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SOME RESULTS ON L-HARMONIC MAPS

F. Gherib, E. Remli and A. Mohammed Cherif

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Corresponding Author: A. Mohammed Cherif

Abstract In this paper, we discuss the stabilities of L-harmonic maps on sphere $\mathbb{S}^n$ with $n > 2$. We also prove that any L-harmonic map from a complete Riemannian manifold (M, g) to Riemannian manifold (N, h) is necessarily constant, with (N, h) admitting a proper homothetic vector field satisfying some conditions.

1 Preliminaries and Notations

We give some definitions. Let (M, g) be a Riemannian manifold. By R^M and Ric^M we denote respectively the Riemannian curvature tensor and the Ricci tensor of (M, g). Thus R^M and Ric^M are defined by

$$R^M(X,Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z, \tag{1.1}$$

$$\mathrm{Ric}^M(X,Y) = g(R^M(X, e_i)e_i, Y), \tag{1.2}$$

where ∇^M is the Levi-Civita connection with respect to g, $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. The divergence of $(0, p)$-tensor α on M is defined by

$$(\mathrm{div}^M \alpha)(X_1, ..., X_{p-1}) = (\nabla^M_{e_i} \alpha)(e_i, X_1, ..., X_{p-1}), \tag{1.3}$$

where $X_1, ..., X_{p-1} \in \Gamma(TM)$, and $\{e_i\}$ is an orthonormal frame. Given a smooth function λ on M, the gradient of λ is defined by

$$g(\mathrm{grad}^M \lambda, X) = X(\lambda), \tag{1.4}$$

the Hessian of λ is defined by

$$(\mathrm{Hess}^M \lambda)(X,Y) = g(\nabla^M_X \mathrm{grad}\, \lambda, Y), \tag{1.5}$$

where $X, Y \in \Gamma(TM)$, the Laplacian of λ is defined by

$$\Delta^M(\lambda) = \mathrm{trace}\, \mathrm{Hess}^M \lambda, \tag{1.6}$$

(for more details, see for example [12]).
A vector field ξ on a Riemannian manifold (M, g) is called a homothetic if $\mathcal{L}_\xi g = 2kg$, for some constant $k \in \mathbb{R}$, where $\mathcal{L}_\xi g$ is the Lie derivative of the metric g with respect to ξ, that is

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2kg(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{1.7}$$

The constant k is then called the homothetic constant. If ξ is homothetic and $k \neq 0$, then it is called proper homothetic while $k = 0$ it is Killing (see [1, 8, 17]). Note that, if a complete Riemannian manifold of dimension ≥ 2 admits a proper homothetic vector field then the manifold

is isometric to the Euclidean space (see [7, 17]).
Consider a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds, $L : M \times N \times \mathbb{R} \to (0, \infty)$, $(x, y, r) \mapsto L(x, y, r)$, be a smooth positive function, for any compact domain D of M the L-energy functional of φ is defined by

$$E_L(\varphi; D) = \int_D L\big(x, \varphi(x), e(\varphi)(x)\big)\, v_g, \tag{1.8}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \frac{1}{2}\, h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{1.9}$$

v_g is the volume element, here $\{e_i\}$ is a orthonormal frame on (M, g). A map is called L-harmonic if it is a critical point of the L-energy functional over any compact subset D of M. L-harmonic maps are solutions of two-order nonlinear elliptic system, challenging to solve even in simple cases like f-harmonic and bi-f-harmonic curves (see [5]).
We denote by $\partial_r = \partial/\partial r$, $L' = \partial_r(L)$, $L'' = \partial_r\big(\partial_r(L)\big)$, and let L'_φ, $L''_\varphi \in C^\infty(M)$ defined by

$$L'_\varphi(x) = L'\big(x, \varphi(x), e(\varphi)(x)\big), \quad L''_\varphi(x) = L''\big(x, \varphi(x), e(\varphi)(x)\big). \tag{1.10}$$

Theorem 1.1 (The first variation of E_L, [10]). *Let $\varphi : (M, g) \to (N, h)$ be a smooth map and let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then*

$$\frac{d}{dt} E_L(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_L(\varphi), v)\, v_g, \tag{1.11}$$

where $v = \dfrac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_L(\varphi) = L'_\varphi\, \tau(\varphi) + d\varphi\big(\operatorname{grad}^M L'_\varphi\big) - (\operatorname{grad}^N L) \circ \varphi, \tag{1.12}$$

and $\tau(\varphi)$ is the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi. \tag{1.13}$$

From the first variation formula (1.11), a map $\varphi : (M, g) \to (N, h)$ is L-harmonic if and only if $\tau_L(\varphi) = 0$.

