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POLAR IDEALS OF A PSEUDO RING

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Abstract. In this paper, we introduce the notion of a polar ideal of a pseudo ring and investigate its related properties. We prove that the polar of an atom forms a maximal ideal. And also, we establish a one-to-one correspondence between the set of atoms and the set of all maximal ideals of an atomic pseudo ring. Further, we prove that every pseudo ring R is isomorphic to the direct product of quotient pseudo rings R/I and $R/I^{\perp}$ for a suitable ideal I.

1 Introduction

It is MH-Stone [7] who has shown that the class of Boolean algebras and the class of Boolean rings with unity are equivalent. The notion of Boolean rings has been generalized in many ways by many authors. p-rings by MeCoy,[5] Regular rings by Neumann,[8] BZS rings by M. Farag [3] and pre-p-rings by A. Yaqub [9] are a few ring theoretic generalizations. Chajda and L$\ddot{a}$nger in [2] introduced the notion of pseudo rings, which are generalizations of Boolean rings. It is clear that a pseudo ring is not even a ring; it is a generalization of a Boolean ring. The notion of ideals is introduced and studied in pseudo rings by the authors in [6]. For some related study see [4], [1]. In this paper, we introduce polar ideals and study their properties. We obtain the relations between polar ideals and maximal ideals. We prove that the polar of an atom is a maximal ideal. Also, we establish that there is a one-to-one correspondence between the set of all atoms and the set of maximal ideals of the atomic pseudo ring.Finally we show that every pseudo ring can be expressed as the product of quotient pseudo rings.

2 Preliminaries

Definition 2.1. [2, Definition 3.1] A pseudo ring is an algebra $R = (R, +, \cdot, 1)$ of type $(2, 2, 0)$ satisfying the following axioms:

P_1. $(xy)z = x(yz)$;

P_2. $xy = yx$;

P_3. $x1 = x$;

P_4. $1 + (1 + x) = x$;

P_5. $x0 = 0$;

P_6. $(1 + x(1 + y))(1 + y) = (1 + y(1 + x))(1 + x)$;

P_7. $1 + (1 + x(1 + y))(1 + y(1 + x)) = x + y$;

where 0 denotes the element 1 + 1.

Remark 2.2. Commutative of $'+'$ follows from (P_2) and (P_7).

Definition 2.3. [2] Define $x \leq y$ for any two elements $x, y \in R$ if and only if x and y satisfy the condition $(y + 1)x = 0$.

Proposition 2.4. *[2] The following proprieties follow directly from the definition.*

N_1. $x(x+1)=0, \quad \forall x \in R.$

N_2. $y(1+0)=y, \ \forall y \in R$ *and* $1+0=1$.

N_3. $x+0=x.$

N_4. *Char* $R=2$.

Remark 2.5. From P_6 of definition 2.1 it can be concluded that $x(1+(y+1)x)=y(1+(x+1)y)$.

Definition 2.6. [6] Let $I \subseteq R$. I become an ideal of R if and only if the following holds.

$$(i) \quad 0 \in I.$$
$$(ii) \quad 1+(x+1)(y+1) \in I \ \ for \ every \ \ x,y \in I.$$
$$(iii) \ \ For \ any \ \ x \in R \ and \ y \in I, x \leq y \Rightarrow x \in I).$$

Proposition 2.7. *[6] I is an ideal of R if and only if the following holds:*

$$(i) \quad 0 \in I,$$
$$(ii) \quad 1+(x+1)(y+1) \in I \ \ for \ every \ \ x,y \in I,$$
$$(iii) \ \ (y+1)x, y \in I \Rightarrow \ x \in I.$$

Definition 2.8. [6] Let P be a proper ideal of R. If P is prime then for all $x, y \in R$ either t $x(y+1) \in P$ or $y(x+1) \in P$.

