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On Weighted Sums of Padovan, Perrin, and Van der Laan Sequences

Prabha Sivaraman Nair

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Corresponding Author: Prabha Sivaraman Nair

Abstract Let $\left(H_n\right)_{n\geq 0}$ be the generalized Padovan sequence defined by $H_n = H_{n-2}+H_{n-3}$ for $n \geq 3$ with initial terms $H_0 = a, H_1 = b$ and $H_2 = c$, where a, b, c are real numbers, not all being zero. This paper aims to provide an explicit formula for computing the sum $\sum_{r=1}^{n} H_r x^r$, and a recursive formula for $\sum_{r=1}^{n} r^m H_r x^r$ for all integers $m \geq 1$, where x is any real number such that $1 - x^2 - x^3 \neq 0$. Furthermore, an exact formula for the infinite sum $\sum_{r=1}^{\infty} H_r x^r$ is provided, along with a recurrence formula for $\sum_{r=1}^{\infty} r^m H_r x^r$ for all integers $m \geq 1$ within the interval of convergence. Additionally, formulas for such sums related to Padovan, Perrin, and Van der Laan sequences are deduced. Finally, algorithms for finding such infinite sums are developed, provided that they converge.

1 Introduction

The Padovan sequence $\left(P_n\right)_{n\geq 0}$ is defined by

$$P_n = P_{n-2} + P_{n-3}, \tag{1.1}$$

for $n \geq 3$ with initial terms $P_0 = P_1 = P_2 = 1$. It is well-known that the sequence $\left(\frac{P_{n+1}}{P_n}\right)_{n\geq 0}$ converges to the number $\rho = \sqrt[3]{\frac{1}{2}+\sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2}-\sqrt{\frac{23}{108}}}$, the real root of $x^3 = x + 1$. The approximate value of ρ is 1.32471795724. The Padovan sequence can be extended to negative subscripts by

$$P_{-n} = P_{-(n-3)} - P_{-(n-1)},$$

for $n \geq 1$. The Perrin sequence $\left(Q_n\right)_{n\geq 0}$ and the Van der Laan sequence $\left(V_n\right)_{n\geq 0}$ are also defined by the same recurrence (1.1) with initial terms $Q_0 = 3, Q_1 = 0, Q_2 = 2$, and $V_0 = 1, V_1 = 0, V_2 = 1$. These numbers are extensively studied by many researchers (see [3, 7, 9, 11, 12, 13, 15, 14, 16]). Some intriguing summation identities about Padovan and Perrin numbers can be found in [2]. Recently, Dişkaya and Menken [8] derived an expression for the Brousseau weighted sum $\sum_{r=1}^{n} r^m P_r$ of the Padovan numbers.

The generalized Padovan sequence $\left(H_n\right)_{n\geq 0}$ is defined by $H_n = H_{n-2}+H_{n-3}$ for $n \geq 3$ with initial terms $H_0 = a, H_1 = b$ and $H_2 = c$, where a, b, c are real numbers with $(a, b, c) \neq (0, 0, 0)$. These numbers can be extended to negative subscripts by $H_{-n} = H_{-(n-3)} - H_{-(n-1)}$, for $n \geq 1$. Note that the ratio $\frac{H_{n+1}}{H_n}$ also converges to ρ as $n \to \infty$. In this paper, we find an explicit formula for the sum

$$\sum_{r=1}^{n} H_r x^r,$$

and a recurrence formula for the sum

$$\sum_{r=1}^{n} r^m H_r x^r,$$

where m is a positive integer and x is any real number with $1 - x^2 - x^3 \neq 0$. Using these formulas, we establish a recurrence scheme for the finite sums

$$\sum_{r=1}^{n} r^m P_r x^r, \ \sum_{r=1}^{n} r^m Q_r x^r, \text{ and } \sum_{r=1}^{n} r^m V_r x^r.$$

Similar types of sums (with $x = 1$) involving the classical Fibonacci numbers were first studied by Brousseau [5, 6]. For some related studies, see [1, 10]. The generating functions of Padovan, Perrin, and Van der Laan numbers can be retrieved as well. We also develop recurrence formulas to find the sum of the power series

$$\sum_{r=1}^{\infty} r^m P_r x^r, \ \sum_{r=1}^{\infty} r^m Q_r x^r, \text{ and } \sum_{r=1}^{\infty} r^m V_r x^r$$

in their interval of convergence. The method we employ is akin to the approach utilized by Brandão and Martins [4] for the classical Fibonacci numbers.

2 Finite Sums

In this section, we find a recurrence formula for the sum $\sum_{r=1}^{n} r^m H_r x^r$ for all integers $m, n \geq 1$, where x is any real number with $1 - x^2 - x^3 \neq 0$. We use that formula (along with the exact formula for $m = 0$) to derive numerous summation identities related to Padovan-type sequences such as Padovan, Perrin, and Van der Laan sequences.

2.1 Explicit formulas

We begin with finding an explicit formula for the sum $\sum_{r=k}^{n} H_r x^r$, for all integers $n \geq k \geq 1$.

