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SOME RESULTS ON PRIME AND PRIMITIVE FUZZY HYPERRINGS

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Abstract
In this paper, we introduce and study prime and primitive fuzzy hyperrings. In this regard, hyperideals of such hyperrings are investigated and extended to fuzzy hyperideals, and some basic properties of these notions are obtained. Let R be a fuzzy hyperring. If R is primitive, then R is semisimple. If R is simple and semisimple, then R is primitive. Additionally, we consider the fundamental relation γ^* on a fuzzy hyperring R with identity, and prove that the fundamental ring R/γ^* of R is also a primitive ring. Finally, we present another definition for prime and primitive fuzzy hyperrings using membership functions, and show that if ξ is a semiprime fuzzy hyperideal of R, then ξ^* is a semiprime fuzzy hyperideal of R.

1 Introduction

The theory of hyperstructures was introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [15]. Marty introduced the notion of a hypergroup, and since then, many researchers have worked on this new topic in modern algebra and developed it further (see [1], [2], [8], [9], [10], and [11]).

The notion of a fuzzy semihypergroup was introduced and studied by Sen, Ameri, and Chowdhury in [17]. This concept was later extended to fuzzy hyperrings, fuzzy hypermodules, and fuzzy hyperalgebras (for more details, see [3, 4, 5, 6, 7, 12, 13, 14]). The purpose of this paper is to study prime and primitive fuzzy hyperrings. In this regard, we introduce the notion of prime and primitive hyperrings (in the sense of Krasner) and then fuzzify these notions to introduce prime and primitive fuzzy hyperrings.

We will proceed to investigate the fuzzy hyperideals of such hyperrings and present some basic results of these notions. In particular, we study the relationship between them. Specifically, we prove that a fuzzy hyperring R is semiprime if and only if R has no nonzero nilpotent fuzzy hyperideal. Additionally, if R is an arbitrary primitive fuzzy hyperring with identity and γ^* is a fundamental relation of R, then the fundamental ring R/γ^* is a primitive ring.

In the last part of the paper, we present another definition for prime and primitive fuzzy hyperrings using membership functions. We show that if ξ is a semiprime fuzzy hyperideal of R, then ξ^* is a semiprime fuzzy hyperideal of R.

2 Preliminaries

In this section, we briefly present some notions and results of fuzzy sets and algebraic hyperstructures which are necessary for the development of our paper. Most of the contents of this section are taken from sources [12, 13, 17], and [18].

Throughout this paper, $F(X)$ denotes the set of all fuzzy subsets of X, and $F^*(X) = F(X)\setminus\{\emptyset\}$ denotes the set of all nonempty fuzzy subsets of S.

Definition 2.1. [17] A *fuzzy hyperoperation* on S is a map $\circ : S \times S \to F^*(S)$, which associate a nonzero fuzzy subset $a \circ b$ with any pair (a, b) of elements of S. The couple $(S, \circ)$ is called a *fuzzy hypergroupoid*. We say that $(S, \circ)$ is *commutative* if for all $a, b \in S$, $a \circ b = b \circ a$. A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy semihypergroup* if for all $a, b, c \in S$, $a\circ(b\circ c) = (a\circ b)\circ c$. A fuzzy semihypergroup $(S, \circ)$ is a *fuzzy hypergroup* if for all $a \in S$, we have $a\circ S = \chi_S = S\circ a$ (*fuzzy reproduction axiom*).

Definition 2.2. [13] Let R be a nonempty set and $\oplus$, $\odot$ be two hyperoperations on R. The triple $(R, \oplus, \odot)$ is called a *fuzzy hyperring* if the following axiom hold:

(i) $(R, \oplus)$ is a commutative fuzzy hypergroup;

(ii) $(R, \odot)$ is a fuzzy semihypergroup;

(iii) $\odot$ is distributive over the addition $\oplus$ i.e., for all a, b, c of R we have $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.

A fuzzy hyperring $(R, \oplus, \odot)$ is called *unitary* if it satisfies the following condition:

For there exists an element $1_R \in R$ such that $1_R \odot a = \chi_{\{a\}}$ all a of R. Moreover, $(R, \oplus, \odot)$ is called *commutative* if for all a, b of R, we have $a \odot b = b \odot a$.

Definition 2.3. [13] Let $(R, \oplus, \odot)$ be a fuzzy hyperring and S be a nonempty subset of R. Then, S is called to be a *fuzzy subhyperring* of R if $(S, \oplus, \odot)$ is itself a fuzzy hyperring.

Definition 2.4. A commutative and unitary fuzzy hyperring $(R, \oplus, \odot)$ is called a *fuzzy integral hyperdomain* if for all a, b of R, $a \odot b = \chi_{\{0\}}$ implies $a = 0$ or $b = 0$.

$a \in R$ is called *left zero divisor* if there is an element $b \neq 0$ in R with $a \odot b = \chi_{\{0\}}$. So The commutative and unitary fuzzy hyperring $(R, +, \cdot)$ is called a Fuzzy Integral Hyperdomain (*FIHD*) if has not any zero divisor.

Definition 2.5. A hyperring $(R, \oplus, \odot)$ is a *fuzzy hyperfield* if $(R, \odot)$ being a fuzzy hypergroup.

Definition 2.6. [13] Let $(R, \oplus, \odot)$ be a fuzzy hyperring and S be a nonempty subset of R. Then, S is called to be a *Fuzzy Subhyperring* of R if $(S, \oplus, \odot)$ is itself a fuzzy hyperring.

Proposition 2.7. *[13] Let I be a fuzzy hyperideal of fuzzy hyperring $(R, \oplus, \odot)$ and $R/I = \{rI : r \in R\}$. Defining the fuzzy hyperoperations $\boxplus$ and $\boxtimes$ on R/I as follows:*

$aI \odot bI = (a \oplus b)I$ and $aI \boxtimes bI = (a \odot b)I$,

we get that $(R/I, \boxplus, \boxtimes)$ is a fuzzy hyperring, too. We call the above fuzzy hyperring $(R/I, \boxplus, \boxtimes)$ the quotient fuzzy hyperring.

Definition 2.8. [13] Fuzzy hyperring $(R, \oplus, \odot)$ is called a *Simple Fuzzy Hyperring* if not has any non trivial fuzzy hyperideal.

Definition 2.9. [13] A non-empty subset I of a fuzzy hyperring $(R, \oplus, \odot)$ is called a *(Right) Left Fuzzy Hyperideal* denoted by $I \trianglelefteq_l R$ $(I \trianglelefteq_r R)$ if

(i) $a, b \in I$ implies $a - b = a \oplus (-b) \in F^*(I)$,

(ii) $r \odot a \in F^*(I)$ $(a \odot r \in F^*(I))$ for every $r \in R$.

A subset I of a fuzzy hyperring R is called *fuzzy hyperideal* denoted by $I \trianglelefteq R$ if it is a right fuzzy hyperideal as well as a left fuzzy hyperideal of R.

In the sequel, for $x \in R$, let $Rx = \{r \odot x : r \in R\}$. It is easy to see $Rx \trianglelefteq R$.

Definition 2.10. [13] Let S and T be non-empty fuzzy subsets of a fuzzy hyperring $(R, \oplus, \odot)$. The fuzzy hypersum $S + T$ is defined by

$$S + T = \{x : x \in s \oplus t \text{ for some } s \in S, t \in T\}.$$

The *fuzy hyperproduct* ST is defined by

$$ST = \{x : x \in \sum_i^{\oplus} s_i \odot t_i, s_i \in S, t_i \in T, n \in \mathbb{Z}^+\}.$$

If S and T are fuzzy hyperideals of R, then $S+T$ and ST are also fuzzy hyperideals of R. By inspiered of fuzzy hypersum used to ([13], definition 4.3), we denoted $x_1 \oplus x_2 \oplus \ldots \oplus x_n$ by $\sum_{1\leq i\leq n}^{\oplus} x_i$ or for short $\sum_i^{\oplus} x_i$.

Definition 2.11. Proper fuzzy hyperideal I of fuzzy hyperring $(R, \oplus, \odot)$ is called *Maximal Fuzzy hyperideal* if R and I being only fuzzy hyperideals of R containing I.

Definition 2.12. [13] Let $(R_1, \oplus_1, \odot_1)$ and $(R_2, \oplus_2, \odot_2)$ be two fuzzy hyperrings. A map $f : R_1 \to R_2$ is a *homomorphism* of fuzzy hyperrings if $f(a \oplus_1 b) \leq f(a) \oplus_2 f(b)$ and $f(a \odot_1 b) \leq f(a) \odot_2 f(b)$, for all a, b of R_1.

Definition 2.13. If $(R, \oplus, \odot)$ is a fuzzy hyperring and S is a subset of R then denote $\langle S\rangle$ = the smallest fuzzy hyperideal of R that contains S We say that $\langle S\rangle$ is the fuzzy hyperideal of R *generated* by the set S. Note We have

$$\langle S\rangle = \{b_1 \odot a_1 \oplus ... \oplus b_k \odot a_k : a_i \in I, b_i \in R, k \geq 0\}$$
or
$$\langle S\rangle = \{\sum_{1\leq i\leq n}^{\oplus}(b_i \odot a_i) : a_i \in I, b_i \in R, k \geq 0\}$$

Definition 2.14. A fuzzy hyperideal I of R is *finitely generated* if $I = \langle a_1, ..., a_n\rangle$ for some $a_1, ..., a_n \in R$. A fuzzy hyperideal IR is a *Principal Fuzzy Hyperideal* $(PFHI)$ if $I = \langle a\rangle$ for some $a \in R$.

Definition 2.15. A fuzzy hyperring $(R, \oplus, \odot)$ is a *Principal Fuzzy Hyperideal Domain* $(PFHID)$ if it is a fuzzy integral hyperdomain such that every fuzzy hyperideal of R is a principal fuzzy hyperideal.

We introduce now the fuzzy hypermodule notion.

Definition 2.16. [12] Let $(R, \oplus, \odot)$ be a fuzzy hyperring. A nonempty set M, endowed with two fuzzy hyperoperations $\boxplus$, $\boxdot$ is called a *Left Fuzzy Hypermodule* over $(R, \oplus, \odot)$ and denoted by FHR_l*-hypermodule*, if the following conditions hold:

(1) $(M, \boxplus)$ is a commutative fuzzy hypergroup;

(2) $\boxdot : R \times M \to F^*(M)$ is defined by $(a, m) \mapsto a \boxdot m \in F^*(M)$ such that for all a, b of M and α, β of R we have

  (i) $\alpha \boxdot (a \boxplus b) = (\alpha \boxdot a) \boxplus (\alpha \boxdot b)$;

  (ii) $(\alpha \oplus \beta) \boxdot a = (\alpha \boxdot a) \oplus (\beta \boxdot b)$;

  (iii) $(\alpha \odot \beta) \boxdot a = \alpha \odot (\beta \boxdot a)$.

We say that an element e of fuzzy semihypergroup $(S, \circ)$ is called *identity (scalar identity)* if for all $r \in R$, we have $(e \boxdot r)(r) > 0$ and $(r \boxdot e)(r) > 0$ (from $(e \boxdot r)(s) > 0$ and $(r \boxdot e)(s) > 0$ it follows $r = s$ respectively).

If both $(R, \oplus)$ and $(M, \boxplus)$ have scalar identities, denoted by 0_R and 0_M, then the fuzzy hypermodule $(M, \boxplus, \boxdot)$ also satisfies the condition: for all a of M, we have $0_R \boxdot a = \chi_{\{0_M\}}$.

Moreover if $(R, \odot)$ has an identity, denoted by 1, then the fuzzy hypermodule $(M, \boxplus, \boxdot)$ is called *unitary* if it satisfies the condition: for all a of M, we have $1_R \boxdot a = \chi_{\{a\}}$.