Definition 2.9. [6] A proper ideal M of R is called maximal if and only if $x \in M$ or $x+1 \in M$ but not both for every $x \in R$

Definition 2.10. [6] Let R_1 and R_2 be two pseudo rings. A pseudo ring homomorphism is a mapping $\varphi : R_1 \longrightarrow R_2$ that meets the following conditions:

$$1) \quad \varphi(1)=1,$$
$$2) \quad \varphi(x \cdot y)=\varphi(x)\cdot\varphi(y),$$
$$3) \quad \varphi(x+y)=\varphi(x)+\varphi(y) \quad for \ every \quad x,y \in R_1.$$

Theorem 2.11. *[6] Let $\varphi : R_1 \to R_2$ is a pseudo ring epimorphism. If I is an ideal of R_1 and $ker(\varphi) \subseteq I$, then $\varphi(I)$ is an ideal of R_2.*

3 Polar ideal of a pseudo ring

We begin with the following

Proposition 3.1. *If R is a pseudo ring, then for all $x, \ y, \ z \in R$ the following properties hold:*

i. $x \leq y$ *if and only if* $1+y \leq x+1$;

ii. *if* $x \leq y \Rightarrow 1+(x+1)(z+1) \leq 1+(y+1)(z+1)$;

iii. *if* $x \leq y$, *then* $xz \leq yz$.

Proof. i. Suppose $x \leq y$, then $(y+1)x=0$. From P_4 it follows that $((x+1)+1)(1+y)=0$. Hence $1+y \leq 1+x$. Conversely, suppose $1+y \leq 1+x$, then $0=((x+1)+1)(1+y)=x(1+y)$. Thus $x \leq y$.

ii. Let $x \leq y$. Take $z=y(1+x)$. Then $y=1+(x+1)(1+z)$ by P_4 and P_6. This implies $x \leq 1+(x+1)(z+1) \Rightarrow y \leq 1+(y+1)(z+1)$ for any $z \in R$. Hence $1+(x+1)(z+1) \leq 1+(y+1)(z+1)$.

iii. Let $x, y, z \in R$, with $x \leq y$. From proposition 3.1 (i), we have $x \leq y \Rightarrow y+1 \leq x+1$, applying the result in (ii) to this result, we obtain $1+((y+1)+1)((z+1)+1) \leq 1+((x+1)+1)((z+1)+1)$, $\ p_4 \ \Rightarrow 1+yz \leq 1+xz \quad \Rightarrow xz \leq yz, \quad \forall z \in R.$

∎

Definition 3.2. A non zero element h of R is called an atom if $(h+1)x = 0$, then either $x = 0 \ \ or \ \ x = h$.

Remark 3.3. For any atom h of R, $h^n = h$ for all $n \in \mathbb{Z}^+$.

Proposition 3.4. *Let h be an atom of R, then* $h \leq x$ *if and only if* $h = hx$.

Proof. Suppose $h \leq x$. Then by Remark 3.3 and Proposition 3.1(iii), $h^2 \leq hx \Rightarrow h \leq hx$. Hence $h = hx$. Conversely suppose $h = hx$. By definition 2.3 it follows that $h = hx \leq x$. □

Proposition 3.5. *For any atoms* $h, h_1, h_2 \in R$*, the following are true:*

i) If x *be any non-zero element of R, either* $h_1 \leq x$ *or* $h_1 \leq x + 1$*;*

ii) Either $h_1 = h_2$ *or* $h_1(1 + (h_2 + 1)h_1) = 0$;

iii) For any $a, b \in R$ *with* $h_1 \leq 1 + (a(b + 1) + 1)(b + 1)$, *either* $h_1 \leq a$ *or* $h_1 \leq b$;

iv) Let $h \leq 1 + (h_1 + 1)(1 + h_2(h_1 + 1))$*, then* $h = h_1$ *or* $h = h_2$.

Proof. i) Consider $(1 + (x + 1))h_1 = xh_1$. Clearly by Definition 2.3, $xh_1 \leq h_1$, by the definition of an atom it follows that $xh_1 = 0 \ \ or \ \ xh_1 = h_1$. If $xh_1 = 0$, then $h_1 \leq x + 1$. If $xh_1 = h_1$, then by Proposition 3.4, $h_1 \leq x$. If $h_1 \leq x$ and $h_1 \leq x + 1$ at the same time,then by Proposition 3.4 and Definition 2.3, $h_1 = h_1 x = 0$. This contradicts the fact that h_1 is an atom. Hence either $h_1 \leq x$ or $h_1 \leq x + 1$ not both.