Theorem 2.1. *If* $1 - x^2 - x^3 \neq 0$*, then for all integers* $n \geq k \geq 1$*, the following identity holds:*

$$\sum_{r=k}^{n} x^r H_r = \frac{x^k\left(H_k + xH_{k+1} + x^2 H_{k-1}\right) - x^{n+1}\left(H_{n+1} + xH_{n+2} + x^2 H_n\right)}{1 - x^2 - x^3}. \tag{2.1}$$

Proof. Let S denote the sum on the left-hand side of (2.1). Then

$$\begin{aligned}(1 - x^2 - x^3)S &= (1 - x^2 - x^3)\sum_{r=k}^{n} x^r H_r \\ &= \sum_{r=k}^{n} x^r H_r - \sum_{r=k}^{n} x^{r+2} H_r - \sum_{r=k}^{n} x^{r+3} H_r.\end{aligned}$$

Applying a suitable change of variable in the last two summations on the right-hand side yields

$$\begin{aligned}(1 - x^2 - x^3)S &= \sum_{r=k}^{n} x^r H_r - \sum_{r=k+2}^{n+2} x^r H_{r-2} - \sum_{r=k+3}^{n+3} x^r H_{r-3} \\ &= x^k H_k + x^{k+1} H_{k+1} + x^{k+2} H_{k+2} + \sum_{r=k+3}^{n} x^r H_r \\ &\quad - x^{k+2} H_k - x^{n+1} H_{n-1} - x^{n+2} H_n - \sum_{r=k+3}^{n} x^r H_{r-2} \\ &\quad - x^{n+1} H_{n-2} - x^{n+2} H_{n-1} - x^{n+3} H_n - \sum_{r=k+3}^{n} x^r H_{r-3}.\end{aligned}$$

Thus,

$$(1-x^2-x^3)S = x^k H_k + x^{k+1} H_{k+1} + x^{k+2}\big(H_{k+2} - H_k\big) - x^{n+1}\big(H_{n-1} + H_{n-2}\big) - x^{n+2}\big(H_n + H_{n-1}\big) - x^{n+3} H_n + \sum_{r=k+3}^{n} x^r\big(H_r - H_{r-2} - H_{r-3}\big).$$

Now, using the Padovan recurrence $H_i = H_{i-2} + H_{i-3}$, we obtain

$$(1-x^2-x^3)S = x^k H_k + x^{k+1} H_{k+1} + x^{k+2} H_{k-1} - x^{n+1} H_{n+1} - x^{n+2} H_{n+2} - x^{n+3} H_n.$$

Thus,

$$S = \frac{x^k\big(H_k + xH_{k+1} + x^2 H_{k-1}\big) - x^{n+1}\big(H_{n+1} + xH_{n+2} + x^2 H_n\big)}{1-x^2-x^3}.$$

□

Setting $k=1$ in (2.1) yields the following corollary:

Corollary 2.2. *If* $1-x^2-x^3 \neq 0$, *then for all integers* $n \geq 1$, *the following identity holds:*

$$\sum_{r=1}^{n} x^r H_r = \frac{x\left(ax^2 + cx + b\right) - x^{n+1}\left(H_{n+1} + xH_{n+2} + x^2 H_n\right)}{1-x^2-x^3}. \tag{2.2}$$

In particular,

$$\sum_{r=1}^{n} x^r P_r = \frac{x\left(x^2 + x + 1\right) - x^{n+1}\left(P_{n+1} + xP_{n+2} + x^2 P_n\right)}{1-x^2-x^3},$$

$$\sum_{r=1}^{n} x^r Q_r = \frac{x\left(3x^2 + 2x\right) - x^{n+1}\left(Q_{n+1} + xQ_{n+2} + x^2 Q_n\right)}{1-x^2-x^3},$$

and

$$\sum_{r=1}^{n} x^r V_r = \frac{x\left(x^2 + x\right) - x^{n+1}\left(V_{n+1} + xV_{n+2} + x^2 V_n\right)}{1-x^2-x^3}.$$

Corollary 2.3. *For all integers* $n \geq 1$, *we have*

$$\sum_{r=1}^{n} H_r = H_n + H_{n+1} + H_{n+2} - (a+b+c). \tag{2.3}$$

In particular,

$$\sum_{r=1}^{n} P_r = P_n + P_{n+1} + P_{n+2} - 3, \tag{2.4}$$

$$\sum_{r=1}^{n} Q_r = Q_n + Q_{n+1} + Q_{n+2} - 5, \tag{2.5}$$

and

$$\sum_{r=1}^{n} V_r = V_n + V_{n+1} + V_{n+2} - 2. \tag{2.6}$$

Proof. It follows by setting $x=1$ in Corollary 2.2. □

Since $H_n + H_{n+1} + H_{n+2} = H_{n+5}$, the identity (2.3) simplifies to

$$\sum_{r=1}^{n} H_r = H_{n+5} - (a+b+c).$$

Therefore,

$$\sum_{r=1}^{n} P_r = P_{n+5} - 3, \quad \sum_{r=1}^{n} Q_r = Q_{n+5} - 5, \quad \text{and} \quad \sum_{r=1}^{n} V_r = V_{n+5} - 2.$$

Corollary 2.4. *For all integers $n \geq 1$, the following identity holds:*

$$\sum_{r=1}^{n}(-1)^r H_r = (-1)^n\big(H_n + H_{n+1} - H_{n+2}\big) - (a+b-c). \tag{2.7}$$

In particular,

$$\sum_{r=1}^{n}(-1)^r P_r = (-1)^n\big(P_n + P_{n+1} - P_{n+2}\big) - 1, \tag{2.8}$$

$$\sum_{r=1}^{n}(-1)^r Q_r = (-1)^n\big(Q_n + Q_{n+1} - Q_{n+2}\big) - 1, \tag{2.9}$$

and

$$\sum_{r=1}^{n}(-1)^r V_r = (-1)^n\big(V_n + V_{n+1} - V_{n+2}\big). \tag{2.10}$$

Proof. It follows by setting $x = -1$ in Corollary 2.2. □

Since $H_n + H_{n+1} - H_{n+2} = H_{n-2}$, the identity (2.7) simplifies to

$$\sum_{r=1}^{n}(-1)^r H_r = (-1)^n H_{n-2} - (a+b-c).$$

Therefore,

$$\sum_{r=1}^{n}(-1)^r P_r = (-1)^n P_{n-2} - 1, \sum_{r=1}^{n}(-1)^r Q_r = (-1)^n Q_{n-2} - 1, \text{ and} \sum_{r=1}^{n}(-1)^r V_r = (-1)^n V_{n-2}.$$