Definition 2.17. [12] Let $(M, \boxplus, \boxdot)$ be a fuzzy hypermodule over a fuzzy hyperring $(R, \oplus, \odot)$. A nonempty subset M' of M is called a *Fuzzy Subhypermodules* if for all x, y of M' and α of R, the following conditions hold: $(x \boxplus y)(t) > 0$, then $t \in M'$; $x \boxplus M' = \chi_{M'}$ and if $(\alpha \boxdot x)(t) > 0$, then $t \in M'$.

Remark 2.18. Let $(M, \boxplus, \boxdot)$ be a fuzzy hypermodule over a fuzzy hyperring $(R, \oplus, \odot)$. for $x \in M$, let $Rx = \{r \boxdot x : r \in R\}$. It is easy to see $Rx \leq M$.

Let M_1 and M_2 be non-empty subsets of *FHR*-hypermodule $(M, \boxplus, \boxdot)$. The sum $M_1 + M_2$ is defined by $M_1 + M_2 = \{x \in b \boxplus c : b \in M_1, c \in M_2\}$ and $M_1 + M_2 \leq M$.

Definition 2.19. [12] Let $(M_1, \boxplus_1, \boxdot_1)$ and $(M_2, \boxplus_2, \boxdot_2)$ be two fuzzy hypermodules over a fuzzy hyperring $(R, \oplus, \odot)$. We say that $f : M_1 \to M_2$ is a *Homomorphism* of fuzzy hypermodules if for all x, y of M_1 and α of R we have:

$$f(x \boxplus_1 y) \leq f(x) \boxplus f(y) \text{ and } f(\alpha \boxdot_1 x) \leq \alpha \boxdot_2 f(x)$$

and denoted by FHM-homomorphism. If (i) and (ii) of this definition, If the equality hold, then f is called $strong$ (or $good$) fuzzy homomorphism and denoted by FHM_S-homomorphism. The class of all FHM-homomorphisms (resp., FHM_S-homomorphisms) from M into N is denoted by $hom_{FHM}(M, N)$ (res., $hom_{FHM_S}(M, N)$).

Definition 2.20. Proper fuzzy subhypermodule N of fuzzy hypermodule M is colled *Maximal Fuzzy Subhypermodule* if N and M being only fuzzy subhypermodules of M containing N.

Remark 2.21. Let $(M, \boxplus, \boxdot)$ be an fuzzy hypermodule on fuzzy hyperring $(R, \oplus, \odot)$. Then for $x \in A$ and $n \in \mathbb{Z}$,

$$m \boxdot x = \begin{cases} \underbrace{x \boxplus x \boxplus \ldots \boxplus x}_{n\ times\ x}; & n>0 \\ 0_A; & n=0 \\ \underbrace{(-x) \boxplus (-x) \boxplus \ldots \boxplus (-x)}_{-n\ times\ x}; & n<0 \end{cases}$$

Definition 2.22. Let $(R, \oplus, \odot)$ be a fuzzy hyperring not necessarily with 1_R and $(M, \boxplus, \boxdot)$ be a fuzzy hypermodule over R and $X \subseteq M$. $\langle X \rangle$ denotes the smallest fuzzy subhypermodule of M containing X or the intersection of all fuzzy subhypermodules of M containing X. The set X is said to be a *Generating Set* for an fuzzy hypermodule M, or X generates M, if $M = \langle X \rangle$. Here, M is called *Finitely Generated* if it has a finite generating set. Let $X = \{x\}$. For simplicity, we use $\langle x \rangle$ instead of $\langle X \rangle$ and is called *Cyclic* fuzzy subhypermodule of M.

It is easy to see that;

$\langle x \rangle = \{a \in (r \boxdot x) \boxplus (m \boxdot x) \boxplus \sum_i^{\boxplus} n_i \boxdot (x \boxplus (-x)) :\ x \in X, r \in R, m \in \mathbb{Z}, n_i \in \mathbb{N}\}$.

Let $Rx = \{r \boxdot x :\ r \in R, x \in M\} = \langle x \rangle$.

Remark 2.23. If $(M, \boxplus, \boxdot)$ is an unitary fuzzy hypermodule on fuzzy hyperring $(R, \oplus, \odot)$ with identity 1_R, Then

(i) $\langle x \rangle = Rx$.

(ii) Letting $X = \{x_i\}_{i \in I} \subseteq M$, $M = \langle X \rangle$ if and only if for every $a \in M$, there exists a finite $j \subseteq I$ such that $a \in \sum_{j \in J}^{\oplus} r_i \boxdot x_j$ which $r_j \in R$ and $x_j \in X$.

Proposition 2.24. *Let N be a fuzzy subhypermodule of fuzzy hypermodule $(M, \boxplus, \boxdot)$ over fuzzy hyperring $(R, \oplus, \odot)$ and $M/N = \{xN :\ x \in M\}$. For every $a, b \in M$ and $r \in R$, defining the fuzzy hyperoperations $\circledast$ and $\circledcirc$ on M/N as follows: $(aN) \circledast (bN) = (a \boxplus b)N$ and $r \circledcirc (aN) = (r \odot a)N$, we get that $(M/N, \circledast, \circledcirc)$ is a fuzzy hypermodule, too. We call the above fuzzy hypermodule $(M/N, \circledast, \circledcirc)$ the quotient fuzzy hypermodule.*

Connections between fuzzy hyperoperations and the above associated hyperoperations have been considered by Sen, Amery and Chowdhury in the context of semihypergroups and hypergroups and by Leoreanu-Fotea and Davvaz in the context of hyperrings. They have shown that if $(M, \boxplus)$ is a fuzzy hypergroup, then $(M, +)$ is a hypergroup (see [17]), while if $(R, \oplus, \odot)$ is a fuzzy hyperring, then $(R, \uplus, \circ)$ is a hyperring (see [13]).

3 categories of fuzzy hypermodules

In the next theorem was established a similar result for fuzzy hypermodules by Leoreanu-Fotea and Davvaz in [13].

Theorem 3.1. *If* $(M, \boxplus, \boxdot)$ *is a fuzzy hypermodule over a fuzzy hyperring* $(R, \oplus, \odot)$*, then* $(M, +, \cdot)$ *is a hypermodule over the hyperring* $(R, \uplus, \circ)$*. ([13], Theorem 3.5)*

If we denote by $\mathcal{HM}$ the class of all hypermodules and by $\mathcal{FHM}$ the class of all fuzzy hypermodules, then we can consider the map $\psi : \mathcal{FHM} \to \mathcal{HM}$, $\psi((M, \boxplus, \boxdot)) = (M, +, \cdot)$. On the other hand, if $(M, +, \cdot)$ is a hypermodule over a hyperring $(R, \uplus, \circ)$, then we define for every $a, b \in M$ and $\alpha, \beta \in R$ the following fuzzy hyperoperations:

$$a \boxplus b = \chi_{a \uplus b} \text{ and } \alpha \boxdot \beta = \chi_{\alpha \circ \beta} \text{ and } \alpha \oplus \beta = \chi_{\alpha+\beta} \text{ and } \alpha \odot \beta = \chi_{\alpha \cdot \beta}.$$

In [13] it is shown that if $(M, +, \cdot)$ is a hypergroup, then $(M, \boxplus)$ is a fuzzy hypergroup, while in [17], it is checked that if $(R, \uplus, \circ)$ is a hyperring, then $(R, \oplus, \odot)$ is a fuzzy hyperring.

Hence, there exists a map $\varphi : \mathcal{FHR} \to \mathcal{HR}$, $\varphi(M, +, \cdot) = (M, \boxplus, \boxdot)$. It is natural to consider and study homomorphisms between fuzzy hypermodules. First, recall that if μ_1, μ_2 are fuzzy sets on M, then we say that μ_1 is smaller than μ_2 and we denote $\mu_1 \leq \mu_2$ if and only if for all $x \in M$, we have $\mu_1(x) \leq \mu_2(x)$. Let $f : M_1 \to M_2$ be a map. If μ is a fuzzy set on M_1, then we define $f(\mu) : M_1 \to [0, 1]$, as follows:$(f(\mu))(t) = \bigvee_{r \in f^{-1}(t)} \mu(r)$, if $f^{-1}(t) \neq \emptyset$, otherwise we consider $(f(\mu))(t) = 0$.

The next two theorem was shown a connection between homomorphisms of fuzzy hypermodules and homomorphisms of hypermodules by Leoreanu-Fotea and Davvaz in [13].

Theorem 3.2. *Let* $(M_1, \boxplus_1, \boxdot_1)$ *and* $(M_2, \boxplus_2, \boxdot_2)$ *be two fuzzy hypermodules over a fuzzy hyperring* $(R, \oplus, \odot)$ *and* $(M_1, +_1, \cdot_1) = \psi(M_1, \boxplus_1, \boxdot_1)$*,* $(M_2, +_2, \cdot_2) = \psi(M_2, \boxplus_2, \boxdot_2)$ *be the associated hypermodules over the corresponding hyperring* $(R, \uplus, \circ) = \psi(R, \oplus, \odot)$*. If* $f : M_1 \to M_2$ *is a homomorphism of fuzzy hypermodules, then* f *is a homomorphism of hypermodules, too. ([13], Theorem 3.7)*

Theorem 3.3. *Let* $(M_1, +_1, \cdot_1)$ *and* $(M_2, +_2, \cdot_2)$ *be two hypermodules over a hyperring* $(R, \uplus, \circ)$ *and let* $(M_1, \boxplus_1, \boxdot_1) = \varphi(M_1, +_1, \cdot_1)$*,* $(M_2, \boxplus_2, \boxdot_2) = \varphi(M_2, +_2, \cdot_2)$ *be the associated fuzzy hypermodules over the fuzzy hyperring* $(R, \oplus, \odot) = \varphi((R, \uplus, \circ))$*. The map* $f : M1 \to M_2$ *is a homomorphism of hypermodules if and only if it is a homomorphism of fuzzy hypermodules. ([13], Theorem 3.8)*

The following theorem established a connection between fuzzy subhypermodules of a fuzzy hypermodule and subhypermodules of the corresponding hypermodule by Leoreanu-Fotea and Davvaz in [13].

Theorem 3.4.

(i) If $(M', \boxplus, \boxdot)$ *is a subfuzzy hypermodule of* $(M, \boxplus, \boxdot)$ *over* $(R, \oplus, \odot)$*, then* $(M', +, \cdot) = \psi(M', \boxplus, \boxdot)$ *is a submodule of* $(M, +, \cdot) = \psi(M, \boxplus, \boxdot)$ *over* $(R, \uplus, \circ) = \psi(R, \oplus, \odot)$*;*

(ii) $(M', +, \cdot)$ *is a submodule of* $(M, +, \cdot)$ *over* $(R, \uplus, \circ)$ *if and only if* $(M', \boxplus, \boxdot) = \varphi(M', +, \cdot)$ *is a fuzzy subhypermodule of* $(M, \boxplus, \boxdot) = \varphi(M, +, \cdot)$ *over* $(R, \oplus, \odot) = \varphi(R, \uplus, \circ)$*.*

Here we prove propositions 3.5 and 3.10 that are two of the most important and widely used propositions by inspier of Leoreanu-Fotea and Davvaz in [13].