ii) Assume $h_1 \neq h_2$. Clearly $h_1(1+(h_2+1)h_1) \leq h_1$. Since h_1, h_2 are atoms, $h_1(1+(h_2+1)h_1) = 0$ or $h_1 = h_1(1 + (1 + h_2)h_1)$. If $h_1 = h_1(1 + (1 + h_2)h_1)$ from Remark 2.5, $h_1(1 + h_1(h_2 + 1)) = h_2(1+h_2(h_1+1)) = h_2$. This contradicts the fact that $h_1 \neq h_2$. Thus $h_1(1+(h_2+1)h_1) = 0$.

iii) Let $h_1 \leq 1 + (a(b+1)+1)(b+1)$. If $b = 0$, then $h_1 \leq 1 + (a+1) = a$. Assume $b \neq 0$. It is clear that $b(1 + (h_1 + 1)b) \leq h_1$, Since h_1 is an atom $b(1 + (h_1 + 1)b) = 0 \ \ or \ \ b(1 + (h_1 + 1)b) = h_1$. If $b(1 + (h_1 + 1)b) = 0$, then by Remark 2.5, Definition 2.3 and Proposition 3.4 respectively, $h_1 \leq b + 1$ by Proposition 3.1 it follows that, $h_1(a(b + 1) + 1) \leq (a(b + 1) + 1)(b + 1)$. Hence $h_1 \leq b + 1 \Rightarrow h_1(a(b + 1) + 1) \leq (a(b + 1) + 1)(b + 1) \Rightarrow h_1 \leq 1 + (a(b + 1) + 1)(b + 1) \leq 1 + h_1(a(b + 1) + 1) \Rightarrow h_1^2(a(b + 1) + 1) = 0 \Rightarrow h_1 \leq a(b + 1) \leq a$. As a result $h_1 \leq a$ and $h_1 \leq b + 1$. If $b(1 + (h_1 + 1)b) = h_1$, then $h_1 \leq b$. The same is true if $a = 0$ and $a \neq 0$. If both $a \neq 0$ and $b \neq 0$ at the same time, by (i) either $h_1 \leq a$ or $h_1 \leq a + 1$ and $h_1 \leq b$ or $h_1 \leq b + 1$. We need to show the case when $h_1 \leq a + 1$ and $h_1 \leq b + 1$. If $h_1 \leq a + 1$, then $h_1 a = 0$. Similarly $h_1 b = 0$. Since $h_1 \leq 1+(a(b+1)+1)(b+1) \Rightarrow h_1(a(b+1)+1)(b+1) = 0 \Rightarrow h_1(b+1) \leq a(b+1)$. By Proposition 3.4 and Proposition 3.1 (iii) $h_1(b+1) \leq h_1 a(b+1) = 0(b+1) = 0 \Rightarrow h_1 \leq b$ and similarly $h_1 \leq a$. This contradicts the fact in (i). Thus $h_1 \leq a + 1$ and $h_1 \leq b + 1$ does not hold at the same time.

iv) Let $h \leq 1 + (h_1 + 1)(1 + h_2(h_1 + 1))$. By Definition 3.2 and (iii), it follows that either $h = h_1$ or $h = h_2$.

□

Definition 3.6. Let X be a non-empty subset of R. Then the set $X^{\perp} = \{y \in R : \ y(1 + (x + 1)y) = 0, \ \forall x \in X\}$ is called a polar of X in R.

Theorem 3.7. *Let X be a non-empty subset of R.* $X^{\perp}$ *is an ideal of R.*

Proof. i. Clearly $0 \in X^{\perp}$.

ii. Let $a, \ b \in X^{\perp}$. By Remark 2.5 and definition of polar, it follows that $x(1 + (a + 1)x) = a(1 + (x + 1)a) = 0$. By Definition 2.3, $x = (a+1)x$. Consider $(1 + (a+1)(b+1))[1 + (x+1)(1 + (a+1)(b+1))] = x(1+x(a+1)(b+1))$ by Remark 2.5. It follows that $x(1+x(a+1)(b+1)) = x(1 + x(b + 1)) = 0$. Hence $1 + (a + 1)(b + 1) \in X^{\perp}$.

iii. Let $a \in R$ and $b \in X^{\perp}$ with $a \leq b$. By Proposition 3.1 (i) and (iii) $x(b+1) \leq x(a+1)$. By similar proposition and steps $x(1 + (a + 1)x) \leq x(1 + (b + 1)x)$. By Remark 2.5 $a(1 + (x + 1)a) = 0$. Hence $a \in X^{\perp}$.