2.2 Recurrence formulas

Let us now derive our main recurrence formula. Let x be any real number. For all integers $m \geq 0$ and $n \geq 1$, we define

$$S_n^{(m)}(x) = \sum_{r=1}^{n} r^m H_r x^r. \tag{2.11}$$

Theorem 2.5. *If $S_n^{(m)}(x)$ is as defined in (2.11) and $1 - x^2 - x^3 \neq 0$, then for all integers $m, n \geq 1$, we have*

$$S_n^{(m)}(x) = \frac{1}{1-x^2-x^3}\Bigg[ax^3 + (-1)^m bx - n^m H_{n+2}x^{n+2} - (n-1)^m H_{n+1}x^{n+1} - (n+1)^m H_n x^{n+3} + \sum_{j=1}^{m}\big(x^3 - (-2)^j\big)\binom{m}{j} S_n^{(m-j)}(x)\Bigg].$$

In particular,

$$\sum_{r=1}^{n} r^m P_r x^r = \frac{1}{1-x^2-x^3}\Bigg[x^3 + (-1)^m x - n^m P_{n+2}x^{n+2} - (n-1)^m P_{n+1}x^{n+1} - (n+1)^m P_n x^{n+3} + \sum_{j=1}^{m}\big(x^3 - (-2)^j\big)\binom{m}{j}\Bigg(\sum_{r=1}^{n} r^{m-j} P_r x^r\Bigg)\Bigg],$$

$$\sum_{r=1}^{n} r^m Q_r x^r = \frac{1}{1-x^2-x^3}\Bigg[3x^3 - n^m Q_{n+2}x^{n+2} - (n-1)^m Q_{n+1}x^{n+1} - (n+1)^m Q_n x^{n+3} + \sum_{j=1}^{m}\big(x^3 - (-2)^j\big)\binom{m}{j}\Bigg(\sum_{r=1}^{n} r^{m-j} Q_r x^r\Bigg)\Bigg],$$

and

$$\sum_{r=1}^{n} r^m V_r x^r = \frac{1}{1-x^2-x^3}\Big[x^3 - n^m V_{n+2}x^{n+2} - (n-1)^m V_{n+1}x^{n+1} - (n+1)^m V_n x^{n+3} + \sum_{j=1}^{m}\left(x^3-(-2)^j\right)\binom{m}{j}\Big(\sum_{r=1}^{n} r^{m-j}V_r x^r\Big)\Big].$$

Proof. We have

$$\begin{aligned} S_n^{(m)}(x) &= 1^m H_1 x + 2^m H_2 x^2 + 3^m H_3 x^3 + \cdots + n^m H_n x^n \\ &= 1^m\Big(\sum_{j=1}^{n} H_j x^j - \sum_{j=2}^{n} H_j x^j\Big) + 2^m\Big(\sum_{j=2}^{n} H_j x^j - \sum_{j=3}^{n} H_j x^j\Big) \\ &\quad + 3^m\Big(\sum_{j=3}^{n} H_j x^j - \sum_{j=4}^{n} H_j x^j\Big) + \cdots + n^m\Big(\sum_{j=n}^{n} H_j x^j\Big). \end{aligned}$$

Now, by regrouping the terms, this becomes

$$\begin{aligned} S_n^{(m)}(x) &= 1^m \sum_{j=1}^{n} H_j x^j + (2^m - 1^m)\sum_{j=2}^{n} H_j x^j + \cdots + (n^m - (n-1)^m)\sum_{j=n}^{n} H_j x^j \\ &= \sum_{r=1}^{n}(r^m - (r-1)^m)\Big(\sum_{j=r}^{n} H_j x^j\Big). \end{aligned}$$

Next, we apply Theorem 2.1 to obtain

$$\begin{aligned} S_n^{(m)}(x) &= \frac{1}{1-x^2-x^3}\Big[\sum_{r=1}^{n}(r^m-(r-1)^m)x^r\left(H_r + xH_{r+1} + x^2H_{r-1}\right) \\ &\quad - x^{n+1}\left(H_{n+1} + xH_{n+2} + x^2H_n\right)\sum_{r=1}^{n}(r^m-(r-1)^m)\Big] \\ &= \frac{1}{1-x^2-x^3}\Big[\sum_{r=1}^{n}(r^m-(r-1)^m)\left(H_r + xH_{r+1} + x^2H_{r-1}\right)x^r \\ &\quad - n^m\left(H_{n+1} + xH_{n+2} + x^2H_n\right)x^{n+1}\Big]. \end{aligned}$$

Thus,

$$S_n^{(m)}(x) = \frac{1}{1-x^2-x^3}\Big[T_n^{(m)}(x) - n^m\left(H_{n+1} + xH_{n+2} + x^2H_n\right)x^{n+1}\Big], \tag{2.12}$$

where

$$\begin{aligned} T_n^{(m)}(x) &= \sum_{r=1}^{n}(r^m-(r-1)^m)\left(H_r + xH_{r+1} + x^2H_{r-1}\right)x^r \\ &= \sum_{r=1}^{n}(r^m-(r-1)^m)H_r x^r + \sum_{r=1}^{n}(r^m-(r-1)^m)H_{r+1}x^{r+1} \\ &\quad + \sum_{r=1}^{n}(r^m-(r-1)^m)H_{r-1}x^{r+2}. \end{aligned}$$