Proposition 3.5. *(i) every (unitary) fuzzy hypermod- ule* $(M, \boxplus, \boxdot)$ *over* $(R, \oplus, \odot)$*,* $(-1_R) \boxdot a = \chi_{\{-a\}}$ *for every* $a \in M$*.*

(ii) In every fuzzy hyperring $(R, \oplus, \odot)$*,* $(-1_R) \odot r = \chi_{\{-r\}}$ *for every* $r \in R$*.*

Proof. (i): Let $(M, \boxplus, \boxdot)$ be a fuzzy hypermodule over $(R, \oplus, \odot)$ and the map $\psi : \mathcal{FHM} \to \mathcal{HM}$, $(M, +, \cdot) = \psi(M, \boxplus, \boxdot)$ be associated hypermodule over the corresponding hyperring $(R, \uplus, \circ) = \psi(R, \boxplus, \boxdot)$ and $a \in M$, and also by Proposition 2.1-(i) in [18] we have $(-1_R) \cdot a = -a$, Thus for every $t \in M$, by Definition 2.14 and theorem 3.2, we obtian;

$$\begin{gathered}((-1_R) \boxdot a)(t) = ((-1_R) \boxdot (1_R \boxdot a))(t) = ((-1_R) \boxdot 1_R) \boxdot a)(t) = \\ \bigvee_{r \in R} ((-1_R) \boxdot 1_R))(r) \wedge (r \boxdot a)(t) = \bigvee_{r \in R} \chi_{\{-1_R \cdot 1_R\}}(r) \wedge \chi_{\{r \cdot a\}}(t) = \\ \bigvee_{r \in M} \chi_{\{-1_R\}}(r) \wedge \chi_{\{r \cdot a\}}(t) = \bigvee_{a \in M} \chi_{\{-1_R \cdot a\}}(t) = \chi_{\{-a\}}(t).\end{gathered}$$

(ii): By the argument similar to the proof of (i) we obtain the result (ii). □

We recall that if $(M_1, \boxplus_1, \boxdot_1)$ and $(M_2, \boxplus_2, \boxdot_2)$ be fuzzy hypermodules over a fuzzy hyperring $(R, \oplus, \odot)$, than A function $f : M_1 \to M_2$ that satisfies the conditions:

(i) $f(\boxplus_1 y) \leq f(x) \boxplus_2 f(y)$;

(ii) $f(r \boxdot_1 x) \leq r \boxdot_2 f(x)$,

for all $r \in R$ and all $x, y \in M_1$, is said to be an (inclusion) homomorphism of fuzzy hypermodules from M_1 into M_2.

Remark 3.6. If (i) and (ii) above definition the equality hold, then f is called a *strong* (or *good*) *FHR*-homomorphism.

Let $f \in Hom_F(A, B)$ and $h \in Hom_F(B, C)$. The composition $h \circ f$ is defined as Equation in the above Definition. Also, for every fuzzy hypermodule A, the homomorphism id_A with definition $id_A(x) = \chi_{\{x\}}$ for all $x \in A$ is the identity morphism as before. Hereafter, ${}_F\mathcal{H}\mathbf{mod}$ (resp., ${}_{F_S}\mathcal{H}\mathbf{mod}$) denotes the category whose objects are all fuzzy hypermodules and whose morphisms from A to B are all F_{mv}-homomorphisms (resp., F_{smv}-homomorphisms) from A into B. Clearly, ${}_{F_S}\mathcal{H}\mathbf{mod}$ is a subcategory of ${}_F\mathcal{H}\mathbf{mod}$, i.e.,

$${}_{F_S}\mathcal{H}\mathbf{mod} \preccurlyeq_F \mathcal{H}\mathbf{mod}.$$

Remark 3.7. (i) Hereafter, we identify a singleton $X = \{a\}$ by its element a. Also, we sometimes write $f(a) = b$ instead of $f(a) = \{b\}$. So every fuzzy single-valued morphism $f \in Hom_F(A, B)$ (resp. $f \in Hom_{F_S}(A, B)$) is an elemen of $hom_F(A, B)$ (resp. $hom_{F_S}(A, B)$), and conversely, every element of $hom_F(A, B)$ (resp. $hom_{F_S}(A, B)$) can be considered as an element of $Hom_F(A, B)$ (resp. $Hom_{F_S}(A, B)$), So

$${}_F h\mathbf{mod} \preccurlyeq_F \mathcal{H}\mathbf{mod}$$
$$\text{(resp., } {}_{F_S} h\mathbf{mod} \preccurlyeq_{F_S} \mathcal{H}\mathbf{mod}).$$

(ii) Let $f, g \in Hom_F(A, B)$. Define the relation $\leq$ on $Hom_F(A, B)$ in which $f \leq g$ means $f(x) \leq g(x)$ for all $x \in A$. Clearly $(Hom_F(A, B), \leq)$ is a poset.

For convnience and distinguishing, we call ${}_F h\mathbf{mod}$ and ${}_{F_S} h\mathbf{mod}$ *primary categories* of fuzzy hypermodules. Also, ${}_F\mathcal{H}\mathbf{mod}$ and ${}_{F_S}\mathcal{H}\mathbf{mod}$ are called *secondary categories* of fuzzy hypermodules.

So far we have considered the morphisms or arrows, as usual, the functions between objects. But one can consider a morphism from A to B as a function from A into $F^*(B)$ called a *Fuzzy multivalued function* from A to B.

For two multivalued functions f and g their composition $g \circ f$ is defined as the following:

$$\forall a \in A, \; (g \circ f)(a) = \bigvee_{b \in f(a)} g(b),$$

and an identity morphism for an object A is defined $id_A(x) : A \to F^*(A)$ by $id_A(x) = \chi_{\{x\}}$ for all $x \in A$.

Definition 3.8. If M and N are two fuzzy hypermodules over fuzzy hyperring $(R, \oplus, \odot)$ then multivalued function f from M into N is a mapping $f : M \to F^*(N)$ satisfying the following conditions:

(i) $f(x \boxplus y) \leq f(x) \boxplus f(y)$;

(ii) $f(r \boxdot x) \leq r \boxdot f(x)$;

for all $r \in R$ and all $x, y \in M$, is said to be a *fuzzy multivalued homomorphism* and denoted by FHR_{mv}-homomorphism or for short F_{mv}-homomorphism.

Remark 3.9. If (i) and (ii) of Definition 3.6, If the equality hold, then f is called *strong* (or *good*) fuzzy nultivalued homomorphism, for short an F_{smv}-homomorphism.

The class of all F_{mv}-Homomorphisms (resp., F_{smv}-Homomorphisms) from M into N is denoted by $Hom_F(M, N)$ (resp., $Hom_{F_S}(M, N)$).

Proposition 3.10. *Let* $(M_1, \boxplus_1, \boxdot_1)$ *and* $(M_2, \boxplus_2, \boxdot_2)$ *are two fuzzy hypermodules over fuzzy hyperring* $(R, \oplus, \odot)$.

(i) For every $f \in hom_F(M_1, M_2)$, $f(0_{M_1}) = \chi_{\{0_{M_2}\}}$.

(ii) For every $f \in hom_F(M_1, M_2)$, $f(-x) = -f(x)$ *and* $f(x \boxplus_1 (-y)) = f(x) + \boxplus_2(-f(y))$.

Proof. (i): Let $(M_1, \boxplus_1, \boxdot_1)$ and $(M_2, \boxplus_2, \boxdot_2)$ are two fuzzy hypermodules over fuzzy hyperring $(R, \oplus, \odot)$ and $f \in hom_F(M_1, M_2)$, and also by theorem 3.2, given the map $\psi : \mathcal{FHM} \to \mathcal{HM}$, $(M_1, +_1, \cdot_1) = \psi(M_1, \boxplus_1, \boxdot_1)$ be associated hypermodule over the corresponding hyperring $(R, \uplus, \circ) = \psi(R, \boxplus, \boxdot)$ and also by Proposition (2.2)-(i) in [18] we have $0_R \cdot a = 0_{M_1}$ and by Definition (2.4)-(4) [18], if $t \in im(f)$, then we obtian $f(0_R \cdot_1 a) = f(0_{M_1}) = \{0_{M_2}\}$.
So for every $a_1 \in M_1$ and $t \in R$, by Definition 2.14;

$$f(0_{M_1})(t) = f(0_R \boxdot_1 a_1)(t) = \bigvee_{r \in f^{-1}(t)} \chi_{\{0_R \cdot_1 a_1\}}(r).$$

Thus

$$f(0_{M_1})(t) = \begin{cases} 1, & f^{-1}(t) \wedge (0_R \cdot_1 a_1) \neq 0 \\ 0, & o.w \end{cases}$$

$$= \begin{cases} 1, & t \in f(0_R \cdot_1 a_1) \\ 0, & o.w \end{cases}$$

$$= \begin{cases} 1, & t \in f(0_{M_1}) \\ 0, & o.w \end{cases}$$

$$= \begin{cases} 1, & t \in f(0_{M_2}) \\ 0, & o.w \end{cases}$$

So $f(0_{M_1}) = \chi_{\{0_{M_2}\}}$.
If $t \notin im(f)$, then proof is tirivale.
(ii): By the argument similar to the proof of (i) we obtain the result (ii). □

Theorem 3.11. *(The Isomorphism Theorems). Let* $(R, \oplus, \odot)$ *be a fuzzy hyperring and* $(M, \boxplus_1, \boxdot_1)$ *and* $(N, \boxplus_2, \boxdot_2)$ *be FHR*$_l - hypermodules$.

(1) If $f : M \to N$ *is an epimorphism with* $Kerf = K$*, then there is a unique isomorphism* $h : M/K \to F^*(N)$ *such that* $h(m + K) = f(m)$ *for all* $m \in M$.
(2) If $K \leq L \leq M$*, then* $L/K \leq M/K$ *and* $M/L \cong (M/K)/(L/K)$.
(3) If $H \leq M$ *and* $K \leq M$*, then* $(H + K)/K \cong H/(H \cap K)$.

where $M_1 \cong M_2$ if the strongly homomorphism $f : M_1 \to M_2$ is mono and epi.

Definition 3.12. Fuzzy hypermodule $(M, \boxplus, \boxdot)$ over fuzzy hyperring $(R, \oplus, \odot)$ is called a $Simple$ fuzzy hypermodule if not has any non trivial fuzzy subhypermodule.

Definition 3.13. Fuzzy hypermodule $(M, \boxplus, \boxdot)$ over fuzzy hyperring $(R, \oplus, \odot)$ is called a $Semisimple$ fuzzy hypermodule if for every Fuzzy subhypermodule K of M, there exists a Fuzzy subhypermodule L of M such that $M = K \bigoplus L$.

Definition 3.14. Let $(R, \oplus, \odot)$ be a fuzzy hyperring. According to the existence maximal fuzzy hyperideal for any fuzzy hypermodule, R has both maximal left fuzzy hyperideal and maximal right fuzzy hyperideal. the intersection of all the maximal left fuzzy hyperideals of R is called *left Jacobson Radica*l R and denoted by $J_l(R)$. and the intersection of all the maximal right fuzzy hyperideals of R is called *right Jacobson Radica*l R and denoted by $J_r(R)$.
It is clear that $J_l(R)$ and $J_r(R)$ are respect to left and right fuzzy hyperideals of R.

Definition 3.15. An element x of a fuzzy hyperring $(R, \oplus, \odot)$ is said to be *nilpotent* if there exists a natural number n such that $a^n = \chi_{\{0\}}$.
Thus a fuzzy hyperideal I of the fuzzy hyperring R is *nil* if every element in I is nilpotent, i.e. for every $a \in I$, there exists a natural number n such that $a^n = \chi_{\{0\}}$. This n is dependent on the element a.

A fuzzy hyperideal I of a fuzzy hyperring $(R, \oplus, \odot)$ is called *nilpotent* if there exists a natural number n such that $I^n = (0)$. Here

$$I^n = \{\sum_{finite} a_{i_1} \odot a_{i_2} \odot ... \odot a_{i_n} : \ a_{i_j} \in I, j = 1, 2, ..., n\}$$
or
$$I^n = \{\sum_{1 \leq j \leq n}^{\odot} a_{i_j} : a_{i_j} \in I\}.$$

Remark 3.16. Clearly, Every nilpotent fuzzy hyperideal is nil since $I^n = \{0\}$ implies $a^n = \chi_{\{0\}}$ for all a. It is possible, however, to have a nil fuzzy hyperideal that is not nilpotent. .

4 Prime and primitive fuzzy hyperrings and fuzzy hyperideals

In this section, we investigeat some of the results and definitions related to the prime and primitive fuzzy hyperrings and fuzzy hyperideals and the relationships of between them and we study some of their properties by inspiered of the results and definitions in [**?**].