□

Theorem 3.8. *If* $X \subseteq R$*, then the following are true.*

i. If $X \cap X^{\perp} \neq \emptyset$, then $X \cap X^{\perp} = \{0\}$.

ii. $X \subseteq X^{\perp\perp}$.

Proof. i. Let $X \cap X^{\perp} \neq \emptyset$. Then there exists $x \in X \ and \ x \in X^{\perp}$ such that $x = x(1+(x+1)x) = 0 \Rightarrow X \cap X^{\perp} = \{0\}$.

ii. Let $x \in X$. It follows that for all $a \in X^{\perp}$, by Remark 2.5, $a(1+(x+1)a) = x(1+(a+1)x) = 0$. Thus $x \in X^{\perp\perp}$.

□

Proposition 3.9. *For every non-zero ideal I of R, $I^{\perp}$ is a proper subset of R.*

Proof. Assume $I^{\perp}$ is not a proper subset of R. It follows that $1 \in I^{\perp} \Rightarrow a = 1 \cdot (1 + (a+1) \cdot 1) = 0$, for all $a \in I$. Implies $a = 0, \ \forall a \in I \Rightarrow I = \{0\}$. This contradicts the fact that $I \neq \{0\}$. Thus $I^{\perp}$ is proper subset of R. □

Proposition 3.10. *Let P be a non-zero prime ideal of R. If $P^{\perp} \neq \{0\}$, then $P = P^{\perp\perp}$.*

Proof. Let $x_0 \in P^{\perp\perp}$. It follows that $a(1+(x_0+1)a) = 0, \ \forall a \in P^{\perp}$. Since $P^{\perp} \neq \{0\}$, in particular let $a \neq 0$. As P is prime, either $x_0(a+1) \in P \ or \ a(x_0+1) \in P$. If $a(x_0+1) \in P$, since $a(1+(x_0+1)a) = 0$, $a = a(1 + (a(x_0+1)+1)a) = 0$ which contradicts the fact that $a \neq 0$. Hence $x_0(a+1) \in P$. If $x_0(a+1) \in P$, then $a(1+(x_0+1)a) = 0 \in P$. This implies that by Remark 2.5 and Proposition 2.7, $x_0 \in P$. Thus by Theorem 3.8 (ii) $P = P^{\perp\perp}$. □

Proposition 3.11. *Let I be an ideal of R. If $I^{\perp}$ is a prime ideal, then I is totally ordered.*

Proof. Let $x, y \in I$. Since $I^{\perp}$ is prime, either $x(y+1) \in I^{\perp} \ \ or \ \ y(x+1) \in I^{\perp}$. Since I is an ideal, if $x(y+1) \in I^{\perp}$, then by Theorem 3.8 (i), $x(y+1) = 0 \Rightarrow x \leq y$. Similarly, if $y(x+1) \in I^{\perp}$, then $y \leq x$. □

Lemma 3.12. $x \in h^{\perp} \Leftrightarrow x = (h+1)x \Leftrightarrow h = (x+1)h$.

Proof. By Definition 3.6, and Definition 2.3, it follows that $x = (h+1)x$. Similarly in addition using Remark 2.5, we have $x \in h^{\perp} \Leftrightarrow h = (x+1)h$. □

Theorem 3.13. *The polar of any atom h in R, denoted by the symbol $h^{\perp}$, is a maximal ideal of R.*

Proof. Follows from Theorem 3.7, Proposition3.5(i) and Definition 2.9. □

R is called an atomic pseudo ring if, for every non-zero element $x \in R$, there is an atom $h \in R$ such that $h \leq x$.

Theorem 3.14. *An atomic pseudo ring R contains an atom $h \notin M$ such that $h^{\perp} = M$ for any maximal ideal M of R.*