Applying a suitable change of variable in the second and third summations on the right-hand side, we get

$$\begin{aligned}T_n^{(m)}(x) &= \sum_{r=1}^{n}\big(r^m-(r-1)^m\big)H_r x^r+\sum_{r=2}^{n+1}\big((r-1)^m-(r-2)^m\big)H_r x^r\\&\quad+\sum_{r=0}^{n-1}\big((r+1)^m-r^m\big)H_r x^{r+3}\\&=\sum_{r=1}^{n}\big(r^m-(r-1)^m+(r-1)^m-(r-2)^m+x^3((r+1)^m-r^m)\big)H_r x^r\\&\quad+\big(n^m-(n-1)^m\big)H_{n+1}x^{n+1}+H_1(-1)^m x-\big((n+1)^m-n^m\big)H_n x^{n+3}+H_0x^3.\end{aligned}$$

This simplifies to

$$\begin{aligned}T_n^{(m)}(x) &= \sum_{r=1}^{n}\big(r^m-(r-2)^m+x^3\big((r+1)^m-r^m\big)\big)H_r x^r+\big(n^m-(n-1)^m\big)H_{n+1}x^{n+1}\\&\quad-\big((n+1)^m-n^m\big)H_n x^{n+3}+ax^3+(-1)^m bx.\end{aligned} \tag{2.13}$$

Using the binomial expansion, we have

$$r^m-(r-2)^m=-\sum_{j=1}^{m}\binom{m}{j}(-2)^j r^{m-j},$$

and

$$(r+1)^m-r^m=\sum_{j=1}^{m}\binom{m}{j}r^{m-j}.$$

Substituting this into (2.13) yields

$$\begin{aligned}T_n^{(m)}(x) &= \sum_{r=1}^{n}\sum_{j=1}^{m}\binom{m}{j}\big(x^3-(-2)^j\big)r^{m-j}H_r x^r+(n^m-(n-1)^m)H_{n+1}x^{n+1}\\&\quad-((n+1)^m-n^m)H_n x^{n+3}+ax^3+(-1)^m bx\\&=\sum_{j=1}^{m}\binom{m}{j}\big(x^3-(-2)^j\big)S_n^{(m-j)}(x)+(n^m-(n-1)^m)H_{n+1}x^{n+1}\\&\quad-((n+1)^m-n^m)H_n x^{n+3}+ax^3+(-1)^m bx.\end{aligned}$$

Substituting this into (2.12), we obtain

$$\begin{aligned}S_n^{(m)}(x) =&\frac{1}{1-x^2-x^3}\Bigg[ax^3+(-1)^m bx-(n+1)^m H_n x^{n+3}-(n-1)^m H_{n+1}x^{n+1}\\&-n^m H_{n+2}x^{n+2}+\sum_{j=1}^{m}\binom{m}{j}\big(x^3-(-2)^j\big)S_n^{(m-j)}(x)\Bigg],\end{aligned}$$

as desired. This completes the proof. □

Setting $x=1$ in Theorem 2.5, we obtain the following recurrence about the Brousseau weighted sums of the generalized Padovan numbers:

Corollary 2.6. *For all integers $n, m \geq 1$, the following identity holds:*

$$\begin{aligned}\sum_{r=1}^{n} r^m H_r =&(n+1)^m H_n+(n-1)^m H_{n+1}+n^m H_{n+2}-a-(-1)^m b\\&-\sum_{j=1}^{m}\big(1-(-2)^j\big)\binom{m}{j}\left(\sum_{r=1}^{n} r^{m-j}H_r\right).\end{aligned}$$

In particular,

$$\begin{aligned}\sum_{r=1}^{n} r^m P_r =&(n+1)^m P_n + (n-1)^m P_{n+1} + n^m P_{n+2} - 1 - (-1)^m \\ &- \sum_{j=1}^{m}(1-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n} r^{m-j}P_r\right),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\sum_{r=1}^{n} r^m Q_r =& (n+1)^m Q_n + (n-1)^m Q_{n+1} + n^m Q_{n+2} - 3 \\ &- \sum_{j=1}^{m}(1-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n} r^{m-j}Q_r\right),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\sum_{r=1}^{n} r^m V_r =&(n+1)^m V_n + (n-1)^m V_{n+1} + n^m V_{n+2} - 1 \\ &- \sum_{j=1}^{m}(1-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n} r^{m-j}V_r\right).\end{aligned} \tag{2.16}$$

Likewise, setting $x = -1$ in Theorem 2.5, we obtain the following recurrence about the alternating weighted sums of the generalized Padovan numbers:

Corollary 2.7. *For all integers $n, m \geq 1$, we have*

$$\begin{aligned}\sum_{r=1}^{n}(-1)^r r^m H_r =&(-1)^n\left[(n+1)^m H_n + (n-1)^m H_{n+1} - n^m H_{n+2}\right] - a - (-1)^m b \\ &- \sum_{j=1}^{m}(1+(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n}(-1)^r r^{m-j}H_r\right).\end{aligned}$$

In particular,

$$\begin{aligned}\sum_{r=1}^{n}(-1)^r r^m P_r =&(-1)^n\left[(n+1)^m P_n + (n-1)^m P_{n+1} - n^m P_{n+2}\right] - 1 - (-1)^m \\ &- \sum_{j=1}^{m}(1+(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n}(-1)^r r^{m-j}P_r\right),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\sum_{r=1}^{n}(-1)^r r^m Q_r =&(-1)^n\left[(n+1)^m Q_n + (n-1)^m Q_{n+1} - n^m Q_{n+2}\right] - 3 \\ &- \sum_{j=1}^{m}(1+(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n}(-1)^r r^{m-j}Q_r\right),\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\sum_{r=1}^{n}(-1)^r r^m V_r =&(-1)^n\left[(n+1)^m V_n + (n-1)^m V_{n+1} - n^m V_{n+2}\right] - 1 \\ &- \sum_{j=1}^{m}(1+(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n}(-1)^r r^{m-j}V_r\right).\end{aligned} \tag{2.19}$$

