \leq\; & (\chi-\psi)^2 \int_0^1 |M(\eta)| \left| \delta''((1-\eta)\psi + \eta\chi) \right| d\eta \\ \leq\; & (\chi-\psi)^2 \left[\int_0^{\frac{1}{3}} \left| \frac{\eta}{2}\left(\frac{1}{4} - \eta \right) \right| \left| \delta''((1-\eta)\psi + \eta\chi) \right| d\eta + \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \frac{\eta}{2} - \frac{\eta^2}{2} - \frac{1}{8} \right| \left| \delta''((1-\eta)\psi + \eta\chi) \right| d\eta \right. \\ & \left. + \int_{\frac{2}{3}}^1 \left| (1-\eta)\left(\frac{\eta}{2} - \frac{3}{8} \right) \right| \left| \delta''((1-\eta)\psi + \eta\chi) \right| d\eta \right] \end{aligned}$$

$$\leq (\chi-\psi)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|^q d\eta\right)^{\frac{1}{q}}\right]$$
$$\leq (\chi-\psi)^2\left[\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left(\left|\delta''(\psi)\right|^q+\eta^s\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)d\eta\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left(\left|\delta''(\psi)\right|^q+\eta^s\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)d\eta\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left(\left|\delta''(\psi)\right|^q+\eta^s\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)d\eta\right)^{\frac{1}{q}}\right]$$
$$= (\chi-\psi)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152}{82944}\left|\delta''(\psi)\right|^q+\frac{27\left(2^{2s+3}s+3^{s+3}-2^{2s+3}\right)}{2^{2s+6}3^{s+3}\left(5s+s^s+6\right)}\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{\left|\delta''(\psi)\right|^q}{648}+\left(\frac{2^{s+1}s^2+7\times 2^{2+s}-9s-3\times 2^{1+s}s-s^2-26}{2\times 3^{s+3}(s+2)(s+3)}\right)\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152}{82944}\left|\delta''(\psi)\right|^q-\frac{1}{(s+1)(s+2)(s+3)}\left(3^{2s+5}s-15\times 2^{3s+4}s-2^{3s+4}s^2+3^{2s+7}+2^{2s+3}3^{s+3}s\right.\right.$$
$$\left.\left.\left.+\frac{1}{2^{2s+6}3^{s+3}}\left(5\times 2^{2s+3}s+3^{s+3}+17\times 2^{3s+6}\right)\varphi\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Corollary 4.9. *If we choose $s=1$ in Theorem 4.8, then we get*

$$\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\delta\right)(k)+\left({}^{CF}I^{\alpha}_{\chi}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right|$$
$$\leq (\chi-\psi)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152}{82944}\left|\delta''(\psi)\right|^q+\frac{1}{3072}\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{\left|\delta''(\psi)\right|^q}{648}+\frac{1}{1296}\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{152}{82944}\left|\delta''(\psi)\right|^q+\frac{125}{82944}\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Corollary 4.10. *If we choose $\varphi(\psi,\chi)=\psi-\chi$ in Corollary 4.9, then we get*

$$\left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\delta\right)(k)+\left({}^{CF}I^{\alpha}_{\chi}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right|$$
$$\leq (\chi-\psi)^2\left[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{125\left|\delta''(\psi)\right|^q}{82944}+\frac{1}{3072}\left|\delta''(\chi)\right|^q\right)^{\frac{1}{q}}+\frac{1}{648}\left(\frac{\left|\delta''(\psi)\right|^q+\left|\delta''(\chi)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\frac{1}{3072}\left|\delta''(\psi)\right|^q+\frac{125\left|\delta''(\chi)\right|^q}{82944}\right)^{\frac{1}{q}}\right].$$

Remark 4.11. If we choose $\alpha=1$ and $\beta(0)=\beta(1)=1$, in Corollary 4.10, then we have [37, Theorem 2.2].

Remark 4.12. If we choose $q=1$ in Theorem 4.8, then we obtain Theorem 4.1.

5 Newton-type inequalities for φ-quasi-convex functions

In this section, we obtain fractional Newton-type inequalities for φ-quasi-convex function.