Theorem 1.2 (The second variation of the E_L, [10]). *Let $\varphi : (M, g) \to (N, h)$ be an L-harmonic map between Riemannian manifolds and $\{\varphi_{t,s}\}_{t,s\in(-\epsilon,\epsilon)}$ be a two-parameter variation with compact support in D. Set*

$$v = \frac{\partial \varphi_{t,s}}{\partial t}\Big|_{t=s=0}, \qquad w = \frac{\partial \varphi_{t,s}}{\partial s}\Big|_{t=s=0}. \tag{1.14}$$

Under the notation above we have the following

$$\frac{\partial^2}{\partial t \partial s} E_L(\varphi_{t,s}; D)\Big|_{t=s=0} = \int_D h(J_{\varphi,L}(v), w)\, v_g, \tag{1.15}$$

where $J_{\varphi,L}(v) \in \Gamma(\varphi^{-1}TN)$ given by

$$\begin{aligned} J_{\varphi,L}(v) \;\; = \;\; & -L'_\varphi \operatorname{trace} R^N(v, d\varphi)d\varphi - \operatorname{trace} \nabla^\varphi L'_\varphi \nabla^\varphi v \\ & + \big(\nabla^N_v \operatorname{grad}^N L\big) \circ \varphi + < \nabla^\varphi v, d\varphi > (\operatorname{grad}^N L') \circ \varphi \\ & - \operatorname{trace} \nabla^\varphi < \nabla^\varphi v, d\varphi > L''_\varphi\, d\varphi. \end{aligned} \tag{1.16}$$

Here $<,>$ denote the inner product on $T^*M \otimes \varphi^{-1}TN$ and R^N is the curvature tensor on (N, h). If M is a compact Riemannian manifold, φ be a L-harmonic map from (M, g) to Riemannian manifold (N, h), and for any vector field v along φ,

$$I^\varphi_L(v, v) \equiv \int_M h(J_{\varphi,L}(v), v)\, v_g \geq 0, \tag{1.17}$$

then φ is called a stable L-harmonic map. Note that, the definition of stable L-harmonic maps is a generalization of stable harmonic maps for $L(x, y, r) = r$ (see [18]), is also a generalization of stable f-harmonic maps with f is a smooth positive function on M, and $L(x, y, r) = f(x)r$ (see [11]). The Liouville type theorem for harmonic and biharmonic maps on Riemannain manifolds in particular on $\mathbb{S}^n$ has been studied by many researchers. In [14], S. Ouakkas gives an example of non-harmonic biharmonic maps. In this paper, we present some Liouville type theorems for L-harmonic maps between two Riemannian manifolds. In particular, we study the case where the codomain of L-harmonic maps has a proper homothetic vector field. We shall extend some results proved in [6, 9, 11, 15, 19].

2 Nonexistence theorems on stable L-harmonic maps

Theorem 2.1. *Any stable L-harmonic map φ from sphere $(\mathbb{S}^n, g)$ $(n > 2)$ to Riemannian manifold (N, h) is constant, where L is a smooth positive function on $\mathbb{S}^n \times N \times \mathbb{R}$ satisfying $L'_\varphi > 0$ and the following inequality*

$$\int_{\mathbb{S}^n} \Big[\text{trace}\, h((\nabla d\varphi)(\cdot, \text{grad}^{\mathbb{S}^n} L'), d\varphi(\cdot)) - |d\varphi|^4 L''_\varphi \Big] v^{\mathbb{S}^n} \geq 0.$$

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^n$. Set

$$\lambda(x) = <\alpha, x>_{\mathbb{R}^{n+1}},$$

for all $x \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ and let $v = \text{grad}^{\mathbb{S}^n} \lambda$. Note that

$$v = <\alpha, e_i> e_i, \; \nabla^{\mathbb{S}^n}_X v = -\lambda X, \text{ for all } X \in \Gamma(T\mathbb{S}^n),$$
$$\text{trace}_g (\nabla^{\mathbb{S}^n})^2 v = \nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v - \nabla^{\mathbb{S}^n}_{\nabla^{\mathbb{S}^n}_{e_i} e_i} v = -v,$$

where $\nabla^{\mathbb{S}^n}$ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric g of the sphere (see [18]). At point x_0, we have

$$\nabla^\varphi_{e_i} L'_\varphi \nabla^\varphi_{e_i} d\varphi(v) = \nabla^\varphi_{\text{grad}^{\mathbb{S}^n} L'_\varphi} d\varphi(v) + L'_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} d\varphi(v), \tag{2.1}$$

the first term of (2.1) is given by

$$\begin{aligned} \nabla^\varphi_{\text{grad}^{\mathbb{S}^n} L'_\varphi} d\varphi(v) &= \nabla^\varphi_v d\varphi(\text{grad}^{\mathbb{S}^n} L'_\varphi) + d\varphi([\text{grad}^{\mathbb{S}^n} L'_\varphi, v]) \\ &= \nabla^\varphi_v d\varphi(\text{grad}^{\mathbb{S}^n} L'_\varphi) + d\varphi(\nabla^{\mathbb{S}^n}_{\text{grad}^{\mathbb{S}^n} L'_\varphi} v) \\ &\quad - d\varphi(\nabla^{\mathbb{S}^n}_v \text{grad}^{\mathbb{S}^n} L'_\varphi), \end{aligned} \tag{2.2}$$