Example 2.8. If we put $m = 1$ into (2.14), we get

$$\sum_{r=1}^{n} rP_r = (n+1)P_n + (n-1)P_{n+1} + nP_{n+2} - 3\sum_{r=1}^{n} P_r.$$

Now using (2.4), this becomes

$$\sum_{r=1}^{n} rP_r = (n-2)P_n + (n-4)P_{n+1} + (n-3)P_{n+2} + 9. \tag{2.20}$$

By setting $m = 2$ in (2.14), we obtain

$$\sum_{r=1}^{n} r^2 P_r = (n+1)^2 P_n + (n-1)^2 P_{n+1} + n^2 P_{n+2} - 2 - \left(6\sum_{r=1}^{n} rP_r - 3\sum_{r=1}^{n} P_r\right).$$

Substituting (2.4) and (2.20) yields

$$\sum_{r=1}^{n} r^2 P_r = (n^2 - 4n + 16)P_n + (n^2 - 8n + 28)P_{n+1} + (n^2 - 6n + 21)P_{n+2} - 65.$$

Similarly, we have

$$\begin{aligned}\sum_{r=1}^{n} r^3 P_r =& (n^3 - 6n^2 + 48n - 170)P_n + (n^3 - 12n^2 + 84n - 298)P_{n+1} \\ &+ (n^3 - 9n^2 + 63n - 225)P_{n+2} + 693,\end{aligned}$$

and so on.
Setting $m = 1, 2, \ldots$ in (2.15) in succession and using (2.5), we get

$$\sum_{r=1}^{n} rQ_r = (n-2)Q_n + (n-4)Q_{n+1} + (n-3)Q_{n+2} + 12,$$

$$\sum_{r=1}^{n} r^2 Q_r = (n^2 - 4n + 16)Q_n + (n^2 - 8n + 28)Q_{n+1} + (n^2 - 6n + 21)Q_{n+2} - 90,$$

$$\begin{aligned}\sum_{r=1}^{n} r^3 Q_r =& (n^3 - 6n^2 + 48n - 170)Q_n + (n^3 - 12n^2 + 84n - 298)Q_{n+1} \\ &+ (n^3 - 9n^2 + 63n - 225)Q_{n+2} + 960,\end{aligned}$$

and so on.
Similarly, setting $m = 1, 2, \ldots$ in (2.16) in succession and using (2.6), we get

$$\sum_{r=1}^{n} rV_r = (n-2)V_n + (n-4)V_{n+1} + (n-3)V_{n+2} + 5,$$

$$\sum_{r=1}^{n} r^2 V_r = (n^2 - 4n + 16)V_n + (n^2 - 8n + 28)V_{n+1} + (n^2 - 6n + 21)V_{n+2} - 37,$$

$$\begin{aligned}\sum_{r=1}^{n} r^3 V_r =& (n^3 - 6n^2 + 48n - 170)V_n + (n^3 - 12n^2 + 84n - 298)V_{n+1} \\ &+ (n^3 - 9n^2 + 63n - 225)V_{n+2} + 395,\end{aligned}$$

and so on.

Example 2.9. When we put $m = 1$ in (2.17), we get

$$\sum_{r=1}^{n} (-1)^r rP_r = (-1)^n \left[(n+1)P_n + (n-1)P_{n+1} - nP_{n+2}\right] + \sum_{r=1}^{n} (-1)^r P_r.$$

Now using (2.8), this becomes

$$\sum_{r=1}^{n}(-1)^r rP_r = (-1)^n\left[(n+2)P_n + nP_{n+1} - (n+1)P_{n+2}\right] - 1. \tag{2.21}$$

If we let $m = 2$ in (2.17), we get

$$\sum_{r=1}^{n}(-1)^r r^2 P_r = (-1)^n\left[(n+1)^2P_n + (n-1)^2P_{n+1} - n^2P_{n+2}\right] - 2 + 2\sum_{r=1}^{n}(-1)^r rP_r - 5\sum_{r=1}^{n}(-1)^r P_r.$$

Applying the identities (2.8) and (2.21), this becomes

$$\sum_{r=1}^{n}(-1)^r r^2 P_r = (-1)^n\left[(n^2+4n)P_n + (n^2-4)P_{n+1} - (n^2+2n-3)P_{n+2}\right] + 1.$$

Similarly, we have

$$\sum_{r=1}^{n}(-1)^r r^3 P_r = (-1)^n\left[(n^3+6n^2-22)P_n + (n^3-12n-6)P_{n+1} - (n^3+3n^2-9n-17)P_{n+2}\right] + 11,$$

and so on.
If we put $m = 1, 2, \ldots$ in (2.18) in succession and use (2.9), we obtain

$$\sum_{r=1}^{n}(-1)^r rQ_r = (-1)^n\left[(n+2)Q_n + nQ_{n+1} - (n+1)Q_{n+2}\right] - 4,$$

$$\sum_{r=1}^{n}(-1)^r r^2 Q_r = (-1)^n\left[(n^2+4n)Q_n + (n^2-4)Q_{n+1} - (n^2+2n-3)Q_{n+2}\right] - 6,$$

$$\sum_{r=1}^{n}(-1)^r r^3 Q_r = (-1)^n\left[(n^3+6n^2-22)Q_n + (n^3-12n-6)Q_{n+1} - (n^3+3n^2-9n-17)Q_{n+2}\right] + 32,$$

and so on.
Similarly, if we put $m = 1, 2, \ldots$ in (2.19) in succession and use (2.10), we obtain

$$\sum_{r=1}^{n}(-1)^r rV_r = (-1)^n\left[(n+2)V_n + nV_{n+1} - (n+1)V_{n+2}\right] - 1,$$

$$\sum_{r=1}^{n}(-1)^r r^2 V_r = (-1)^n\left[(n^2+4n)V_n + (n^2-4)V_{n+1} - (n^2+2n-3)V_{n+2}\right] - 3,$$

$$\sum_{r=1}^{n}(-1)^r r^3 V_r = (-1)^n\left[(n^3+6n^2-22)V_n + (n^3-12n-6)V_{n+1} - (n^3+3n^2-9n-17)V_{n+2}\right] + 5,$$

and so on.