Definition 4.1. A proper fuzzy hyperideal P of a fuzzy hyperring $(R, \oplus, \odot)$ is called a *prime* fuzzy hyperideal if for all fuzzy hyperideals I and J of R, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Definition 4.2. A proper fuzzy hyperideal P of a commutative and unitary fuzzy hyperring $(R, \oplus, \odot)$ is called *semiprime* fuzzy hyperideal if for every fuzzy hyperideals I of R, $I^2 \subseteq P$ implies that $I \subseteq P$.

Definition 4.3. A proper fuzzy hyperideal M of a fuzzy hyperring $(R, \oplus, \odot)$ is called a *maximal* fuzzy hyperideal if R and M being only fuzzy hyperideals of R containing M.

Lemma 4.4. *Let $(R, \oplus, \odot)$ be a commutative and unitary fuzzy hyperring. Then a proper fuzzy hyperideal P is prime if for every $a, b \in R$, the following condition are satisfied:*

$$a \odot b \in F^*(P) \ \ \Rightarrow \ \ a \in P \ \ or \ \ b \in P. \qquad (*)$$

Conversely if P is a prime hyperideal and R is commutative, then P satisfies condition $(*)$.

Proof. If A and B are fuzzy hyperideals such that $AB \subset P$ and $A \not\subset P$, then there exists an element $a \in A - P$. For every $b \in B$, $a \odot b \in F^*(AB) \subset F^*(P)$, whence $a \in P$ or $b \in P$. Since $a \notin P$, we must have $b \in P$ for all $b \in B$; that is, $B \subset P$. Therefore, P is prime.

Conversely, if P is any fuzzy hyperideal and $a \odot b \in F^*(P)$, then the principal fuzzy hyperideal $a \odot b$ is contained in P by Definition 2.14. If R is commutative, then remark 2.23 implies that $\langle a \rangle \langle b \rangle \subset \langle a \odot b \rangle$, whence $\langle a \rangle \langle b \rangle \subset P$. If P is prime, then either $\langle a \rangle \subset P$ or $\langle b \rangle \subset P$, whence $a \in P$ or $b \in P$. □

Theorem 4.5. *If I is a fuzzy hyperideal in a fuzzy hyperring R, then there is a one-to-one correspondence between the set of all fuzzy hyperideals of R which contain I and the set of all fuzzy hyperideals of R/I, given by $J \to J/I$. Hence every fuzzy hyperideal in R/I is of the form J/I, where J is a fuzzy hyperideal of R which contains I.*

Proof. The proof is similar to the proof of that come in the theory of modules, by some manipulations. □

Proposition 4.6. *Let P be a fuzzy hyperideal of a fuzzy hyperring R. Then P is fuzzy prime if and only if R/P is a prime fuzzy hyperring.*

Proof. If R/K is prime, let $\pi : R \to R/K$ be the canonical epimorphism. If I and J are fuzzy hyperideals of R such that $IJ \subset K$, then $\pi(l)$, $\pi(J)$ are fuzzy hyperideals of R/K such that $\pi(I)\pi(J) = \pi(IJ) = \chi_{\{0\}}$. Since R/K is prime, either $\pi(I) = \{0\}$ or $\pi(J) = \{0\}$; that is, $I \subset K$ or $J \subset K$. Therefore, K is a prime fuzzy hyperideal. The converse is an easy consequence of Theorem 4.5 and Definition 4.1. □

Proposition 4.7. *If $(R, \oplus, \odot)$ is a commutative and unitary fuzzy hyperring, then any maximal fuzzy hyperideal of R is a Prime fuzzy hyperideal.*

Proof. Let $(R, \oplus, \odot)$ be commutative and unitary fuzzy hyperring and I be maximal fuzzy hyperideal of fuzzy hyperring R and for every $a, b \in R$, $a \odot b \in F^*(I)$. If $a \notin I$ then $I \subset (I, a) = \{(r \odot a) \oplus u : u \in I, r \in R\} \subset F^*(R)$, but since I is a maximal fuzzy hyperideal of R so $R = (I, a)$, therefore $(r \odot a) \oplus u = \chi_{\{1\}}$, so by Theorem 3.3, for $t \in R$ we have;

$$((r \odot a) \oplus u)(t) = \chi_{\{1\}}(t) \Rightarrow \bigvee_{p \in R} \chi_{\{r \cdot a\}}(p) \wedge \chi_{\{p+u\}}(t) = \chi_{\{1\}}(t) \Rightarrow \bigvee \chi_{\{(r \cdot a)+u\}}(t) = \chi_{\{1\}}(t)$$

Therefore $(r \cdot a) + u = 1$ and for $b \in R$ we have $((r \cdot a) \cdot b) + (u \cdot b) = b$, thus $(r \cdot (a \cdot b)) + (u \cdot b) = b$. Now we obtian;

$$\bigvee \chi_{\{(r \cdot (a \cdot b)) + (u \cdot b)\}}(t) = \chi_{\{b\}}(t) \Rightarrow \bigvee_{p \in R} \chi_{\{(r \cdot (a \cdot b))\}}(p) \wedge \chi_{\{p + (u \cdot b)\}}(t) = \chi_{\{b\}}(t) \Rightarrow$$
$$r \odot (a \odot b) \oplus (u \odot b) = \chi_{\{b\}}$$

thus $b \in I$. □

Proposition 4.8. *Let $(R, \oplus, \odot)$ be commutative and unitary fuzzy hyperring and P be proper hyperideals of fuzzy hyperring R. P is prime if and only if R/P is fuzzy integral hyperdomain.*

Proof. By proposition 2.7, $(R/P, \boxplus, \boxtimes)$ is a commutative hyperring with identity $1_R + P$ and zero element $\{0\} + P = P$. If P is prime, then $1_R + P \neq P$ since $P \neq R$. Furthermore, R/P has no zero divisors since

$$(a + P) \boxtimes (b + P) = P \Rightarrow (a \odot b) + P = P \Rightarrow a \odot b \in F^*(P) \Rightarrow a \in P \text{ or } b \in P \Rightarrow a + P = P \text{ or } b + P = P.$$

Therefore, R/P is a fuzzy integral hyperdomain.

Conversely, if $(R/P, \boxplus, \boxtimes)$ is a fuzzy integral hyperdomain, then $1_R + P \neq 0 + P$, whence $1_R \notin P$. Therefore, $P \neq R$. Since R/P has no zero divisors,

$$a \odot b \in F^*(P) \Rightarrow (a \odot b) + P = P \Rightarrow (a + P) \boxtimes (b + P) = P \Rightarrow a + P = P \text{ or } b + P = P \Rightarrow a \in P \text{ or } b \in P.$$

Therefore, P is prime by lamma 4.4. □

Theorem 4.9. *If $(R, \oplus, \odot)$ is a commutative fuzzy hyperring such that $R^2 = R$ (in particular if R has an identity), then every maximal fuzzy hyperideal M in R is prime.*

Proof. Suppose $a \odot b \in F^*(M)$ but $a \notin M$ and $b \notin M$. Then each of the fuzzy hyperideals $M + \langle a \rangle$ and $M + \langle b \rangle$ properly contains M. By maximality $M + \langle a \rangle = R = M + \langle b \rangle$. Since R is commutative and $a \odot b \in F^*(M)$, remark 2.23 implies that $\langle a \rangle \langle b \rangle \subset \langle a \odot b \rangle \subset F^*(M)$. Therefore,

$$R = R^2 = (M + \langle a \rangle)(M + \langle b \rangle) \subset M^2 + \langle a \rangle M + M \langle b \rangle + \langle a \rangle \langle b \rangle \subset M.$$

This contradicts the fact that $M \neq R$ (since M is maximal). Therefore, $a \in M$ or $b \in M$, whence M is prime by lamma 4.4. □

Proposition 4.10. *Let M be a hyperideal in a fuzzy hyperring $(R, \oplus, \odot)$ with identity $1_R \neq 0$.*

(i) M is maximal if and only if R/M is fuzzy hyperfield.

(ii) If the quotient fuzzy hyperring R/M is a division fuzzy hyperring, then M is maximal.

Proof. (i): If M is maximal, then M is prime (Theorem 4.9), whence R/M is a fuzzy integral hyperdomain by proposition 4.8. Thus we need only show that if $a + M \neq M$, then $a + M$ has a multiplicative inverse in $(R/M, \boxplus, \boxtimes)$. Now $a + M \neq M$ implies that $a + M$, whence M is properly contained in the fuzzy hyperideal $M + \langle a \rangle$. Since M is maximal, we must have $M + \langle a \rangle = R$. Therefore, since R is commutative, $\chi_{\{1_R\}} = m \oplus (r \odot a)$ for some $m \in M$ and $r \in R$, by remark 2.23. Now by Theorem 3.3, for $t \in R$ we have;

$$\chi_{\{1_R\}}(t) = m \oplus (r \odot a)(t) \Rightarrow \chi_{\{1_R\}}(t) = \bigvee_{p\in R} \chi_{\{m+p\}}(t) \wedge \chi_{\{r\cdot a\}}(p) \Rightarrow \chi_{\{1_R\}}(t) = \bigvee \chi_{\{m+(r\cdot a)\}}(t) \Rightarrow 1_R = m + (r\cdot a) \Rightarrow 1_R + (-r\cdot a) = m \Rightarrow \chi_{\{1_R+(-r\cdot a)\}}(t) = \chi_{\{m\}}(t) \Rightarrow \bigvee_{p\in R} \chi_{\{1_R+p\}}(t) \wedge \chi_{\{-r\cdot a\}}(p) = \chi_{\{m\}}(t) \Rightarrow (1_R \oplus (-r\odot a))(t) = \chi_{\{m\}}(t)$$

Thus $1_R \oplus (-r\odot a) = \chi_{\{m\}} \in F^*(M)$, whence by Proposition 2.7 $\chi_{\{1_R+M\}} = (r\odot a) + M = (r+M) \boxtimes (a+M)$.
Thus $r+M$ is a multiplicative inverse of $a+M$ in R/M, whence R/M is a fuzzy hyperfield.

(ii): If $(R/M, \boxplus, \boxtimes)$ is a division fuzzy hyperring, then $1_R + M \neq \{0\} + M$, whence $1_R \notin M$ and $M \neq R$. If N is a fuzzy hyperideal such that $M \subset N$, let $a \in N - M$. Then $a + M$ has a multiplicative inverse in R/M, say by Proposition 2.7 $(a+M) \boxtimes (b+M) = \chi_{\{1_R+M\}}$. Consequently, $(a\odot b) + M = \chi_{\{1_R+M\}}$ and $(a\odot b) \oplus (-1_R) = \chi_{\{c\}} \in F^*(M)$. But $a \in N$ and $M \subset N$ imply that $1_R \in N$. Thus $N = R$. Therefore, M is maximal. □

Theorem 4.11. *If $f : R \to S$ is a homomorphism of fuzzy hyperrings, then the kernel of f is a fuzzy hyperideal in $(R, \oplus, \odot)$. Conversely if I is a fuzzy hyperideal in R, then the map $\pi : R \to R/I$ given by $r \mapsto r + I$ is an epimorphism of fuzzy hyperrings with kernel I.*

Proof. $Kerf$ is a fuzzy subhypergroup of R. If $x \in Kerf$ and $r \in R$, then $f(r \odot_R x) = f(r) \odot_S f(x) = f(r) \odot_S \{0\} = \chi_{\{0\}}$, whence $r \odot_R x \in F^*(Kerf)$. Similarly, $x \odot_R r \in F^*(Kerf)$. Therefore, $Kerf$ is a fuzzy hyperideal. So the map π is an epimorphism of fuzzy hypergroups with kernel I.