the seconde term of (2.1) is given by

$$\begin{aligned} L'_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} d\varphi(v) &= L'_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_v d\varphi(e_i) + L'_\varphi \nabla^\varphi_{e_i} d\varphi([e_i, v]) \\ &= L'_\varphi R^N(d\varphi(e_i), d\varphi(v)) d\varphi(e_i) + L'_\varphi \nabla^\varphi_v \nabla^\varphi_{e_i} d\varphi(e_i) \\ &\quad + L'_\varphi d\varphi([e_i, [e_i, v]]) + 2L'_\varphi \nabla^\varphi_{[e_i, v]} d\varphi(e_i), \end{aligned} \tag{2.3}$$

from the definition of tension field, we get

$$\begin{aligned} L'_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i} d\varphi(v) &= -L'_\varphi R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + L'_\varphi \nabla^\varphi_v \tau(\varphi) \\ &\quad + L'_\varphi \nabla^\varphi_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + L'_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - L'_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_v e_i) + 2L'_\varphi \nabla^\varphi_{[e_i, v]} d\varphi(e_i) \\ &= -L'_\varphi R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + \nabla^\varphi_v L'_\varphi \tau(\varphi) - v(L'_\varphi) \tau(\varphi) \\ &\quad + L'_\varphi \nabla^\varphi_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + L'_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - L'_\varphi d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_v e_i) + 2L'_\varphi \nabla^\varphi_{[e_i, v]} d\varphi(e_i), \end{aligned} \tag{2.4}$$

by equations (2.1), (2.2), (2.4), and the L-harmonicity condition of φ, we have

$$\begin{aligned}\nabla^{\varphi}_{e_i} L'_{\varphi} \nabla^{\varphi}_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\text{grad}^{\mathbb{S}^n} L'_{\varphi}} v) - d\varphi(\nabla^{\mathbb{S}^n}_{v} \text{grad}^{\mathbb{S}^n} L'_{\varphi}) \\ &\quad - L'_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) \\ &\quad + \nabla^{\varphi}_{v} (\text{grad}^N L) \circ \varphi - v(L'_{\varphi}) \tau(\varphi) \\ &\quad + L'_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{v} \nabla^{\mathbb{S}^n}_{e_i} e_i) + L'_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - L'_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{v} e_i) + 2 L'_{\varphi} \nabla^{\varphi}_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i), \end{aligned} \tag{2.5}$$

by the definition of Ricci tensor, we get

$$\begin{aligned}\nabla^{\varphi}_{e_i} L'_{\varphi} \nabla^{\varphi}_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\text{grad}^{\mathbb{S}^n} L'_{\varphi}} v) - d\varphi(\nabla^{\mathbb{S}^n}_{v} \text{grad}^{\mathbb{S}^n} L'_{\varphi}) \\ &\quad - L'_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + \nabla^{\varphi}_{v} (\text{grad}^N L) \circ \varphi \\ &\quad - v(L'_{\varphi}) \tau(\varphi) + L'_{\varphi} d\varphi(\text{Ricci}^{\mathbb{S}^n} v) + L'_{\varphi} d\varphi(\text{trace}(\nabla^{\mathbb{S}^n})^2 v) \\ &\quad + L'_{\varphi} \nabla^{\varphi}_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i), \end{aligned} \tag{2.6}$$

from the property $\nabla^{\mathbb{S}^n}_{X} v = -\lambda X$, we obtain

$$\begin{aligned}\nabla^{\varphi}_{e_i} L'_{\varphi} \nabla^{\varphi}_{e_i} d\varphi(v) &= -\lambda d\varphi(\text{grad}^{\mathbb{S}^n} L'_{\varphi}) - d\varphi(\nabla^{\mathbb{S}^n}_{v} \text{grad}^{\mathbb{S}^n} L'_{\varphi}) \\ &\quad - L'_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) \\ &\quad + \nabla^{\varphi}_{v} (\text{grad}^N L) \circ \varphi - v(L'_{\varphi}) \tau(\varphi) \\ &\quad + L'_{\varphi} d\varphi(\text{Ricci}^{\mathbb{S}^n} v) + L'_{\varphi} d\varphi(\text{trace}(\nabla^{\mathbb{S}^n})^2 v) \\ &\quad - \lambda L'_{\varphi} \tau(\varphi). \end{aligned} \tag{2.7}$$