2.3 Sums with rising and falling powers

By substituting specific values for x, a wide range of new identities associated with Padovan, Perrin, and Van der Laan numbers can be discovered. Let's examine a few instances.

If we set $x = 2$ in (2.2), we obtain

$$11 \cdot \sum_{r=1}^{n} 2^r H_r = 2^{n+1}\big(4H_n + H_{n+1} + 2H_{n+2}\big) - 2\big(4a + b + 2c\big). \qquad \text{(rising powers of 2)}$$

In particular, we have the following identities:

$$11 \cdot \sum_{r=1}^{n} 2^r P_r = 2^{n+1}\big(4P_n + P_{n+1} + 2P_{n+2}\big) - 14,$$

$$11 \cdot \sum_{r=1}^{n} 2^r Q_r = 2^{n+1}\big(4Q_n + Q_{n+1} + 2Q_{n+2}\big) - 32,$$

$$11 \cdot \sum_{r=1}^{n} 2^r V_r = 2^{n+1}\big(4V_n + V_{n+1} + 2V_{n+2}\big) - 12.$$

Likewise, if we set $x = \frac{1}{2}$ in (2.2) and then multiply through by 2^n, we obtain

$$5 \cdot \sum_{r=1}^{n} 2^{n-r} H_r = (a + 4b + 2c) \cdot 2^n - \big(H_n + 4H_{n+1} + 2H_{n+2}\big), \qquad \text{(falling powers of 2)}$$

and hence

$$5 \cdot \sum_{r=1}^{n} 2^{n-r} P_r = 7 \cdot 2^n - \big(P_n + 4P_{n+1} + 2P_{n+2}\big),$$

$$5 \cdot \sum_{r=1}^{n} 2^{n-r} Q_r = 7 \cdot 2^n - \big(Q_n + 4Q_{n+1} + 2Q_{n+2}\big),$$

$$5 \cdot \sum_{r=1}^{n} 2^{n-r} V_r = 3 \cdot 2^n - \big(V_n + 4V_{n+1} + 2V_{n+2}\big).$$

3 Infinite Sums

For each non-negative integer m, consider the power series

$$\sum_{r=1}^{\infty} r^m H_r x^r, \tag{3.1}$$

where $\big(H_n\big)_{n\geq 0}$ is the generalized Padovan sequence. By applying the ratio test, it can be concluded that the power series (3.1) will converge (absolutely) for all non-negative integers m and for all x within the interval $\left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, where $\rho = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{23}{108}}}$. Note that $1 - x^2 - x^3 \neq 0$ for all $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$.

Theorem 3.1. *If $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, then we have*

$$\sum_{r=1}^{\infty} H_r x^r = \frac{x\big(ax^2 + cx + b\big)}{1 - x^2 - x^3}. \tag{3.2}$$

In particular,

$$\sum_{r=1}^{\infty} P_r x^r = \frac{x + x^2 + x^3}{1 - x^2 - x^3},$$

$$\sum_{r=1}^{\infty} Q_r x^r = \frac{2x^2+3x^3}{1-x^2-x^3},$$

and

$$\sum_{r=1}^{\infty} V_r x^r = \frac{x^2+x^3}{1-x^2-x^3}.$$

Proof. Since the power series on the left-hand side of (3.2) converges for all $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, it follows that $H_n x^n \to 0$ as $n \to \infty$. Now, the identity (3.2) follows by letting $n \to \infty$ in (2.2). □

Note that the generating function $f(x) = \dfrac{(c-a)x^2+bx+a}{1-x^2-x^3}$ (see [14, Lemma 2.1]) for the generalized Padovan numbers can be retrieved by adding $H_0 = a$ on both sides of (3.2).

Theorem 3.2. *If $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, then for all integers $m > 0$, we have*

$$\sum_{r=1}^{\infty} r^m H_r x^r = \frac{1}{1-x^2-x^3}\left[ax^3+(-1)^m bx+\sum_{j=1}^{m}(x^3-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{\infty} r^{m-j} H_r x^r\right)\right]. \quad (3.3)$$

In particular,

$$\sum_{r=1}^{\infty} r^m P_r x^r = \frac{1}{1-x^2-x^3}\left[x^3+(-1)^m x+\sum_{j=1}^{m}(x^3-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{\infty} r^{m-j} P_r x^r\right)\right],$$

$$\sum_{r=1}^{\infty} r^m Q_r x^r = \frac{1}{1-x^2-x^3}\left[3x^3+\sum_{j=1}^{m}(x^3-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{\infty} r^{m-j} Q_r x^r\right)\right],$$

and

$$\sum_{r=1}^{\infty} r^m V_r x^r = \frac{1}{1-x^2-x^3}\left[x^3+\sum_{j=1}^{m}(x^3-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{\infty} r^{m-j} V_r x^r\right)\right].$$