Proof. Let M be a maximal ideal of R with $x \notin M$. It follows that $x \neq 0$. By Proposition 3.5(i), for each atom $h \in R$ either $h \leq x$ or $h \leq x+1$. For the case $h \leq x+1$, it follows that $hx = 0$. Thus x is an atom different from h or there is an atom $h' \neq h$ such that $h' \leq x$. If x is an atom, let $y \in x^{\perp}$. Then $y(1+(x+1)y) = 0 \in M$. Since M is maximal, by Definition 2.3, Definition 2.9 and Proposition 2.7, $y(x+1) \leq x+1 \in M$ it follows $y \in M$. Hence by Corollary 3.13, we have $x^{\perp} = M$.
If x is not an atom, then there is an atom $h' \in R$ such that $h' \leq x$. Thus the following case will demonstrate it. For the case $h \leq x$, let $y \in h^{\perp}$. It follows $y(1+(h+1)y) = 0 \in M$. Since maximal ideal is prime, either $y(h+1) \in M$ or $h(y+1) \in M$. Thus by Remark 2.5 and Proposition 2.7, either $y \in M$ or $h \in M$. Since $h \notin M$, it follows $y \in M$. Hence $h^{\perp} = M$. □

Corollary 3.15. *Let At(R) be a set of all atoms of an atomic R and $Id_M(R)$ be a set of all maximal ideals of atomic pseudo ring R. Then there is a one-to-one correspondence between At(R) and $Id_M(R)$.*

Note. The intersection of any family of ideals of R is an ideal. Let H be a subset of R. Then the intersection of all ideals $I \supseteq H$ is the smallest ideal containing H and is denoted by $\prec H \succ$.

Lemma 3.16. *Let the map $\varphi : R_1 \to R_2$ is an epimorphism, then the following properties holds.*

a. If h is an atom of R_1, then $[h]$ is an atom of $R_1/ker(\varphi)$. Where $[h]$ is the equivalence class determined by h with respect to $ker(\varphi)$.

b. *If [h] is an atom of $R_1/ker(\varphi)$, then $\varphi(h)$ is also an atom of R_2.*

c. *If M is the maximal ideal of R_1 and $ker(\varphi) \subseteq M$, then $\varphi(M)$ is the maximal ideal of R_2.*

Proof. Let the map $\varphi : R_1 \to R_2$ be an epimorphism.

a. Suppose h is an atom of R_1. Let $([h]+[1])[t] = [0]$ with $[h] \neq [t]$. As a result we have $\varphi((h+1)t) = 0 \Rightarrow (h+1)t \in ker(\varphi)$. If $(x+1)t = 0 \in ker(\varphi)$, then either $x \neq t$ *or* $x = t$. If $x \neq t$, then $t = 0$. This implies $[t] = [0]$. If $x = t \Rightarrow [x] = [t]$ contradiction.Therefore $[x]$ is an atom of $R_1/ker(\varphi)$.

b. Clearly $f : R_1/ker(\varphi) \to R_2$ defined by $f([x]) = \varphi(x)$ is an isomorphism. Hence $[x]$ an atom of $R_1/ker(\varphi) \Rightarrow \varphi(x)$ is an atom of R_2.

c. Let M be maximal ideal of R_1 and $ker(\varphi) \subseteq M$. By Theorem 2.11 $\varphi(M)$ is an ideal of R. Assume $\varphi(M)$ is not maximal ideal of R_2. Then there is J ideal of R_2 such that $\varphi(M) \subsetneq J \subseteq R_2 \Rightarrow \exists y \in J$ and $y \notin \varphi(M)$. Since y is in R_2 and φ is an onto map, there is t in R_1 such that $\varphi(t) = y$ and t is not in M. Since M is maximal ideal of $R_1 \Rightarrow \prec M \cup \{t\} \succ = R_1 \Rightarrow 1 \in \prec M \cup \{t\} \succ \Rightarrow 1 \in \varphi(\prec M \cup \{t\} \succ) \Rightarrow 1 \in J \Rightarrow J = R_2$.

□

Theorem 3.17. *The polar of the homomorphic image of an atom is a maximal ideal if it contains its kernel.*

Proof. Follows Theorem 3.13 and lemma 3.16. □

Corollary 3.18. *If $\varphi : R_1 \to R_2$ be onto homomorphism with $ker\ \varphi \subseteq h^{\perp}$ for an atom $h \in R_1$, then $\varphi(h^{\perp}) = (\varphi(h))^{\perp}$.*

Definition 3.19. Two ideals I and J of R are said to be adjacent if $I \cap J^{\perp} \neq \{0\}$ *and* $J \cap I^{\perp} \neq \{0\}$.