From (1.16), and equation (2.7) we have

$$\begin{aligned} J^{\varphi}_{f}(d\varphi(v)) &= \lambda d\varphi(\text{grad}^{\mathbb{S}^n} L'_{\varphi}) + d\varphi(\nabla^{\mathbb{S}^n}_{v} \text{grad}^{\mathbb{S}^n} L'_{\varphi}) + v(L'_{\varphi}) \tau(\varphi) \\ &\quad - L'_{\varphi} d\varphi(\text{Ricci}^{\mathbb{S}^n} v) - L'_{\varphi} d\varphi(\text{trace}(\nabla^{\mathbb{S}^n})^2 v) + \lambda L'_{\varphi} \tau(\varphi) \\ &\quad + < \nabla^{\varphi} d\varphi(v), d\varphi > (\text{grad}^N L') \circ \varphi \\ &\quad - \text{trace } \nabla^{\varphi} < \nabla^{\varphi} d\varphi(v), d\varphi > L''_{\varphi} d\varphi, \end{aligned} \tag{2.8}$$

since $\text{trace}_g(\nabla^{\mathbb{S}^n})^2 v = -v$ and $\text{Ricci}^{\mathbb{S}^n} v = (n-1)v$ (see [1, 18]), we conclude

$$\begin{aligned} h(J^{\varphi}_{f}(d\varphi(v)), d\varphi(v)) &= \lambda h(d\varphi(\text{grad}^{\mathbb{S}^n} L'_{\varphi}), d\varphi(v)) \\ &\quad + h(d\varphi(\nabla^{\mathbb{S}^n}_{v} \text{grad}^{\mathbb{S}^n} L'_{\varphi}), d\varphi(v)) \\ &\quad + v(L'_{\varphi}) h(\tau(\varphi), d\varphi(v)) \\ &\quad - (n-2) L'_{\varphi} h(d\varphi(v), d\varphi(v)) \\ &\quad + \lambda L'_{\varphi} h(\tau(\varphi), d\varphi(v)) \\ &\quad + < \nabla^{\varphi} d\varphi(v), d\varphi > d\varphi(v)(L') \\ &\quad - h(\text{trace } \nabla^{\varphi} < \nabla^{\varphi} d\varphi(v), d\varphi > L''_{\varphi} d\varphi, d\varphi(v)), \end{aligned} \tag{2.9}$$

by (2.9) and the L-harmonicity condition of φ, it follows that

$$\begin{aligned}\operatorname{trace}_\alpha h(J_f^\varphi(d\varphi(v)),d\varphi(v)) &= h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j}\operatorname{grad}^{\mathbb{S}^n}L'_\varphi),d\varphi(e_j))\\ &\quad +h(\tau(\varphi),d\varphi(\operatorname{grad}^{\mathbb{S}^n}L'_\varphi))\\ &\quad -(n-2)L'_\varphi|d\varphi|^2\\ &\quad +\operatorname{trace}_\alpha h(\nabla^\varphi_{e_j}d\varphi(v),d\varphi(e_j))d\varphi(v)(L')\\ &\quad -\operatorname{trace}_\alpha h(\nabla^\varphi_{e_j}<\nabla^\varphi d\varphi(v),d\varphi>L''_\varphi d\varphi(e_j),d\varphi(v)),\end{aligned} \tag{2.10}$$

since at x_0, $\tau(\varphi)=\nabla^\varphi_{e_j}d\varphi(e_j)$, we have the following formula

$$\begin{aligned}h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j}\operatorname{grad}^{\mathbb{S}^n}L'_\varphi),d\varphi(e_j)) &+ h(\tau(\varphi),d\varphi(\operatorname{grad}^{\mathbb{S}^n}L'_\varphi))\\ &= \operatorname{div}^{\mathbb{S}^n}h(d\varphi(\cdot),d\varphi(\operatorname{grad}^{\mathbb{S}^n}L'_\varphi))\\ &\quad -\operatorname{trace}h((\nabla d\varphi)(\cdot,\operatorname{grad}^{\mathbb{S}^n}L'_\varphi),d\varphi(\cdot)),\end{aligned} \tag{2.11}$$

and note that by the definition of $\nabla d\varphi$ and the property $\nabla^{\mathbb{S}^n}_X v=-\lambda X$, we have

$$\operatorname{trace}_\alpha h(\nabla^\varphi_{e_j}d\varphi(v),d\varphi(e_j))d\varphi(v)(L') = \operatorname{trace}h((\nabla d\varphi)(\cdot,d\varphi(e_i)(L')e_i),d\varphi(\cdot)), \tag{2.12}$$