Proof. From Theorem 2.5, we have

$$\sum_{r=1}^{n} r^m H_r x^r = \frac{1}{1-x^2-x^3}\left[ax^3+(-1)^m bx - n^m H_{n+2}x^{n+2} - (n-1)^m H_{n+1}x^{n+1} \right.$$
$$\left. - (n+1)^m H_n x^{n+3} + \sum_{j=1}^{m}(x^3-(-2)^j)\binom{m}{j}\left(\sum_{r=1}^{n} r^{m-j} H_r x^r\right)\right]. \quad (3.4)$$

We will prove (3.3) by letting $n \to \infty$ in (3.4). Since the power series on the left-hand side of (3.3) converges for all integers $m \geq 0$ and $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, it follows that $n^m H_n x^n \to 0$ as $n \to \infty$. Then

$$\begin{aligned} n^m H_{n+2}x^{n+2} &= \big((n+2)-2\big)^m H_{n+2}x^{n+2} \\ &= \sum_{j=0}^{m}\binom{m}{j}(-2)^j (n+2)^{m-j} H_{n+2}x^{n+2} \to 0, \end{aligned}$$

as $n \to \infty$. Likewise,

$$(n-1)^m H_{n+1}x^{n+1} \to 0, \quad \text{and} \quad (n+1)^m H_n x^{n+3} \to 0,$$

as $n \to \infty$. Therefore, the identity (3.3) follows by letting $n \to \infty$ in (3.4). □

Corollary 3.3. *If $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, then*

$$\sum_{r=1}^{\infty} rH_r x^r = \frac{bx + 2cx^2 + (3a+b)x^3 + 2bx^4 + (c-a)x^5}{(1-x^2-x^3)^2}.$$

In particular,

$$\sum_{r=1}^{\infty} rP_r x^r = \frac{x + 2x^2 + 4x^3 + 2x^4}{(1-x^2-x^3)^2},$$

$$\sum_{r=1}^{\infty} rQ_r x^r = \frac{4x^2 + 9x^3 - x^5}{(1-x^2-x^3)^2},$$

and

$$\sum_{r=1}^{\infty} rV_r x^r = \frac{2x^2 + 3x^3}{(1-x^2-x^3)^2}.$$

Proof. Setting $m = 1$ in (3.3) and using (3.2), we get

$$\begin{aligned}\sum_{r=1}^{\infty} rH_r x^r &= \frac{1}{1-x^2-x^3}\left[(x^3+2)\left(\sum_{r=1}^{\infty} H_r x^r\right) + ax^3 - bx\right] \\ &= \frac{1}{1-x^2-x^3}\left[(x^3+2)\left(\frac{ax^3+cx^2+bx}{1-x^2-x^3}\right) + ax^3 - bx\right] \\ &= \frac{bx + 2cx^2 + (3a+b)x^3 + 2bx^4 + (c-a)x^5}{(1-x^2-x^3)^2}.\end{aligned}$$

This completes the proof. □

3.1 An algorithm for finding the infinite sums

For all integers $m \geq 0$, we define

$$S_P^{(m)}(x) = \sum_{r=1}^{\infty} r^m P_r x^r,$$

$$S_Q^{(m)}(x) = \sum_{r=1}^{\infty} r^m Q_r x^r,$$

and

$$S_V^{(m)}(x) = \sum_{r=1}^{\infty} r^m V_r x^r.$$

Then, by Theorem 3.1, we obtain the following algorithms to find the sums $S_P^{(m)}(x)$, $S_Q^{(m)}(x)$, and $S_V^{(m)}(x)$ for all $x \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$:

$$\begin{cases} S_P^{(0)}(x) = \dfrac{x+x^2+x^3}{1-x^2-x^3}, \\ S_P^{(i)}(x) = \dfrac{1}{1-x^2-x^3}\left[x^3 + (-1)^i x + \displaystyle\sum_{j=1}^{i} \binom{i}{j} \left(x^3 - (-2)^j\right) S_P^{(i-j)}(x)\right], i = 1, 2, \ldots, m; \end{cases}$$

$$\begin{cases} S_Q^{(0)}(x) = \dfrac{2x^2+3x^3}{1-x^2-x^3}, \\ S_Q^{(i)}(x) = \dfrac{1}{1-x^2-x^3}\left[3x^3 + \displaystyle\sum_{j=1}^{i} \binom{i}{j} \left(x^3 - (-2)^j\right) S_Q^{(i-j)}(x)\right], i = 1, 2, \ldots, m; \end{cases}$$

and

$$\begin{cases} S_V^{(0)}(x) = \dfrac{x^2+x^3}{1-x^2-x^3}, \\ S_V^{(i)}(x) = \dfrac{1}{1-x^2-x^3}\left[x^3 + \sum_{j=1}^{i}\binom{i}{j}\left(x^3-(-2)^j\right)S_V^{(i-j)}(x)\right], i = 1,2,\ldots,m. \end{cases}$$

For example, when $x = \frac{1}{2} \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, we obtain

$$\sum_{r=1}^{\infty}\frac{rP_r}{2^r} = \frac{104}{25}, \sum_{r=1}^{\infty}\frac{rQ_r}{2^r} = \frac{134}{25}, \sum_{r=1}^{\infty}\frac{rV_r}{2^r} = \frac{56}{25};$$

$$\sum_{r=1}^{\infty}\frac{r^2P_r}{2^r} = \frac{2576}{125}, \sum_{r=1}^{\infty}\frac{r^2Q_r}{2^r} = \frac{3546}{125}, \sum_{r=1}^{\infty}\frac{r^2V_r}{2^r} = \frac{1464}{125};$$

$$\sum_{r=1}^{\infty}\frac{r^3P_r}{2^r} = \frac{94016}{625}, \sum_{r=1}^{\infty}\frac{r^3Q_r}{2^r} = \frac{130286}{625}, \sum_{r=1}^{\infty}\frac{r^3V_r}{2^r} = \frac{53624}{625};$$