Theorem 5.1. *Under the assumptions of Lemma 3.1. If $|\delta''|$ is φ-quasi-convex on $[\psi,\chi]$, then the following fractional inequality holds:*

$$\begin{aligned} & \left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\ & \left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\ \leq\; & \frac{(\chi-\psi)^2}{192}\max\left\{\left|\delta''(\psi)\right|,\left|\delta''(\psi)\right|+\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right\}. \end{aligned}$$

Proof. By using the Lemma 3.1, we have

$$\begin{aligned} & \left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\ & \left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\ \leq\; & (\chi-\psi)^2\int_0^1|M(\eta)|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta \\ \leq\; & (\chi-\psi)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta\right. \\ & \left.+\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta\right] \\ \leq\; & (\chi-\psi)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\max\left\{\left|\delta''(\psi)\right|,\left|\delta''(\psi)\right|+\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right\}d\eta\right. \\ & +\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\max\left\{\left|\delta''(\psi)\right|,\left|\delta''(\psi)\right|+\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right\}d\eta \\ & \left.+\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\max\left\{\left|\delta''(\psi)\right|,\left|\delta''(\psi)\right|+\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right\}d\eta\right] \\ =\; & \frac{(\chi-\psi)^2}{192}\max\left\{\left|\delta''(\psi)\right|,\left|\delta''(\psi)\right|+\varphi\left(\left|\delta''(\psi)\right|,\left|\delta''(\chi)\right|\right)\right\}. \end{aligned}$$

This completes the proof. □

Theorem 5.2. *Under the assumptions of Lemma 3.1, if $|\delta''|^q$ is φ-quasi-convex on $[\psi,\chi]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\begin{aligned} & \left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\ & \left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\ \leq\; & \frac{(\chi-\psi)^2}{192}\max\left\{\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. By using the Lemma 3.1, we have

$$\begin{aligned} & \left|\frac{1}{8}\left[\delta(\psi)+3\delta\left(\frac{2\psi+\chi}{3}\right)+3\delta\left(\frac{\psi+2\chi}{3}\right)+\delta(\chi)\right]\right. \\ & \left.-\frac{\beta(\alpha)}{\alpha(\chi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\delta\right)(k)+\left({}^{CF}I_{\chi}^{\alpha}\delta\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\delta(k)\right| \\ \leq\; & (\chi-\psi)^2\int_0^1|M(\eta)|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta \\ \leq\; & (\chi-\psi)^2\left[\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left|\delta''((1-\eta)\psi+\eta\chi)\right|d\eta\right. \end{aligned}$$

$$
\begin{aligned}
& +\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left|\delta''\left((1-\eta)\psi+\eta\chi\right)\right|d\eta\Bigg] \\
\leq\ & (\chi-\psi)^2\Bigg[\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\left|\delta''\left((1-\eta)\psi+\eta\chi\right)\right|^q d\eta\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\left|\delta''\left((1-\eta)\psi+\eta\chi\right)\right|^q d\eta\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\left|\delta''\left((1-\eta)\psi+\eta\chi\right)\right|^q d\eta\right)^{\frac{1}{q}}\Bigg] \\
\leq\ & (\chi-\psi)^2\Bigg[\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|\frac{\eta}{2}\left(\frac{1}{4}-\eta\right)\right|\right. \\
& \times\max\left\{\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right\}d\eta\Big)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\eta}{2}-\frac{\eta^2}{2}-\frac{1}{8}\right|\right. \\
& \times\max\left\{\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right\}d\eta\Big)^{\frac{1}{q}} \\
& +\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|(1-\eta)\left(\frac{\eta}{2}-\frac{3}{8}\right)\right|\right| \\
& \times\left|\max\left\{\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right\}d\eta\right)^{\frac{1}{q}}\Bigg] \\
=\ & (\chi-\psi)^2\Bigg[\left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\left(\frac{19}{10368}\right)\max\left\{\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right\}\right)^{\frac{1}{q}} \\
& +\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\left(\frac{1}{648}\right)\max\left\{\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right\}\right)^{\frac{1}{q}} \\
& \left(\frac{19}{10368}\right)^{1-\frac{1}{q}}\left(\left(\frac{19}{10368}\right)\max\left\{\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right\}\right)^{\frac{1}{q}}\Bigg] \\
=\ & \frac{(\chi-\psi)^2}{192}\max\left\{\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\psi)\right|^q+\varphi\left(\left|\delta''(\psi)\right|^q,\left|\delta''(\chi)\right|^q\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This completes the proof. □