Since by Proposition 2.7 $\pi(a\odot b) = (a\odot b) + I = (a+I) \boxtimes (b+I) = \pi(a) \boxtimes \pi(b)$ for all $a, b \in R$, π is also an epimorphism of fuzzy hyperrings. □

Corollary 4.12. *The following conditions on a commutative fuzzy hyperring R with identity $1_R \neq 0$ are equivalent.*

(i) *R is a fuzzy hyperfield;*

(ii) *R has no proper fuzzy hyperideals;*

(iii) *$\{0\}$ is a maximal fuzzy hyperideal in R ;*

(iv) *every nonzero homomorphism of fuzzy hyperrings $R \to S$ is a monomorphism.*

Proof. This result may be proved directly or as follows. $R \cong R/\{0\}$ is a fuzzy hyperfield if and only if $\{0\}$ is maximal by Theorem 4.10. But clearly $\{0\}$ is maximal if and only if R has no proper fuzzy hyperideals. Finally, for every fuzzy hyperideal $I(\neq R)$ the canonical map $\pi : R \to R/I$ is a nonzero homomorphism with kernel I (Theorem 4.11). Since π is a monomorphism if and only if $I = \{0\}$, (iv) holds if and only if R has no proper fuzzy hyperideals. □

Proposition 4.13. *If R is a commutatice fuzzy hyperring with identity and P is a hyperideal which is maxilmal in the set of all fuzzy hyperideals of R which are not finitely generated, then P is prime.*

Proposition 4.14. *A commutative fuzzy hyperring R with identity is Noetherian if and only if every prime fuzzy hyperideal of R is finitely generated.*

Proof. Let S be the set of all fuzzy hyperideals of R which are not finitely generated. If S is nonempty, then use Zorn's Lemma to find a maximal element P of S. P is prime by Proposition 3.18 and hence finitely generated by hypothesis. This is a contradiction unless $S = \emptyset$. Therefore, R is Noetherian by proposition 4.12. □

Proposition 4.15. *Let I and J be fuzzy hyperideals of a commutative fuzzy hyperring $(R, \oplus, \odot)$, such that $I \subseteq J$. Then the fuzzy hyperideal J/I of the quotient fuzzy hyperring R/I is prime if and only if J is a prime fuzzy hyperideal of R.*

Proof. The statement follows from the third isomorphism theorem for fuzzy hyperrings, saying that $(R/I)/(J/I) \cong R/J$ and from Proposition 4.8. □

Theorem 4.16. *In a nonzero fuzzy hyperring R with identity maximal (resp. left) fuzzy hyperideals always exist. In fact every (resp. left) fuzzy hyperideal in R (except R itself) is contained in a maxinral (resp. left) fuzzy hyperideal.*

Proof. The proof is similar to the proof of that come in the theory of modules, by some manipulations. □

Definition 4.17. A left fuzzy hyperideal I in a fuzzy hyperring $(R, \oplus, \odot)$ is *regular* (or *modular*) if there exists $e \in R$ such that $r \oplus (-r \odot e) \in F^*(I)$ for every $r \in R$. Similarly, a right fuzzy hyperideal J is regular if there exists $e \in R$ such that $r \oplus (-r \odot e) \in F^*(J)$ for every $r \in R$.

Remark 4.18. Every left fuzzy hyperideal in a fuzzy hyperring R with identity is regular (let $e = 1_R$).

Theorem 4.19. *A left fuzzy hypermodule $(M, \boxplus, \boxdot)$ over a fuzzy hyperring $(R, \oplus, \odot)$ is simple if and only if M is isomorphic to R/I for some regular maximal left fuzzy hyperideal I.*

Proof. The discussion preceding Definition 4.44 shows that if M is simple, then $M = Ra \cong R/I$ where the maximal left fuzzy hyperideal I is the kernel of θ. Since $M = Ra$, $\chi_{\{a\}} = e \boxdot a$ for some $e \in R$. Now by Theorem 3.3, for $t \in R$ we have;

$$\chi_{\{a\}}(t) = (e \boxdot a)(t) \Rightarrow \chi_{\{a\}}(t) = \chi_{\{e \cdot a\}}(t) \Rightarrow a = e \cdot a \Rightarrow r \cdot a = (r \cdot e) \cdot a$$

Consequently, for any $r \in R$, $r \boxdot a = (r \odot e) \boxdot a$ or $(r \oplus (-r \odot e)) \boxdot a = \chi_{\{0\}}$, whence $r \oplus (-r \odot e) \in F^*(Ker(\theta)) = F^*(I)$. Therefore I is regular.

Conversely let I be a regular maximal left fuzzy hyperideal of R such that $M \cong R/I$. In view of the discussion preceding Definition 4.44 it suffices to prove that $R(R/I) \neq \{0\}$. If this is not the case. then for all $r \in Rr(e + I) = I$, whence $r \odot e \in F^*(I)$. Since $r \oplus (-r \odot e) \in F^*(I)$, we have $r \in I$. Thus $R = I$, contradicting the maximality of I. □

Definition 4.20. If N be a subset of a left fuzzy hypermodule $(M, \boxplus, \boxdot)$ over a fuzzy hyperring $(R, \oplus, \odot)$, Then

$$Ann(N) = \{r \in R : \ r \boxdot b = \chi_{\{0\}}, \ \forall b \in N\}$$

is a left fuzzy hyperideal of R. If N is a fuzzy subhypermodule of M, then $Ann(N)$ is a fuzzy hyperideal.

$Ann(N)$ is called the (left) *Annihilator* of N. The right annihilator of a right fuzzy hypermodule is defined analogously.

Definition 4.21. A (left) fuzzy hypermodule M is *faithful* if $Ann(M) = \{0\}$. A fuzzy hyperring R is (left) *primitive* if there exists a simple faithful left fuzzy hypermodule.

Proposition 4.22. *A simple fuzzy hyperring R with identity is primitive.*

Proof. R contains a maximal left fuzzy hyperideal I by Theorem 4.43. Since R has an identity, I is regular, whence R/I is a simple fuzzy hypermodule by Theorem 4.19. Since $Ann(R/I)$ is a fuzzy hyperideal of R that does not contain 1_R, $Ann(R/I) = \{0\}$ by simplicity. Therefore R/I is faithful. □

Proposition 4.23. *A commutative fuzzy hyperring $(R, \oplus, \odot)$ is primitive if and only if R is a fuzzy hyperfield.*

Proof. A fuzzy hyperfield is primitive by Proposition 4.22.

Conversely, let $(M, \boxplus, \boxdot)$ be a faithful simple left fuzzy hypermodule over fuzzy hyperring $(R, \oplus, \odot)$. Then $M \cong R/I$ for some regular maximal left fuzzy hyperideal I of R. Since R is commutative, I is in fact a fuzzy hyperideal and $I \subseteq Ann(R/I) = Ann(M) = \{0\}$. Since $I = \{0\}$ is regular, there is an $e \in R$ such that $r = r \odot e (= e \odot r)$ for all $r \in R$. Thus R is a commutative fuzzy hyperring with identity. Since $I = \{0\}$ is maximal, R is a fuzzy hyperfield by Corollary 4.12. □

Theorem 4.24. *(Wedderburn-Artin) If R is a simple left Artinian fuzzy hyperring, then R is primitive.*

Proof. We first observe that $I = \{r \in R : \ Rr = \{0\}\}$ is a fuzzy hyperideal of R, whence $I = R$ or $I = \{0\}$. Since $R^2 \neq \{0\}$, we must have $I = \{0\}$. Since R is left Artinian the set of all nonzero left fuzzy hyperideals of R contains a minimal left fuzzy hyperideal J. J has no proper fuzzy subhypermodules, (an fuzzy subhypermodule of J is a left fuzzy hyperideal of R). We claim that the left $Ann(J)$ in R is zero. Otherwise $Ann(J) = R$ by simplicity and $Ru = \{0\}$ for every nonzero $u \in J$. Consequently, each such nonzero u is contained in $I = \{0\}$, which is a contradiction. Therefore $Ann(J) = \{0\}$ and $RJ \neq \{0\}$. Thus J is a faithful simple fuzzy hypermodule, whence R is primitive. □

Definition 4.25. An element a in a fuzzy hyperring $(R, \oplus, \odot)$ is said to be left *quasi-regular* if there exists $r \in R$ such that $r \oplus a \oplus (r \odot a) = \chi_{\{0\}}$. The element r is called a left *quasi-inverse* of a. A (right, left or two-sided) fuzzy hyperideal I of R is said to be left quasi-regular if every element of I is left quasi-regular. Similarly, $a \in R$ is said to be right quasi-regular if there exists $r \in R$ such that $a \oplus r \oplus (a \odot r) = \chi_{\{0\}}$. Right quasi-inverses and right quasi-regular fuzzy hyperideals are defined analogously.

Theorem 4.26. *If R is a fuzzy hyperring, then there is a fuzzy hyperideal $J(R)$ of R such that:*

(i) $J(R)$ is the intersection of all the left annihilators of simple left fuzzy hypermodules over fuzzy hyperring R;

(ii) $J(R)$ is the intersection of all the regular maximal left fuzzy hyperideals of R;

(iii) $J(R)$ is the intersection of all the left primitive fuzzy hyperideals of R;

(iv) $J(R)$ is a left quasi-regular left fuzzy hyperideal which contains every left quasi-regular left fuzzy hyperideal of R,

(v) Statements (i)-(iv) are also true if "left" is replaced by "right"'.

Proof. The proof is similar to the proof of that come in the theory of modules, by some manipulations. □

Lemma 4.27. *A fuzzy hyperideal P of a fuzzy hyperring $(R, \oplus, \odot)$ is left primitive if and only if P is the left annihilator of a simple left fuzzy hypermodule.*

Proof. If P is a left primitive fuzzy hyperideal, let M be a simple faithful fuzzy hypermodule over fuzzy hyperring R/P. Verify that $(M, \boxplus, \boxdot)$ is an fuzzy hypermodule over fuzzy hyperring R, with $r \boxdot a$ ($r \in R$, $a \in M$) defined to be $(r + P) \boxdot a$. Then $RM = (R/P)M \neq \{0\}$ and every fuzzy suhyperbmodule of M is a fuzzy subhypermodule of M over fuzzy hyperring R/P, whence M is a simple R-hypermodule. If $r \in R$, then $rM = \{0\}$ if and only if $(r+P)M = \{0\}$. But $(r + P)M = \{0\}$ if and only if $r \in P$ since M is a faithful fuzzy hypermodule over fuzzy hyperring R/P. Therefore P is the left annihilator of the simple fuzzy hypermodule M.

Conversely suppose that P is the left annihilator of a simple fuzzy hypermodule $(N, \boxplus, \boxdot)$ over fuzzy hyperring R. Verify that N is a simple fuzzy hypermodule over fuzzy hyperring $(R/P, \boxplus, \boxtimes)$ with $(r + P) \boxdot b = r \boxdot b$ for $r \in R$, $b \in N$. Furthermore if $(r + P)N = \{0\}$, then $rN = \{0\}$, whence $r \in Ann(N) = P$ and $r + P = \{0\}$ in R/P. Consequently, N is a faithful fuzzy hypermodule over fuzzy hyperring R/P. Therefore R/P is a left primitive fuzzy hyperring, whence P is a left primitive fuzzy hyperideal of R. □

Lemma 4.28. *Let I be a left fuzzy hyperideal of a fuzzy hyperring $(R, \oplus, \odot)$. If I is left quasi-regular, then I is right quasi-regular.*

Proof. If I is left quasi-regular and $a \in I$, then there exists $r \in R$ such that $r \circledcirc a = r \oplus a \oplus (r \odot a) = \chi_{\{0\}}$. Now for $t \in R$ we have;

$$(r \circledcirc a)(t) = (r \oplus a \oplus (r \odot a))(t) = \chi_{\{0\}}(t) \Rightarrow \bigvee_{p \in R} \chi_{\{r+a\}}(p) \wedge \chi_{\{p+(r \cdot a)\}}(t) = \chi_{\{0\}}(t) \Rightarrow \bigvee \chi_{\{r+a+(r \cdot a)\}}(t) = \chi_{\{0\}}(t).$$

thus $r + a + (r \cdot a) = 0$, so $r = -a - (r \cdot a)$. Then we have;

$$\bigvee \chi_{\{-a-(r \cdot a)\}}(t) = \chi_{\{r\}}(t) \Rightarrow \bigvee_{p \in R} \chi_{\{-a+(-p))\}}(t) \wedge \chi_{\{r \cdot a\}}(p) = \chi_{\{r\}}(t).$$

Then $\chi_{\{r\}} = (-a) \oplus (-r \odot a) \in F^*(I)$, so there exists $s \in R$ such that $s \circledcirc r = s \oplus r \oplus (s \odot r) = \chi_{\{0\}}$, whence s is right quasi-regular. The fuzzy hyperoperation $\circledcirc$ is easily seen to be associative. Consequently

$$\chi_{\{a\}} = \{0\} \circledcirc a = (s \circledcirc r) \circledcirc a = s \circledcirc (r \circledcirc a) = s \circledcirc \{0\} = \chi_{\{s\}},$$

i.e., $a = s$. Therefore a, and hence I, is right quasi-regular. □

Definition 4.29. A fuzzy hyperring R is said to be *Jacobson semisimple* (or *J-semisimple*) if its Jacobson radical $J(R) = \{0\}$. R is said to be a radical fuzzy hyperring if $J(R) = R$.