Proposition 3.20. *I and J are adjacent ideals of R if and only if $I^{\perp} \neq J^{\perp}$.*

Proof. Suppose I and J are adjacent ideals of R. Thus $I \cap J^{\perp} \neq \{0\}$ and $J \cap I^{\perp} \neq \{0\}$. Let $a \in I \cap J^{\perp} \Rightarrow a \in I$ and $a \in J^{\perp}$. Hence $a(1 + (x+1)a) = 0$ for all $x \in J$. In particular if $a \in J$, then it follow from Theorem 3.8 $a = 0$. This contradicts the fact that $I \cap J^{\perp} \neq \{0\}$. Conversely suppose $I^{\perp} \neq J^{\perp}$. Assume I and J are not adjacent. This implies $I \cap J^{\perp} = \{0\}$ *and* $J \cap I^{\perp} = \{0\}$. Let $x \in I^{\perp}$. If $x \notin J^{\perp}$, then $x(1 + (a+1)x) = 0$ for all $a \in I$ and $x(1 + (y+1)x) \neq 0$ for some $y \in J$. Since J and $I^{\perp}$ are ideals $y(1 + (x+1)y) \in J$ and $x(1 + (y+1)x) \in I^{\perp}$. It follows that $x(1+(y+1)x) \in J \cap I^{\perp} = \{0\} \Rightarrow x(1+(y+1)x) = 0$. This contradicts the fact that $x(1+(y+1)x) \neq 0$. Thus I and J are adjacent. □

Similar results also hold in the annihilator ideal graph of a lattice for example see [4].

Theorem 3.21. *Every polar of a non-empty subset of R is a metric ideal.*

Proof. Let X be a non-empty subset of R and $a \in X^{\perp}$ with $a \circledast 0 \leq y \circledast 0$, for some $y \in R$. This implies $a + 1 \leq y + 1 \Leftrightarrow x(1 + (y+1)x) \leq x(1 + (a+1)x),\ \ for\ all\ \ x \in X$. It follows from Remark 2.5 and Definition 3.6, $y(1 + (x+1)y) = 0$ for all $x \in X$. Thus $y \in X^{\perp}$. Therefore by Definition 4.1 [6]), $X^{\perp}$ is a metric ideal. □

Proposition 3.22. *Let I and J be a metric ideal of R. Then the following holds:*

i. $(I \cup J)^{\perp} = I^{\perp} \cap J^{\perp}$;

ii. $I^{\perp} \cup J^{\perp} \subseteq (I \cap J)^{\perp}$.

Proof. i. Let $x \in (I \cup J)^{\perp}$. The definition of polar implies that $x(1 + (a+1)x) = 0$ for all $a \in I \cup J$. In particular $x(1 + (a+1)x) = 0$ for all $a \in I$. Hence $x \in I^{\perp}$. And similarly $x \in J^{\perp}$. Thus $x \in I^{\perp} \cap J^{\perp}$. Conversely let $x \in I^{\perp} \cap J^{\perp}$. Implies $x \in I^{\perp}$ and $x \in J^{\perp}$. It follows that $x(1+(a+1)x) = 0$ for all $a \in I$ and $x(1+(b+1)x) = 0$ for all $b \in J$. Hence $x(1+(a+1)x) = 0$ for all $a \in I \cup J$. Thus $(I \cup J)^{\perp} = I^{\perp} \cap J^{\perp}$.

ii. Let $x \in I^{\perp} \cup J^{\perp}$. It follows either $x \in I^{\perp}$ or $x \in J^{\perp}$. If $x \in I^{\perp}$, then $x(1 + (a+1)x) = 0$ for all $a \in I^{\perp}$. In particular $x(1 + (a+1)x) = 0$ for all $a \in (I \cap J)$. Hence $x \in (I \cap J)^{\perp}$. Similarly if $x \in J^{\perp}$, then $x \in (I \cap J)^{\perp}$. Thus $I^{\perp} \cup J^{\perp} \subseteq (I \cap J)^{\perp}$.

□

Lemma 3.23. *For any* $x, y, a, b \in R$, $(x+1)(a+1)(1+(y+1)(b+1)) \leq 1+(1+(x+1)y)(1+(a+1)b)$.