6 Application to special means

(a) *The Arithmetic mean:*

$$A=A(\psi,\chi):=\frac{\psi+\chi}{2},\ \psi,\chi\in\mathbb{R}.$$

(b) *The Logarithmic mean:*

$$L=L(\psi,\chi):=\frac{\chi-\psi}{\ln|\chi|-\ln|\psi|},\ \psi,\chi\in\mathbb{R},\ \psi\neq\chi.$$

(c) *The Generalized Logarithmic-mean:*

$$L_r=L_r(\psi,\chi):=\left[\frac{\chi^{r+1}-\psi^{r+1}}{(r+1)(\chi-\psi)}\right]^{\frac{1}{r}}\ r\in\mathbb{R}\backslash\{-1,0\},\ \psi,\chi\in\mathbb{R},\ \psi\neq\chi.$$

Proposition 6.1. *Let $\psi,\chi\in\mathbb{R}$ with $0<\psi<\chi$ and $n\geq 2$. Then we have*

$$
\begin{aligned}
& \left|\frac{1}{4}A(\psi^n,\chi^n)+\frac{3}{8}\left(A^n(\psi,\chi)+A^n(\psi,\chi)\right)-L_n^n(\psi,\chi)\right| \\
\leq\ & \frac{(\chi-\psi)^2}{384}n(n-1)\left(\psi^{n-1}+\chi^{n-1}\right).
\end{aligned}
$$

Proof. The assertion follows from Theorem 4.1, where $\delta(x)=x^n$, $\alpha=s=1$ with $\beta(0)=\beta(1)=1$, and $\varphi(\psi,\chi)=\psi-\chi$. □

Proposition 6.2. *Let* $\psi, \chi \in \mathbb{R}$ *with* $0 < \psi < \chi$. *Then we have*

$$\begin{aligned} & \left| \frac{1}{4} H^{-1}(\psi, \chi) + \frac{3}{8} \left(A^{-1}(\psi, \chi) + A^{-1}(\psi, \chi) \right) - L^{-1}(\psi, \chi) \right| \\ \leq \quad & \frac{H^{-1}(\chi - \psi)^2}{192} \left(\psi^3 + \chi^3 \right). \end{aligned}$$

Proof. The assertion follows from Theorem 4.1, where $\delta(x) = \frac{1}{x}$, $\alpha = s = 1$ with $\beta(0) = \beta(1) = 1$, and $\varphi(\psi, \chi) = \psi - \chi$. □

Proposition 6.3. *Let* $\psi, \chi \in \mathbb{R}$ *with* $0 < \psi < \chi$. *Then we have*

$$\begin{aligned} & \left| \frac{1}{4} H^{-1}(\psi, \chi) + \frac{9}{16} \left(A^{-1}(2\psi, \chi) + A^{-1}(\psi, 2\chi) \right) - L^{-1}(\psi, \chi) \right| \\ \leq \quad & \frac{(\chi - \psi)^2}{192} \max \left\{ \frac{2}{\psi^3}, \frac{2}{\chi^3} \right\}. \end{aligned}$$

Proof. The assertion follows from Theorem 5.1, where $\delta(x) = \frac{1}{x}$, $\alpha = 1$ and $\beta(0) = \beta(1) = 1$ where $|\delta''| = \frac{2}{x^3}$ and $\varphi(\psi, \chi) = \psi - \chi$. □

7 Conclusions

The fractional calculus developing efficient and accurate numerical methods for solving equations is essential. These methods allow researchers to apply fractional calculus techniques to real-world problems, and facilitate the implementation of generalized results. In this article, we have established the new identity via Caputo-Fabrizio fractional integral operator. Employing this new identity of Newton-type inequalities for the s-φ-*convex function. Simpson-type inequality is said to be Simpson's second formula if it consists of four points. Moreover, we also include an application for special means. In the future, we might think about using modified convexity and other fractional integral operators.*

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Author information

A. Munir, H. Budak, I. Faiz, L. Rathour, H. Kara, A. Kashuri, Department of Mathematics, COMSATS University Islamabad, Sahiwal Campus, Sahiwal 57000, Pakistan, Pakistan.*
E-mail: munirarslan999@gmail.com