Theorem 4.30. *Let R be a fuzzy hyperring.*

(i) If R is primitive, then R is semisimple.

(ii) If R is simple and semisimple, then R is primitive.

(iii) If R is simple, then R is either a primitive semisimple or a radical fuzzy hyperring.

Proof. (i): R has a faithful simple left fuzzy hypermodule M over fuzzy hyperring R, whence $J(R) \subset Ann(M) = \{0\}$.

(ii): $R \neq \{0\}$ by simplicity. There must exist a simple left fuzzy hypermodule M over fuzzy hyperring R; (otherwise by Theorem 4.26-(i) $J(R) = R \neq \{0\}$, contradicting semisimplicity). The left annihilator $Ann(M)$ is a fuzzy hyperideal of R by Definition 4.25 and $Ann(M) \neq R$ (since $RM \neq \{0\}$). Consequently $Ann(M) = \{0\}$ by simplicity, whence M is a simple faithful fuzzy hypermodule over fuzzy hyperring R. Therefore R is primitive.

(iii): If R is simple then the fuzzy hyperideal $J(R)$ is either R or zero. In the former case R is a radical fuzzy hyperring and in the latter R is semisimple and primitive by (ii). □

Definition 4.31. A fuzzy hyperideal P of a fuzzy hyperring R is said to be left (resp. right) *primitive* fuzzy hyperideal if the quotient fuzzy hyperring R/P is a left (resp. right) primitve fuzzy hyperring.

Definition 4.32. A fuzzy hyperideal P of a commutative fuzzy hyperring $(R, \oplus, \odot)$ is said to be left (resp. right) *Semiprimitive* fuzzy hyperideal if for every $a, b \in R$, $a \odot b \in F^*(P)$, $a \in P$ imply there exists $n \in$ such that $b^n \in F^*(P)$.

Definition 4.33. A fuzzy hyperring $(R, \oplus, \odot)$ is said to be a *subdirect product* of the family of fuzzy hyperrings $\{R_i : \ i \in I\}$ if R is a fuzzy subhyperring of the direct product $\prod R_i$ such that $\pi_k(R) = R_k$ for every $i \in I$, $k \in I$, where $\pi_k : \prod R_i \to R_k$ is the canonical epimorphism.

Theorem 4.34. *If $f : R \to S$ is a homomorphism of fuzzy hyperrings, then the kernel of f is a fuzzy hyperideal in R. Conversely if I is a fuzzy hyperideal in R, then the map $\pi : R \to R/I$ given by $r \mapsto r + I$ is an epimorphism of fuzzy hyperrings with kernel I.*

Proposition 4.35. *A nonzero fuzzy hyperring R is semisimple if and only if R is isomorphic to a subdirect product of primitive fuzzy hyperrings.*

Proof. Suppose R is nonzero semisimple and let $\mathcal{P}$ be the set of all left primitive fuzzy hyperideals of R. Then for each $P \in \mathcal{P}$, R/P is a primitive fuzzy hyperring (Definition 4.31). By Theorem 4.26-(iii), $\{0\} = J(R) = \bigcap_{P \in \mathcal{P}} P$. For each P let $\lambda_P : R \to R/P$ and $\pi_P : \prod_{Q \in \mathcal{P}} R/Q \to R/P$ be the respective canonical epimorphisms. The map $\phi : R \to \prod_{P \in \mathcal{P}} R/P$ given by $r \mapsto \{\lambda_P(r)\}_{P \in \mathcal{P}} = \{r + p\}_{P \in P}$ is a monomorphism of fuzzy hyperrings such that $\pi_P \phi(R) = R/P$ for every $P \in \mathcal{P}$.

Conversely suppose there is a family of primitive fuzzy hyperrings $\{R_i : \ i \in I\}$ and a monomorphism of fuzzy hyperrings $\phi : R \to \prod R_i$ such that $\pi_P \phi(R) = R_k$ for each $k \in I$. Let ψ_k be the epimorphism $\pi_P \phi$. Then $R/Ker\psi_k$ is isomorphic to the primitive fuzzy hyperring R_k (by first isomorphism theorem), whence $Ker\psi_k$ is a left primitive fuzzy hyperideal of R (Definition 4.31). Therefore $J(R) \subset \bigcap_{k \in I} Ker\psi_k$ by Theorem 4.26-(iii). However, if $r \in R$ and $\psi_k(r) = \{0\}$, then the k-th component of $\phi(r)$ in $\prod R_i$ is zero. Thus if $r \in \bigcap_{k \in I} Ker\psi_k$, we must have $\phi(r) = \{0\}$. Since ϕ is a monomorphism, $r = 0$. Therefore $J(R) \subset \bigcap_{k \in I} Ker\psi_k = \{0\}$, whence R is semisimple. □

Remark 4.36. Propositions 4.23 and 4.35 imply that a nonzero commutative semisimple fuzzy hyperring is a subdirect product of fuzzy hyperfields.

Definition 4.37. The prime radical $P(R)$ of a hyperring R is the intersection of all prime hyperideals of R. lf R has no-prime hyperideals, then $P(R) = R$. A hyperring R such that $P(R) = \{0\}$ is said to be *semiprime*.

Theorem 4.38. *(Recursion Theorem) If S is a set, $a \in S$ and for each $n \in N$, $f_n : S \to S$ is a function, then there is a unique function $\varphi : N \to R$ such $P(R) = R$. A hyperring R such that $P(R) = 0$ is said to be semiprime. that $\varphi(0) = a$ and $\varphi(n+1) = f(\varphi(n))$ for every $n \in N$.*

Proposition 4.39. *A fuzzy hyperring R is semiprime if and only if R has no nonzero nilpotent fuzzy hyperideals.*

Proof. If I is a nilpotent fuzzy hyperideal and K is any prime fuzzy hyperideal, then for some n, $I^n = \{0\} \subset K$, whence $I \subset K$. Therefore $I \subset P(R)$. Consequently, if R is semiprime, so that $P(R) = \{0\}$, then the only nilpotent fuzzy hyperideal is the zero fuzzy hyperideal.

Conversely, suppose that R has no nonzero nilpotent fuzzy hyperideals. We must show that $P(R) = \{0\}$. It suffices to prove that for every nonzero element a of R there is a prime fuzzy hyperideal K such that $a \in K$, whence $a \in P(R)$. We first observe that $Ann(R) \cap R$ is a nilpotent fuzzy hyperideal of R since

$$(Ann(R) \cap R)(Ann(R) \cap R) \subset Ann(R)R = \{0\}.$$

Consequently, $Ann(R) = Ann(R) \cap R = \{0\}$. Similarly $Ann_r(R) = \{0\}$. If b is any nonzero element of R, we claim that $RbR \neq \{0\}$. Otherwise $Rb \subset Ann(R) = 0$, whence $Rb = \{0\}$. Thus $b \in Annr(R) = \{0\}$, which is a contradiction. Therefore RbR is a nonzero fuzzy hyperideal of R and hence not nilpotent. Consequently $bRb \neq \{0\}$ (otherwise $(RbR)^2 \subset RbRbR = \{0\}$). For each nonzero $b \in R$ choose $f(b) \in bRb$ such that $f(b) \neq \{0\}$. Then by the Recursion Theorem (4.38) of the Introduction there is a function $\varphi : N \to R$ such that $\varphi(0) = a$ and $\varphi(n+1) = f(\varphi(n))$.
Let $a_n = \varphi(n)$ so that $a_{n+l} = f(a_n) \neq \{0\}$. Let $S = \{a_i : i \geq 0\}$. Use Zorn's Lemma to find a fuzzy hyperideal K that is maximal with respect to the property $K \cap S = \{0\}$ (since $0 \notin S$ there is at least one fuzzy hyperideal disjoint from S).
Since $a = a_0 \in S, a \in K$ and $K \neq R$. To complete the proof we need only show that K is prime. If A and B are fuzzy hyperideals of R such that $A \not\subset K$ and $B \not\subset K$, then $(A + K) \cap S \neq \emptyset$ and $(B + K) \cap S \neq \emptyset$ by rnaximality. Consequently for some i,j, $a_i \in A + K$ and $a_j \in B + K$. Choose $m > max\{i, j\}$. Since $a_{n+1} = f(a_n) \in a_n R a_n$ for each n, it follows that $a_m \in (a_i R a_i) \cap (a_j R a_j) \subset (A + K) \cap (B + K)$. Consequently,

$$a_{m+1} = f(a_m) \in a_m R a_m \subset (A + K)(B + K) \subset AB + K.$$

Since $a_{m+1} \notin K$, we must have $AB \not\subset K$. Therefore K is a prime fuzzy hyperideal. □

Remark 4.40. A fuzzy hyperring R is said to be a prime fuzzy hyperring if the zero fuzzy hyperideal is a prime fuzzy hyperideal (that is, if I, J are fuzzy hyperideals such that $IJ = \{0\}$, then $I = \{0\}$ or $J = \{0\}$).

Proposition 4.41. *A fuzzy hyperring R is semiprime if and only if R is isomorphic to a subdirect product of prime fuzzy hyperrings.*

Proof. This proposition is simply Proposition 4.35 with the words "semisimple" and "primitive" changed to "semiprime" and "prime" respectively. With this change and the use of Proposition 4.6 in place of Definition 4.31 , the proof of Proposition 4.35 carries over verbatim to the present case. □

Example 4.42. Let $(R, +, \cdot)$ be a ring with identity and G be a normal subgroup of the multiplicative e semigroup $(R^{\times}, \cdot)$, where $R^{\times} = R \setminus \{0\}$. Take $\bar{R} = R/G = \{aG : a \in R\}$ with the fuzzy hyperaddition and multiplication given by:

$$aG \boxplus bG = \chi_{\{aG+bG\}} \text{ and } aG \boxdot bG = \chi_{\{abG\}}.$$

Then $(R/G, \boxplus, \boxdot)$ is a fuzzy hyperring with identity, which is called the *quotient fuzzy hyperring* of R by G, and for short it is denoted by $\bar{R}$ and an element aG is written as $\bar{a}$. Moreover, if R is a field, then $(R, \boxplus, \boxdot)$ is a fuzzy hyperfield.

For a fuzzy hyperring R, by $Spec(R)$ we mean the set of all prime fuzzy hyperideal of R. Consider the mapping $\phi_G : R \to \bar{R}$, by $\phi_G(a) = aG$. It is called the canonical map or canonical projection.

In the next lemma we give some basic properties of the quotient fuzzy hyperrings.