$$\begin{aligned}-\operatorname{trace}_\alpha & h(\nabla^\varphi_{e_j}<\nabla^\varphi d\varphi(v),d\varphi>L''_\varphi d\varphi(e_j),d\varphi(v))\\ &= -\operatorname{trace}_\alpha\operatorname{div}^{\mathbb{S}^n}\big[<\nabla^\varphi d\varphi(v),d\varphi>L''_\varphi h(d\varphi(\cdot),d\varphi(v))\big]\\ &\quad +\operatorname{trace}_\alpha<\nabla^\varphi d\varphi(v),d\varphi>^2L''_\varphi\\ &= -\operatorname{trace}_\alpha\operatorname{div}^{\mathbb{S}^n}\big[<\nabla^\varphi d\varphi(v),d\varphi>L''_\varphi h(d\varphi(\cdot),d\varphi(v))\big]\\ &\quad +\operatorname{trace}h((\nabla d\varphi)(\cdot,\operatorname{grad}^{\mathbb{S}^n}e(\varphi)),d\varphi(\cdot))L''_\varphi+|d\varphi|^4L''_\varphi,\end{aligned} \tag{2.13}$$

from the stable L-harmonic condition, the divergence theorem, and equations (2.10), (2.11), (2.12), (2.13), with

$$\operatorname{grad}^{\mathbb{S}^n}L'_\varphi=\operatorname{grad}^{\mathbb{S}^n}L'+d\varphi(e_i)(L')e_i+L''_\varphi\operatorname{grad}^{\mathbb{S}^n}e(\varphi),$$

we get the following

$$\begin{aligned}0\le\operatorname{trace}_\alpha I_f^\varphi(d\varphi(v),d\varphi(v)) &+ \int_{\mathbb{S}^n}\Big[\operatorname{trace}h((\nabla d\varphi)(\cdot,\operatorname{grad}^{\mathbb{S}^n}L'),d\varphi(\cdot))\\ &- |d\varphi|^4L''_\varphi\Big]v^{\mathbb{S}^n}\\ &= -(n-2)\int_{\mathbb{S}^n}L'_\varphi|d\varphi|^2v^{\mathbb{S}^n}\le 0.\end{aligned}$$

Consequently, $|d\varphi|=0$, that is φ is constant, because $n>2$. The proof is completed. □

Using Theorem 2.1 we obtain:

- Any stable harmonic map φ from sphere $(\mathbb{S}^n,g)$ $(n>2)$ to Riemannian manifold (N,h) is constant (see [18, 19]).
- Any stable f-harmonic map φ from sphere $(\mathbb{S}^n,g)$ $(n>2)$ to Riemannian manifold (N,h) is constant, where f is a smooth positive function on $\mathbb{S}^n$ satisfying the following inequality

$$\int_{\mathbb{S}^n}\operatorname{trace}h((\nabla d\varphi)(\cdot,\operatorname{grad}^{\mathbb{S}^n}f),d\varphi(\cdot))v^{\mathbb{S}^n}\ge 0,$$

(see [11, 15]).

Using the similar technique we have:

Theorem 2.2. *Let (M,g) be a compact Riemannian manifold. When $n>2$, any stable L-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where L is a smooth positive function on $M \times \mathbb{S}^n \times \mathbb{R}$, with $L'_\varphi > 0$ and*

$$\int_M \left[\Delta^{\mathbb{S}^n}(L)\circ\varphi + |d\varphi|^4 L''_\varphi\right] v^M \leq 0.$$

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in M. When the same data of previous proof, we have

$$\nabla^\varphi_{e_i} L'_\varphi \nabla^\varphi_{e_i}(v\circ\varphi) = \nabla^\varphi_{\operatorname{grad}^M L'_\varphi}(v\circ\varphi) + L'_\varphi \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(v\circ\varphi), \tag{2.14}$$

the first term of (2.14) is given by

$$\nabla^\varphi_{\operatorname{grad}^M L'_\varphi}(v\circ\varphi) = -(\lambda\circ\varphi)d\varphi(\operatorname{grad}^M L'_\varphi), \tag{2.15}$$

the seconde term of (2.14) is given by

$$\begin{aligned} L'_\varphi \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(v\circ\varphi) &= -L'_\varphi \nabla^\varphi_{e_i}(\lambda\circ\varphi)d\varphi(e_i) \\ &= -L'_\varphi d\varphi(\operatorname{grad}^M(\lambda\circ\varphi)) - (\lambda\circ\varphi)L'_\varphi \tau(\varphi), \end{aligned} \tag{2.16}$$

by the definition of gradient operator, we get

$$-L'_\varphi d\varphi(\operatorname{grad}^M(\lambda\circ\varphi)) = -L'_\varphi < d\varphi(e_i), v\circ\varphi > d\varphi(e_i), \tag{2.17}$$