Proof. Consider $(x+1)(a+1)(1+(y+1)(b+1))[(x+1)y+1)((a+1)r+1)]$. By Definition 2.1, $(x+1)(a+1)(1+(y+1)(b+1))[(x+1)y+1)((a+1)b+1)] = (x+1)((x+1)y+1)(a+1)((a+1)b+1)[1+(y+1)(b+1)] = (y+1)((y+1)x+1)(b+1)((b+1)a+1)[1+(y+1)(b+1)] = ((y+1)x+1)((b+1)a+1)(b+1)(y+1)[1+(y+1)(b+1)] = 0$. □

Lemma 3.24. *Let I be an ideal of R. If* $x+y, a+b \in I$, *then* $(x+1)(a+1)+(y+1)(b+1) \in I$.

Proof. Let $x+y, a+b \in I$. By Definition 2.6, $1+(x+y+1)(a+b+1) \in I$. By Lemma 3.23, $(x+1)(a+1)(1+(y+1)(b+1)) \leq 1+(1+(x+1)y)(1+(a+1)b)$. And similarly $(y+1)(b+1)(1+(x+1)(a+1)) \leq 1+(1+(y+1)x)(1+(b+1)a)$. Thus by Proposition 3.1, $(x+1)(a+1)+(y+1)(b+1) \leq 1+(x+y+1)(a+b+1)$. □

Lemma 3.25. *If* $x+y, a+b \in I$, *Then* $x \cdot a + y \cdot b \in I$.

Proof. Since $x+1+y+1 = x+y \in I$ and $a+1+b+1 \in I$. By Lemma 3.24, it follows that $x \cdot a + y \cdot b \in I$. □

Definition 3.26. Let I be an ideal of R. Define $a \in R$, $a/I = \{x \in R : x+a \in I\}$ and hence for any $a, b \in R$, $a/I = b/I$ if and only if $a+b \in I$.

Notation: $a/I = a+I = \bar{a}$.

Theorem 3.27. *If I is an ideal of R, then R/I is a pseudo ring with* $x/I+y/I = x+y/I$ $x/I \cdot y/I = x \cdot y/I$.

Proof. First, we need to show the operations are well-defined. Define $a/I + b/I = a+b/I$ and $a/I \cdot b/I = a \cdot b/I$. Let $x, y, a, b \in R$ such that $x/I = a/I$ and $y/I = b/I$. By Remark 3.1 [6] and Definition 3.26, $x+a+y+b \in I$. It follows that $x/I+y/I = x+y/I = a+b/I = a/I+b/I$. Since $x/I = y/I$ and $a/I = b/I$, implies by Definition 3.26, $x+y, a+b \in I$. By Lemma 3.25 $x \cdot a + y \cdot b \in I$. Hence $x \cdot a/I = y \cdot b/I$. Verifying the axioms P_1 through P_7 of a pseudo ring is straight forward. Hence R/I is a pseudo ring. □

Theorem 3.28. *Every pseudo ring R is embedded into a direct product of the quotient pseudo rings* R/I *and* $R/I^{\perp}$ *for any ideal I of R.*

Proof. Define $\varphi : R \to R/I \times R/I^{\perp}$ by $\varphi(x) = (x/I, x/I^{\perp})$. Clearly $\varphi(1) = (1/I,\ 1/I^{\perp})$. Consider $\varphi(x+y) = (x+y/I, x+y/I^{\perp}) = (x/I, x/I^{\perp}) + (y/I, y/I^{\perp}) = \varphi(x)+\varphi(y)$.
$\varphi(x \cdot y) = (x \cdot y/I, x \cdot y/I^{\perp}) = (x/I \cdot y/I, x/I^{\perp} \cdot y/I^{\perp}) = (x/I, x/I^{\perp}) \cdot (y/I, y/I^{\perp}) = \varphi(x) \cdot \varphi(y)$.
Hence φ is homomorphism.
Let $x \in ker(\varphi)$. It follows that $\varphi(x) = (x/I, x/I^{\perp}) = (0/I, 0/I^{\perp})$. Implies $x \in I$ and $x \in I^{\perp}$. By Theorem 3.8, $x = 0$. Thus φ is one-to-one. □

Theorem 3.29. *Let I be an ideal of R. Define* $\varphi : R \to R/I \times R/I^{\perp}$ *by* $\varphi(x) = (x/I, x/I^{\perp})$ *is an isomorphism.*