Lemma 4.43. *Let R be a ring with identity and G be a normal subgroup of multiplicative semigroup $(R^* = R \setminus \{0\}, .)$. Then for the quotient fuzzy hyperring $\bar{R} = (R/G, \boxplus, \boxdot)$ the following statements are satisfied:*

(i) *The canonical map $\phi_G : R \to \bar{R}$ is a good epimorphism of fuzzy hyperrings.*

(ii) *If I is an ideal of R, then $\phi(I) = \bar{I} = IG$ is a hyperideal of $\bar{R}$.*

(iii) *If J is a fuzzy hyperideal of $\bar{R}$, then ${\phi_N}^{-1}(J)$ is an ideal of R.*

(iv) *There is a one to on correspondence between the fuzzy hyperideals of $(R/G, \oplus, \odot)$ and the ideals of R disjoint from G. More precisely, every fuzzy hyperideal of $\tilde{J}$ of $\bar{R}$ is of the form $\bar{J} = JG$, where J is an ideal of R with disjoint from G.*

(v) *There is a one to on correspondence between the prime fuzzy hyperideals of $(R/G, \oplus, \odot)$ and the prime ideals of R which are disjoint from G. In fact, every prime fuzzy hyperideal of $\bar{R}$ is of the form $\bar{P} = PG$, where P is a prime ideal of R and disjoint with G.*

Example 4.44. (i) $\mathbf{K} = \{0, 1\}$ is a fuzzy hyperfield with fuzzy hyperoperation and multiplication given in the following tables:

$\oplus$	0	1
0	$\chi_{\{0\}}$	$\chi_{\{1\}}$
1	$\chi_{\{1\}}$	$\chi_{\{\mathbf{K}\}}$

$\odot$	0	1
0	$\chi_{\{0\}}$	$\chi_{\{0\}}$
1	$\chi_{\{0\}}$	$\chi_{\{1\}}$

Then $\mathbf{K}$ is primitive, since every fuzzy hypermodules over $\mathbf{K}$ is a fuzzy vector hyperspace and fuzzy vector hyperspace of dimension 1 are simple over $\mathbf{K}$. In particular $\mathbf{K}$ is a simple $\mathbf{K}$-vector hyperspace.

(ii) (Sign fuzzy hyperfield) The set $\mathbf{S} = \{-1, 0, 1\}$ is is a a fuzzy hyperfield under fuzzy hyperoperation and multiplication given by following tables:

$\boxplus$	-1	0	1
-1	$\chi_{\{-1\}}$	$\chi_{\{-1\}}$	$\chi_{\{\mathbf{S}\}}$
0	$\chi_{\{-1\}}$	$\chi_{\{0\}}$	$\chi_{\{1\}}$
1	$\chi_{\{\mathbf{S}\}}$	$\chi_{\{1\}}$	$\chi_{\{1\}}$

$\boxdot$	-1	0	1
-1	$\chi_{\{1\}}$	$\chi_{\{0\}}$	$\chi_{\{-1\}}$
0	$\chi_{\{0\}}$	$\chi_{\{0\}}$	$\chi_{\{0\}}$
1	$\chi_{\{-1\}}$	$\chi_{\{0\}}$	$\chi_{\{1\}}$

By the same reason is a primitive fuzzy hyperring, since every fuzzy hyperfield is primitive.

Let R be a ring with identity and M be a unitary module over R. Define a relation $\sim$ on V as follows:

$$x \sim y \Leftrightarrow x = ty \ \ , \exists t \in G.$$

Let $\bar{\bar{M}}$ be the set of all equivalence classes of M modulo $\sim$. Define a fuzzy hyperoperation $\dotplus$ on $\bar{M}$ as follows:

$$\bar{x} \overset{\circ}{+} \bar{y} = \overline{a + x} = \chi_{\{x+y\}}.$$

Then $(\bar{M}, \overset{\circ}{+})$ becomes a canonical fuzzy hypergroup. Let $\overline{RF}$ be the quotient fuzzy hyperfield over $\bar{R}$ by G. Now define the external composition from $\bar{R} \times \bar{M}$ to $\bar{R}$ as follows:

$$\bar{a} \overset{\circ}{\times} \bar{x} = \overline{a \cdot x} = \chi_{\{x \cdot y\}} \ \ , \bar{a} \in \bar{R}, \bar{x} \in \bar{R}.$$

This composition satisfies the axioms of fuzzy hypermodule, and so $\bar{M}$ becomes a fuzzy vector hyperspace. This hypervector space is called quotient vector hyperspace of M $\bar{R}$. Under the above construction we have the next result.

Theorem 4.45. . *Let R be a fuzzy hyperring with identity and M be an unitary fuzzy $R-$hypermodule. Then the following are satisfied:*

(i) *Every fuzzy $\bar{R}-$hypermodules K is an $R-$module under construction by scalar under the canonical mapping $\Pi = \Pi_G : M \to \bar{R}$, given by $rx = \phi_G(r)x = \bar{r}x$. Moreover, the map $M \to \bar{M}$, by $m = \bar{m}$ is a fuzzy $\bar{R}-$hypermodules homomorphism.*

(ii) *If N is an $R-$submodule of M, then $\phi(N) = \bar{N} = GN$ is a fuzzy $\bar{R}-$subhypermodule of K.*

(iii) *If K is a fuzzy $\bar{R}-$hypermodules of K, then $\phi^{-1}(N)$ is an $R-$submodule of M.*

(iv) *There is a one to one correspondence between the fuzzy $\bar{R}-$subhypermodules of $\bar{M}$ and the $R-$submodules of M containing G.*

(v) *If M is (resp. Artinian) Noetherian $R-$modules, then $\bar{M}$ is so.*

(vi) *If R is primitive, then $\bar{R}$ is too.*

(vii) *If R is semisimple, so is $\bar{R}$.*

Theorem 4.46. *Let R be a fuzzy hyperring with identity and N be a normal subgroup of $(R^*, .)$. Then*

(i) *If P is a (resp. simple) semiprime ideal of R, then $\bar{P}$ is (resp. simple) primitive fuzzy hyperideal of $\bar{R}$.*

(ii) *If R is primitive then $\bar{R}$ is so.*

In this step we introduce one of the important relation on a fuzzy hyperring $(R, \oplus, \odot)$. Let $\mathcal{U}$ denotes the set of all finite sum of finite products of elements of R. Note that an element $u \in \mathcal{FU}$, may be a sum of only one element. Define relation γ on R as follows:

$$a\gamma b \iff \mu_f \in \mathcal{FU}(R)\ s.t.\ \mu_f(x) > 0, \mu_f(y) > 0,$$

in fact there exist $n, k_i \in \mathbb{N}$ and $x_{ij} \in R$, such that $\mu_f = \Sigma^{\oplus}_{1\leq i\leq n}\Pi^{\odot}_{1\leq j\leq k_i} x_{ji}$, where $\sum_i^{\oplus}$ and $\prod_i^{\odot}$ are finite fuzzy hypersum and finite fuzzy hyperproduct of the fuzzy hyperring $(R, \oplus, \odot)$ respectivly. As above, we obtain

$$\textstyle\sum^{\oplus}_{1\leq i\leq n}\prod^{\odot}_{1\leq j\leq k_i} x_{ji})(x) > 0 \text{ and } \sum^{\oplus}_{1\leq i\leq n}\prod^{\odot}_{1\leq j\leq k_i} x_{ji})(y) > 0.$$

.

Clearly, γ relation is reflexive and symmetric. Let γ^* denotes the *transitive closure* of γ. Consider the quotient R/γ^*, and define operations $+$ and $\times$ on it as follow:

$$\gamma^*(a) + \gamma^*(b) = \gamma^*(c) = \{c : \ c \in \gamma^*(a) \oplus \gamma^*(b)\} = \{c : \ (\gamma^*(a) \oplus \gamma^*(b))(c) > 0\};$$
$$\gamma^*(a) \times \gamma^*(b) = \gamma^*(d) = \{d : \ d \in \gamma^*(a) \odot \gamma^*(b)\} = \{d : \ (\gamma^*(a) \odot \gamma^*(b))(d) > 0\}.$$

Then γ^* is the smallest equivalence relation of R, such that the quotient set R/γ^* is a ring, which is called the *fuzzy fundamental relation* of R, and R/γ^* is called fundamental ring of R.

Remark 4.47. As it is well known the fuzzy fundamental relation an important role in the theory of algebraic fuzzy hyperstructures. At the following we pose two important question relevant to fundamental relation on an fuzzy R-hypermodule and primitive property of R.

Question: Consider an arbitrary fuzzy hyperring with identity R, is there the smallest strongly regular relation say ρ on R such that the quotient ring R/ρ be a primitive ring?

Now, let R be an arbitrary primitive fuzzy hyperring with identity and R/γ^* be a fundamental ring of R, i.e., R has a faithful simple left fuzzy hyperideale I of fuzzy hyperring R, whence $Ann(I) = \{0\}$. So we have, I/γ^* is a faithful simple left fuzzy hyperideale of fuzzy hyperring R/γ^*.

Proposition 4.48. *If R be an arbitrary primitive fuzzy hyperring with identity, then the fundamental ring R/γ^* is a primitive ring.*

5 Prime and Primitive Fuzzy Hyperring by membership functions

In the section, we will obtain the some result of prime and primitive fuzzy hyperring by membership functions in the sense of Sen, Ameri and Chowdhury in [4].

Let X be a set. A *fuzzy subset* S of X is characterized by a membership function $\mu_S : X \to [0,1]$ which associate with point $x \in X$ it's grade or degree of membership $\mu_S(x) \in [0,1]$.
Here, $F(X)$ denote the set of all fuzzy subset of X and $F^*(X) = F(X) \setminus \{\emptyset\}$.

Definition 5.1. Let R be a fuzzy hyperring (with identity) and $\xi \in FI(R)$, where $FI(R)$ denoted the set of all fuzzy hyperideal of R. Then ξ is called *prime* fuzzy hyperideal of R if ξ is non-constant and $\mu \circ \nu \subseteq \xi$; μ, $\nu \in FI(R)$, implies $\mu \subseteq \xi$ or $\nu \subseteq \xi$.

Note that for $x, x_i, y, y_i, z \in R$, $\mu \circ \nu$ is defin as follows;

$$(\mu \circ \nu)(z) = \bigvee_{z \in xy} \mu(x) \wedge \nu(y),$$

but,

$$(\mu\nu)(z) = \bigvee_{z \in \sum_i x_i y_i} (\bigwedge_i \mu(x_i) \wedge \nu(y_i)).$$

Theorem 5.2. *Let R be a fuzzy hyperring with identity. Then $\xi \in FI(R)$ is prime fuzzy hyperideal of R if and only if ξ is non-constant and $\mu\nu \subseteq \xi$; μ, $\nu \in FI(R)$, implies $\mu \subseteq \xi$ or $\nu \subseteq \xi$.*

Proof. Let $\xi \in FI(R)$ is prime fuzzy hyperideal of R and $\mu\nu \subseteq \xi$; μ, $\nu \in FI(R)$. By the theorem (3.1.18) in [16], we have; $\mu \circ \nu \subseteq \xi$, thus by hypothesis, implies $\mu \subseteq \xi$ or $\nu \subseteq \xi$ and conversely. □

Note that If $\mu \in FI(R)$, we let $\mu_* = \{x \in R : \ \mu(x) = \mu(0)\}$, and also for every $x \in R$ we have; $\mu(0) \geq \mu(x)$.