substituting the formulas (2.15), (2.16), (2.17) into (2.14) gives

$$\begin{aligned} \nabla^\varphi_{e_i} L'_\varphi \nabla^\varphi_{e_i}(v\circ\varphi) &= -(\lambda\circ\varphi)d\varphi(\operatorname{grad}^M L'_\varphi) - L'_\varphi < d\varphi(e_i), v\circ\varphi > d\varphi(e_i) \\ &\quad -(\lambda\circ\varphi)L'_\varphi \tau(\varphi), \end{aligned} \tag{2.18}$$

from the L-harmonicity condition of φ, and equation (2.18), we have

$$\begin{aligned} < \nabla^\varphi_{e_i} L'_\varphi \nabla^\varphi_{e_i}(v\circ\varphi), v\circ\varphi > &= -L'_\varphi < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi > \\ &\quad -(\lambda\circ\varphi) < (\operatorname{grad}^{\mathbb{S}^n} L)\circ\varphi, v\circ\varphi >, \end{aligned} \tag{2.19}$$

since the sphere $\mathbb{S}^n$ has constant curvature, we obtain

$$\begin{aligned} < L'_\varphi R^{\mathbb{S}^n}(v\circ\varphi, d\varphi(e_i))d\varphi(e_i), v\circ\varphi >&= L'_\varphi |d\varphi|^2 < v\circ\varphi, v\circ\varphi > \\ &\quad -L'_\varphi < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi >, \end{aligned} \tag{2.20}$$

by the definition of Jacobi operator and equations (2.19), (2.20), we get

$$\begin{aligned} < J^\varphi_f(v\circ\varphi), v\circ\varphi > &= 2L'_\varphi < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi > \\ &\quad -L'_\varphi |d\varphi|^2 < v\circ\varphi, v\circ\varphi > \\ &\quad +(\lambda\circ\varphi) < (\operatorname{grad}^{\mathbb{S}^n} L)\circ\varphi, v\circ\varphi > \\ &\quad + < (\nabla^{\mathbb{S}^n}_{v\circ\varphi}\operatorname{grad}^{\mathbb{S}^n} L)\circ\varphi, v\circ\varphi > \\ &\quad + < \nabla^\varphi v\circ\varphi, d\varphi >< (\operatorname{grad}^{\mathbb{S}^n} L')\circ\varphi, v\circ\varphi >, \\ &\quad - < \nabla^\varphi_{e_i} < \nabla^\varphi v\circ\varphi, d\varphi > L''_\varphi d\varphi(e_i), v\circ\varphi >, \end{aligned} \tag{2.21}$$

since $< \nabla^\varphi v\circ\varphi, d\varphi >= -(\lambda\circ\varphi)|d\varphi|^2$, by equation (2.21), we find that

$$\operatorname{trace}_\alpha < J^\varphi_f(v\circ\varphi), v\circ\varphi > = (2-n)L'_\varphi |d\varphi|^2 + \Delta^{\mathbb{S}^n}(L)\circ\varphi + |d\varphi|^4 L''_\varphi, \tag{2.22}$$

where $\Delta^{\mathbb{S}^n}(L)\circ\varphi = \text{trace}_\alpha(\text{Hess}^{\mathbb{S}^n} L)(v\circ\varphi, v\circ\varphi)$, and $\text{Hess}^{\mathbb{S}^n} L$ is the hessian of the function L on $\mathbb{S}^n$, from (2.22) we have

$$\begin{aligned} \text{trace}_\alpha I_f^\varphi(v\circ\varphi, v\circ\varphi) &= (2-n)\int_M L'_\varphi |d\varphi|^2 v^M \\ &\quad + \int_M \Big[\Delta^{\mathbb{S}^n}(L)\circ\varphi + |d\varphi|^4 L''_\varphi\Big] v^M. \end{aligned} \tag{2.23}$$

Hence Theorem **??** follows from (2.23) and the stable f-harmonicity condition of φ with $n>2$, $L'_\varphi > 0$ and $\int_M \big[\Delta^{\mathbb{S}^n}(L)\circ\varphi + |d\varphi|^4 L''_\varphi\big]v^M \leq 0$. □

From Theorem 2.2, we deduce:

- Let (M,g) be a compact Riemannian manifold. When $n>2$, any stable harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant (see [6, 13, 18]).
- Let (M,g) be a compact Riemannian manifold. When $n>2$, any stable f-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where f is a smooth positive function on $M\times\mathbb{S}^n$, with $\int_M e(\varphi)[\Delta^{\mathbb{S}^n}(f)\circ\varphi]v^M \leq 0$ (see [15]).