$$\sum_{r=1}^{\infty}\frac{r^4P_r}{2^r} = \frac{4565408}{3125}, \sum_{r=1}^{\infty}\frac{r^4Q_r}{2^r} = \frac{6323418}{3125}, \sum_{r=1}^{\infty}\frac{r^4V_r}{2^r} = \frac{2601912}{3125};$$

and so on. Similarly, if $x = -\frac{1}{2} \in \left(\frac{-1}{\rho}, \frac{1}{\rho}\right)$, then we have

$$\sum_{r=1}^{\infty}\frac{(-1)^r rP_r}{2^r} = -\frac{24}{49}, \sum_{r=1}^{\infty}\frac{(-1)^r rQ_r}{2^r} = -\frac{6}{49}, \sum_{r=1}^{\infty}\frac{(-1)^r rV_r}{2^r} = \frac{8}{49};$$

$$\sum_{r=1}^{\infty}\frac{(-1)^r r^2P_r}{2^r} = -\frac{272}{343}, \sum_{r=1}^{\infty}\frac{(-1)^r r^2Q_r}{2^r} = -\frac{558}{343}, \sum_{r=1}^{\infty}\frac{(-1)^r r^2V_r}{2^r} = -\frac{40}{343};$$

$$\sum_{r=1}^{\infty}\frac{(-1)^r r^3P_r}{2^r} = -\frac{3840}{2401}, \sum_{r=1}^{\infty}\frac{(-1)^r r^3Q_r}{2^r} = -\frac{18894}{2401}, \sum_{r=1}^{\infty}\frac{(-1)^r r^3V_r}{2^r} = -\frac{4600}{2401};$$

$$\sum_{r=1}^{\infty}\frac{(-1)^r r^4P_r}{2^r} = -\frac{29024}{16807}, \sum_{r=1}^{\infty}\frac{(-1)^r r^4Q_r}{2^r} = -\frac{485790}{16807}, \sum_{r=1}^{\infty}\frac{(-1)^r r^4V_r}{2^r} = -\frac{168424}{16807};$$

and so on.

4 Conclusion

In this article, we presented a new recurrence formula for the finite sum $\sum_{r=1}^{n} r^m H_r x^r$ associated with the generalized Padovan sequence, for all integers $m, n \geq 1$, provided $1 - x^2 - x^3 \neq 0$. This formula yields numerous identities for sums involving the Padovan, Perrin, Van der Laan, and many more sequences. Furthermore, it provides a method for calculating the sum of the Padovan-type power series. The approach used to obtain these findings can also be applied to similar series. In future work, we intend to find an explicit polynomial expression for the sum $\sum_{r=1}^{n} r^m H_r$ for all integers $m \geq 0$.

References

[1] K. Adegoke, R. Frontczak, and T. Goy, Some special sums with squared Horadam numbers and generalized Tribonacci numbers, *Palest. J. Math.* **11**:1 (2022), 61–73.

[2] K. Adegoke, Summation identities involving Padovan and Perrin numbers, *Palest. J. Math.* **11**:1 (2022), 633–650.

[3] C. Ballantine and M. Merca, Padovan numbers as sums over partitions into odd parts, *J. Inequal. Appl.*, **2016** (2016), Article 1.

[4] A. J. V. Brandão and J. L. Martins, Recurrence formulas for Fibonacci sums, arXiv preprint arXiv:0805.2707 (2008). Available at https://arxiv.org/abs/0805.2707.

[5] A. Brousseau, Problem H-17, *Fibonacci Quart*. **1**:2 (1963), 55.

[6] A. Brousseau, Summation of $\sum_{k=1}^{n} k^m F_{k+r}$ finite difference approach, *Fibonacci Quart.,* **5**:1 (1967), 91–98.

[7] O. Deveci and E. Karaduman, On the Padovan p-numbers, *Hacet. J. Math. Stat.* **46**:4 (2017), 579–592.

[8] O. Dişkaya and H. Menken, On the weighted Padovan and Perrin sums, *J. Sci. Arts* **23**:4 (2023), 817–826.

[9] K. Kaygisiz and A. Şahin, Calculating terms of associated polynomials of Perrin and Cordonnier numbers, *Notes Numb. Theory Disc. Math.* **20**:1 (2014), 10–18.

[10] S. H. J. Petroudi, M. Pirouz and M. Shadab, A study of Pell-Narayana sequence: Properties and circulant matrix aspects, *Palest. J. Math.* **12**:3 (2023), 103–114.

[11] A. Şahin, On the generalized Perrin and Cordonnier matrices, *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.* **66**:1 (2017), 242–253.

[12] A. G. Shannon, P. G. Anderson, and A. F. Horadam, Properties of Cordonnier, Perrin and Van der Laan numbers, *Int. J. Math. Educ. Sci. Technol.* **37**:7 (2006), 825–831.

[13] A. G. Shannon, A. F. Horadam, and P. G. Anderson, The auxiliary equation associated with the plastic number, *Notes Numb. Theory Disc. Math.* **12**:1 (2006), 1–12.

[14] Y. Soykan, On generalized Padovan numbers, *Int. J. Adv. Appl. Math. and Mech.* **10**:4 (2023), 72–90.

[15] F. Yilmaz and D. Bozkurt, Some properties of Padovan sequence by matrix methods, *Ars Combin.* **104** (2012), 149–160.

[16] N. Yilmaz and N. Taskara, Matrix sequences in terms of Padovan and Perrin numbers, *J. Appl. Math.* **2013** (2013), Article ID 941673.

Author information

Prabha Sivaraman Nair, Department of Mathematics, Baby John Memorial Govt. College, Chavara, Kollam, Kerala, India.
E-mail: prabhamaths@gmail.com

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