Theorem 5.3. *If $\xi \in FI(R)$ is prime fuzzy hyperideal of R, then ξ_* is prime fuzzy hyperideal of R.*

Proof. Let $\xi \in FI(R)$ is prime fuzzy hyperideal of R and $A, B \subseteq \xi_*$. ξ_* is non-constant, since ξ is non-constant, i.e., $\xi \neq \xi(0)$ then $\xi_* \neq \xi_*(0)$. Then 1_A, $1_B \in FI(R)$, ($1_A \neq y = 1$, i.e., $y \in A$, in fact $1_A = \chi_A$) and $1_A \circ 1_B \subseteq 1_{\xi_*} \subseteq \xi$. Now, since ξ is prime fuzzy hyperideal of R, so either $1_A \subseteq \xi$ or $1_B \subseteq \xi$, thus $A \subseteq \xi_*$ or $B \subseteq \xi_*$. (Since if $1_A \subseteq \xi$, then for $x \in A$, $1_A(x) \leq \xi_A(x)$, i.e., $1 \leq \xi_A(x)$, so $\xi_A(0) \leq \xi_A(x)$. Thus $\xi_A(x) = \xi_A(0)$, this implies $x \in \xi_*$.) □

Theorem 5.4. *If $\xi \in F(R)$, then ξ is prime fuzzy hyperideal of R if and only if ξ_* is prime fuzzy hyperideal of R and $\xi(0) = 1$, $\xi(R) = \{1, c\}$, where $c \neq 1$.*

Proof. Let $\xi \in FI(R)$ is prime fuzzy hyperideal of R. Then by the theorem 5.3, ξ_* is prime fuzzy hyperideal of R. We now show that $\xi(0) = 1$. We suppose that $\xi(0) \leq 1$, since ξ_* is non-constant, there exists $x \in R$ such that $\xi(x) \leq \xi(0)$.
Let μ, $\nu \in F(R)$ be defined by $\mu(x) = 1$ if $x \in \xi_*$ and $\mu(x) = 0$ if $x \notin \xi_*$ (In fact $\mu = \chi_{\xi_*}$) and $\nu(x) = \xi(0)$, for $x \in R$. clearly μ, $\nu \in FI(R)$ and $\mu \circ \nu \subseteq \xi$. Since $\mu(0) = 1 > \xi(0)$ and $\nu(x) = \xi(0) > \xi(x)$, then for $x \in \xi_*$, $\mu \nsubseteq \xi$ and $\nu \nsubseteq \xi$. This is a conteradiction since ξ is prime. Hence $\xi(0) = 1$.
We now show that $|\xi(R)| = 2$. Let $x, y \in R \backslash \xi_*$ and $\xi(x) = c$. Then $c_{\langle x \rangle} = \langle x_c \rangle \subseteq \xi$ (note that x_c is a fuzzy pointe), this is $x_c(y) = 1$ i.e. $y = c$ and $x_c(y) = 0$ i.e. $y \neq c$. Clearly, $1_{\langle x \rangle}, c_R \in F(R)$, $1_{\langle x \rangle} \nsubseteq \xi$. Thus $\xi(x) = c = c_R(y) \leq \xi(y)$. Similarly, $\xi(y) \leq \xi(x)$. Hence $\xi(x) = \xi(y) = c$. Thus $|\xi(R)| = 2$ and $\xi = 1_{\xi_*} \cup c_R$, where ξ_* is prime fuzzy hyperideal of R and $c \neq 1$.
Conversely, suppose ξ satisfies the given condition. Then ξ is not constant. Assium that there exist μ, $\nu \in FI(R)$ such that $\mu \circ \nu \subseteq \xi$, but $\mu \nsubseteq \xi$ and $\nu \nsubseteq \xi$.
Then there exist $x, y \in R$ such that $\mu(x) \nleq \xi(x)$ and $\nu(y) \nleq \xi(y)$. Then $\xi(x) = \xi(y) = c$, $x, y \in R \backslash \xi_*$, $\mu(x) \nleq c$ and $\nu(y) \nleq c$ since $x, y \in R \backslash \xi_*$ and ξ_* is prime fuzzy hyperideal of R, $xy \subseteq \xi_*$. Thus $\xi(z) = c$, $\forall z \in xy$.
Now,

$$\mu(x) \wedge \nu(y) \leq (\mu \circ \nu)(z) \leq \xi(z) = c \ ,\forall z \in xy.$$

Thus $\mu(x) \leq c$ and $\nu(y) \leq c$ a conteradiction. □

Theorem 5.5. *If $\xi \in F(R)$, then ξ is prime fuzzy hyperideal of R if and only if ξ is non-constant and $x_a \circ y_b \subseteq \xi$, $x, y \in R$, $a, b \in [0,1]$, then either $x_a \subseteq \xi$ or $y_b \subseteq \xi$.*

Proof. Let $\xi \in F(R)$ is prime fuzzy hyperideal of R and $x_a \circ y_b \subseteq \xi$, $x, y \in R$, $a, b \in [0,1]$. So by the theorem (3.1.35) in [16], for $z \in xy$;

$$(x_a \circ y_b)(z) \subseteq (\langle x_a\rangle \circ \langle y_b\rangle)(z) = (\langle x_a\rangle\langle y_b\rangle)(z) = (\langle (xy)_{a\wedge b}\rangle)(z) = a \wedge b \leq 1_\xi(z) \subseteq \xi(z).$$

Now, since ξ is prime fuzzy hyperideal of R, so either $\langle x_a\rangle \subseteq \xi$ or $\langle y_b\rangle \subseteq \xi$, thus $x_a \subseteq \xi$ or $y_b \subseteq \xi$.

Conversely, let $\mu \circ \nu \subseteq \xi$; μ, $\nu \in FI(R)$. Now, we give, $\mu = x_a$ and $\nu = y_b$, $x, y \in R$, $a, b \in [0,1]$, so by hypothesis, we obtian, $x_a \subseteq \xi$ or $y_b \subseteq \xi$, i.e., $\mu \subseteq \xi$ or $\nu \subseteq \xi$. □

Theorem 5.6. *Let R be a fuzzy hyperring with identity. Every maximal fuzzy hyperideal of R is a prime fuzzy hyperideal.*

Proof. Let $\xi \in FI(R)$ is maximal fuzzy hyperideal of R and $x_a \circ y_b \subseteq \xi$, $x, y \in R$, $a, b \in [0,1]$. We suppose, $x_a \nsubseteq \xi$, so for $z \in \xi$ such that $z \notin x_a$, we have; $\langle x_a\rangle \subseteq \langle x_a, z\rangle \subseteq \xi$, by maximalty $\langle x_a, z\rangle = \xi$. Therefore there exist $m, n \in Z$ such that $mx + xn + z = 1$, so for $y \in R$, $mxy + xny + zy = y$. thus $z \in y_b$, i.e. $y_b \subseteq \xi$. □

Example 5.7. (i). For every prime hyperideal P, χ_P is a prime fuzzy hyperideal. Since, if $A, B \subseteq p$, then for every $x, y \in P$ and $z \in xy$, we have;

$$(1_A \circ 1_B)(z) \leq 1_P(z) = \chi_P(z) \Rightarrow \bigvee_{z\in xy} \chi_A(x) \wedge \chi_B(y) \leq \chi_P(z) \Rightarrow \chi_{AB}(z) \leq \chi_P(z)$$
$$\Rightarrow AB \subseteq P.$$

By hypothesis, we have; $A \subseteq P$ or $B \subseteq P$. Thus $\chi_A \subseteq \chi_P$ or $\chi_B \subseteq \chi_P$.

(ii). Let R be a hyperring and P be a prime hyperideal of R. Define a fuzzy subset ξ of R as follow:

$$\xi(x) = \begin{cases} 1; & x \in P \\ \alpha; & x \notin P, \alpha \in [0,1] \end{cases}.$$

then ξ is prime fuzzy hyperideal. Since, let $\mu \circ \nu \subseteq \xi$; μ, $\nu \in FI(R)$. So for every $x, y \in P$ and $z \in xy \subseteq P$, we have;

$$\bigvee_{z\in xy} \mu(x) \wedge \nu(y) \leq \xi(z).$$

Now, since P is prime hyperideal of R, so $x \in P$ or $y \in P$, thus $\mu(x) = 1$, $\nu(y) = 0$ or $\mu(x) = 0$, $\nu(y) = 1$, i.e., $\nu(z) = 0 \leq \alpha = \xi(z)$ or $\mu(z) = 0 \leq \alpha = \xi(z)$. Therefore, $\mu \subseteq \xi$ or $\nu \subseteq \xi$.

Definition 5.8. A hyperring R is called *prime hyperring*, if for all non-zero hyperideales A and B of R, $AB \neq 0$.

Definition 5.9. A hyperring R is called *prime fuzzy hyperring*, if for non-zero fuzzy hyperideales μ and ν of R, $\mu \circ \nu \neq 1_0$ $(= \widetilde{0})$.

Proposition 5.10. *R is a prime fuzzy hyperring if and only if all non-zero fuzzy hyperideales μ and ν of R, $\mu\nu \neq 0$.*

Proof. Let R be a prime fuzzy hyperring and μ and ν be non-zero fuzzy hyperideales of R. So $\mu \circ \nu \neq 1_0$, thus for $x_i, y_i \in P$ and $z \in x_i y_i$, we have; $\bigvee_{z\in x_i y_i} \mu(x_i) \wedge \nu(y_i) \neq 1_0(z)$. Therefore, $\bigvee_{z\in x_i y_i} (\bigwedge_i \mu(x_i) \wedge \nu(y_i)) \neq 1_0(z)$, then $\mu\nu \neq 0$ and conversely. □

Definition 5.11. A proper fuzzy hyperideal P of commutative fuzzy hyperring with identity is called a *semiprime* fuzzy hyperideal if for every $I \in FI(R)$, $I \circ I \subseteq P$ implies $I \subseteq P$.

Theorem 5.12. *Let P be proper fuzzy hyperideal. Then P is semiprime fuzzy hyperideal if and only if for every $\mu \in FI(R)$ and $\mu^2 = \mu \circ \mu \subseteq P$ implies $\mu \subseteq P$.*

Proof. Let P be proper semiprime fuzzy hyperideal and for every $\mu \in FI(R)$ and $\mu \circ \mu \subseteq P$. We give $\mu = 1_I$, $I \in FI(R)$. So we have; $1_I \circ 1_I \subseteq P$, then for every $x, y \in P$ and $z \in xy$, we obtian;

$$(1_I \circ 1_I)(z) \leq 1_P(z) = \chi_P(z) \Rightarrow \bigvee_{z\in xy} \chi_I(x) \wedge \chi_I(y) \leq \chi_P(z) \Rightarrow \chi_{I^2}(z) \leq \chi_P(z) \Rightarrow I^2 \subseteq P.$$

By hypothesis, we have; $I \subseteq P$. Thus $\chi_I \subseteq P$, i.e., $\mu \subseteq P$. □

Remark 5.13. Clearly, every prime fuzzy hyperideal is fuzzy semiprime, but the convers is not true, because let P is semiprime fuzzy hyperideal which is not prime. Then χ_P is a semiprime fuzzy hyperideal, which is not prime fuzzy hyperideal.

Theorem 5.14. *If ξ is semiprime fuzzy hyperideal of R, then ξ_* is semiprime fuzzy hyperideal of R.*

Proof. Let ξ is semiprime fuzzy hyperideal of R and $A^2 \subseteq \xi_*$. Then $1_A \circ 1_A \subseteq 1_{\xi_*} \subseteq \xi$. Now, since ξ is semiprime fuzzy hyperideal of R, so $1_A \subseteq \xi$, thus $A \subseteq \xi_*$. □

Theorem 5.15. *Let $\xi \in F(R)$. Then ξ is semiprime fuzzy hyperideal of R if and only if ξ_* is semiprime fuzzy hyperideal of R and $\xi(0) = 1$, $\xi(R) = \{1, \alpha\}$, for $\alpha \in [0, 1]\backslash\{1\}$.*

Theorem 5.16. *Let $\xi \in F(R)$. Then ξ is semiprime fuzzy hyperideal of R if and only if ξ is non-constant and $x_\alpha \circ x_\beta \subseteq \xi$, $x \in R$, $\alpha, \beta \in [0, 1]$, then either $x_\alpha \subseteq \xi$ or $x_\beta \subseteq \xi$.*

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