3 Homothetic vector fields and L-harmonic maps

Theorem 3.1. *Let (M,g) be a compact orientable Riemannian manifold without boundary, (N,h) a Riemannian manifold admitting a proper homothetic vector field ξ with homothetic constant $k>0$, and let L be a smooth positive function on $M\times N\times\mathbb{R}$ such that $L'>0$ and $\xi(L)\geq 0$. Then, any L-harmonic map φ from (M,g) to (N,h) is constant.*

Proof. We set

$$\omega(X) = h(\xi\circ\varphi, L'_\varphi d\varphi(X)), \quad \forall X\in\Gamma(TM), \tag{3.1}$$

let $\{e_i\}$ be a normal orthonormal frame at $x\in M$, we have

$$\begin{aligned} \text{div}^M\omega &= e_i\big[h(\xi\circ\varphi, L'_\varphi d\varphi(e_i))\big] \\ &= h(\nabla^\varphi_{e_i}(\xi\circ\varphi), L'_\varphi d\varphi(e_i)) + h(\xi\circ\varphi, \nabla^\varphi_{e_i} L'_\varphi d\varphi(e_i)) \\ &= h(\nabla^\varphi_{e_i}(\xi\circ\varphi), L'_\varphi d\varphi(e_i)) + h(\xi\circ\varphi, L'_\varphi\tau(\varphi) + d\varphi(\text{grad}^M L'_\varphi)) \end{aligned} \tag{3.2}$$

by equation (3.2) and the L-harmonicity of φ, we get:

$$\begin{aligned} \text{div}^M\omega &= h(\nabla^\varphi_{e_i}(\xi\circ\varphi), L'_\varphi d\varphi(e_i)) + h(\xi\circ\varphi, (\text{grad}^N L)\circ\varphi) \\ &= L'_\varphi h(\nabla^N_{d\varphi(e_i)}\xi, d\varphi(e_i)) + h(\xi\circ\varphi, (\text{grad}^N L)\circ\varphi) \end{aligned}$$

since ξ is a homothetic vector field with homothetic constant k, we find that

$$\begin{aligned} \text{div}^M\omega &= L'_\varphi k h(d\varphi(e_i), d\varphi(e_i)) + h(\xi\circ\varphi, (\text{grad}^N L)\circ\varphi) \\ &= kL'_\varphi |d\varphi|^2 + \xi(L)\circ\varphi. \end{aligned}$$

Theorem 3.1 follows from the last equation, and the divergence theorem, with $L'>0$ and $\xi(L)\geq 0$. □

From Theorem 3.1, we get the following result.

Corollary 3.2 ([9]). *Let (M,g) be a compact orientable Riemannian manifold without boundary, and (N,h) be a Riemannian manifold admitting a proper homothetic vector field ξ with homothetic constant $k\neq 0$. Then, any harmonic map φ from (M,g) to (N,h) is constant.*

In the case of non-compact Riemannian manifold, we obtain the following result.

Theorem 3.3. *Let (M,g) be a complete non-compact orientable Riemannian manifold, (N,h) a Riemannian manifold admitting a proper homothetic vector field ξ with homothetic constant $k>0$, and let L be a smooth positive function on $M\times N\times\mathbb{R}$ such that $L'>0$ and $\xi(L)\geq 0$. If $\varphi:(M,g)\to(N,h)$ is L-harmonic map satisfying*

$$\int_M L'_\varphi|\xi\circ\varphi|^2 v^g<\infty,$$

then φ is constant.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\omega(X)=h(\xi\circ\varphi,\rho^2L'_\varphi d\varphi(X)),\quad \forall X\in\Gamma(TM),$$

and let $\{e_i\}$ be a normal orthonormal frame at $x\in M$, we have

$$\begin{aligned}\operatorname{div}^M\omega &= e_i[h(\xi\circ\varphi,\rho^2L'_\varphi d\varphi(e_i))]\\ &= h(\nabla^\varphi_{e_i}(\xi\circ\varphi),\rho^2L'_\varphi d\varphi(e_i))+h(\xi\circ\varphi,\nabla^\varphi_{e_i}\rho^2(L'_\varphi d\varphi(e_i)))\\ &= h(\nabla^\varphi_{e_i}(\xi\circ\varphi),\rho^2L'_\varphi d\varphi(e_i))+h(\xi\circ\varphi,e_i(\rho^2)L'_\varphi d\varphi(e_i))\\ &\quad +h(\xi\circ\varphi,\rho^2\nabla^\varphi_{e_i}L'_\varphi d\varphi(e_i)),\end{aligned}$$

so that

$$\begin{aligned}\operatorname{div}^M\omega &= h(\nabla^\varphi_{e_i}(\xi\circ\varphi),\rho^2L'_\varphi d\varphi(e_i))+h(\xi\circ\varphi,2\rho e_i(\rho)L'_\varphi d\varphi(e_i))\\ &\quad +h(\xi\circ\varphi,\rho^2[L'_\varphi\tau(\varphi)+d\varphi(\operatorname{grad}^M L'_\varphi)])\end{aligned}\tag{3.3}$$