Proof. Clearly $\varphi(1) = (1/I,\ 1/I^{\perp})$. Consider $\varphi(x+y) = (x+y/I, x+y/I^{\perp}) = (x/I, x/I^{\perp}) + (y/I, y/I^{\perp}) = \varphi(x)+\varphi(y)$.
$\varphi(x \cdot y) = (x \cdot y/I, x \cdot y/I^{\perp}) = (x/I \cdot y/I, x/I^{\perp} \cdot y/I^{\perp}) = (x/I, x/I^{\perp}) \cdot (y/I, y/I^{\perp}) = \varphi(x) \cdot \varphi(y)$.
Hence φ is homomorphism.
Let $x \in ker(\varphi)$. It follows that $\varphi(x) = (x/I, x/I^{\perp}) = (0/I, 0/I^{\perp})$. Implies $x \in I$ and $x \in I^{\perp}$. By Theorem 3.8, $x = 0$. Thus φ is one-to-one. Let $(a/I, b/I^{\perp}) \in R/I \times R/I^{\perp}$. If $b \in I, and\ \ a \in I^{\perp}$ take $y = a+b \in R$, such that $\varphi(y) = (a/I,\ b/I^{\perp})$. Suppose for $a \notin I$ or $b \notin I^{\perp}$. Assume $\{x \in R : x+a \in I\} \cap \{z \in R : z+b \in I^{\perp}\} = \{0\}$. This contradicts the facts $a \notin I\ \ or\ \ b \notin I^{\perp}$. Hence $\{x \in R : x+a \in I\} \cap \{z \in R : z+b \in I^{\perp}\} \neq \{0\}$. Thus there is $y \in R$ such that $y/I = a/I$ and $y/I^{\perp} = b/I^{\perp}$. Hence $\varphi(y) = (y/I, y/I^{\perp}) = (a/I, b/I^{\perp})$. Thus φ is onto.

□

Theorem 3.30. *Let* $\mathcal{H} = \{I : I \cup I^{\perp} = R\ \ for\ some\ ideal\ I\ \ of\ \ R\}$. *Then R is isomorphic to the direct product of a quotient pseudo rings* R/I *and* $R/I^{\perp}$ *for any* $I \in \mathcal{H}$.

Proof. Clearly $I = \{0\} \in \mathcal{H}$. Thus $\mathcal{H}$ is non empty. Also, it is Clear that $\varphi(1) = (1/I,\ 1/I^{\perp})$. Consider $\varphi(x+y) = (x+y/I, x+y/I^{\perp}) = (x/I, x/I^{\perp}) + (y/I, y/I^{\perp}) = \varphi(x)+\varphi(y)$.
$\varphi(x \cdot y) = (x \cdot y/I, x \cdot y/I^{\perp}) = (x/I \cdot y/I, x/I^{\perp} \cdot y/I^{\perp}) = (x/I, x/I^{\perp}) \cdot (y/I, y/I^{\perp}) = \varphi(x) \cdot \varphi(y)$.

Hence φ is homomorphism.
Let $x \in ker(\varphi)$. It follows that $\varphi(x) = (x/I, x/I^{\perp}) = (0/I, 0/I^{\perp})$. Implies $x \in I$ and $x \in I^{\perp}$. By Theorem 3.8, $x = 0$. Thus φ is one-to-one. Let $(a/I, b/I^{\perp}) \in R/I \times R/I^{\perp}$. If $b \in I, and \ \ a \in I^{\perp}$ take $y = a + b \in R$, such that $\varphi(y) = (a/I,\ b/I^{\perp})$. Suppose for $a \notin I$ or $b \notin I^{\perp}$. Assume $\{x \in R : x + a \in I\} \cap \{z \in R : z + b \in I^{\perp}\} = \{0\}$. This contradicts the facts $a \notin I \ \ or \ \ b \notin I^{\perp}$. Hence $\{x \in R : x + a \in I\} \cap \{z \in R : z + b \in I^{\perp}\} \neq \{0\}$. Thus there is $y \in R$ such that $y/I = a/I$ and $y/I^{\perp} = b/I^{\perp}$. Hence $\varphi(y) = (y/I, y/I^{\perp}) = (a/I, b/I^{\perp})$. Thus φ is onto.

□

Theorem 3.31. *Every pseudo ring is decomposed into the quotient pseudo rings R/I and $R/I^{\perp}$ for some ideal I of R.*

Proof. Follows from Theorem 3.30. □

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