by equation (3.3), and L-harmonicity condition of φ, we get

$$\begin{aligned}\operatorname{div}^M\omega &= \rho^2L'_\varphi h(\nabla^N_{d\varphi(e_i)}\xi,d\varphi(e_i))+2\rho e_i(\rho)L'_\varphi h(\xi\circ\varphi,d\varphi(e_i))\\ &\quad +\rho^2h(\xi\circ\varphi,(\operatorname{grad}^N L)\circ\varphi)\end{aligned}$$

since ξ is a homothetic vector field with homothetic constant k, we find that

$$\begin{aligned}\operatorname{div}^M\omega &= k\rho^2L'_\varphi h(d\varphi(e_i),d\varphi(e_i))+2\rho e_i(\rho)L'_\varphi h(\xi\circ\varphi,d\varphi(e_i))\\ &\quad +\rho^2\xi(L)\circ\varphi,\end{aligned}$$

that is,

$$\operatorname{div}^M\omega = k\rho^2L'_\varphi|d\varphi|^2+2\rho e_i(\rho)L'_\varphi h(\xi\circ\varphi,d\varphi(e_i))+\rho^2\xi(L)\circ\varphi,\tag{3.4}$$

by the Young's inequality, we have

$$-2\rho e_i(\rho)h(\xi\circ\varphi,d\varphi(e_i))\leq\epsilon\rho^2|d\varphi|^2+\frac{1}{\epsilon}e_i(\rho)^2|\xi\circ\varphi|^2,$$

for all $\epsilon>0$, multiplying the last inequality by L'_φ, we find that

$$-2L'_\varphi\rho e_i(\rho)h(\xi\circ\varphi,d\varphi(e_i))\leq\epsilon L'_\varphi\rho^2|d\varphi|^2+\frac{1}{\epsilon}L'_\varphi e_i(\rho)^2|\xi\circ\varphi|^2,\tag{3.5}$$

from (3.4), (3.5), we deduce the inequality

$$\begin{aligned}k\rho^2L'_\varphi|d\varphi|^2-\operatorname{div}^M\omega &+ \rho^2\xi(L)\circ\varphi\\ &\leq \epsilon L'_\varphi\rho^2|d\varphi|^2+\frac{1}{\epsilon}L'_\varphi e_i(\rho)^2|\xi\circ\varphi|^2,\end{aligned}\tag{3.6}$$

we set $\epsilon=\frac{k}{2}$, by (3.6), we have

$$\begin{aligned}\frac{k}{2}\rho^2L'_\varphi|d\varphi|^2-\operatorname{div}^M\omega &+ \rho^2\xi(L)\circ\varphi\\ &\leq \frac{2}{k}L'_\varphi e_i(\rho)^2|\xi\circ\varphi|^2,\end{aligned}\tag{3.7}$$

by the divergence theorem, and (3.7), we have

$$\frac{k}{2}\int_M \rho^2 L'_\varphi |d\varphi|^2 v^g + \int_M \rho^2[\xi(L)\circ\varphi]v^g \leq \frac{2}{k}\int_M L'_\varphi e_i(\rho)^2|\xi\circ\varphi|^2 v^g. \tag{3.8}$$

Now, consider the cut-off smooth function $\rho = \rho_R$ such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(\rho, R)$, $\rho = 0$ on $M \setminus B(\rho, 2R)$ and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$ (see [16]), from (3.8) we get:

$$\frac{k}{2}\int_M \rho^2 L'_\varphi |d\varphi|^2 v^g + \int_M \rho^2[\xi(L)\circ\varphi]v^g \leq \frac{8}{kR^2}\int_M L'_\varphi|\xi\circ\varphi|^2 v^g, \tag{3.9}$$

since $\int_M L'_\varphi|\xi\circ\varphi|^2 v^g < \infty$, when $R \to \infty$ we obtain:

$$\frac{k}{2}\int_M L'_\varphi |d\varphi|^2 v^g + \int_M [\xi(L)\circ\varphi]v^g = 0. \tag{3.10}$$

Consequently, $|d\varphi| = 0$, that is φ is constant. □

From Theorem 3.3, we deduce:

Corollary 3.4 ([9]). *Let (M, g) be a complete non-compact orientable Riemannian manifold, and (N, h) be a Riemannian manifold admitting a proper homothetic vector field ξ. If $\varphi : (M, g) \to (N, h)$ is harmonic map satisfying $\int_M |\xi\circ\varphi|^2 v^g < \infty$, then φ is constant.*

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Author information

F. Gherib, Department of Mathematics, Faculty of Exact Sciences, Mascara University, Algeria.
E-mail: fatiha.gherib@univ-mascara.dz

E. Remli, Department of Mathematics, Faculty of Exact Sciences, Mascara University, Algeria.
E-mail: ambarka.ramli@univ-mascara.dz

A. Mohammed Cherif, Department of Mathematics, Faculty of Exact Sciences, Mascara University, Algeria.
E-mail: a.mohammedcherif@univ-mascara.dz

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