Palestine Journal of Mathematics

Vol 14(1)(2025) , 356–366

On Some New Approaches in Statistical Convergence

Mridul Kumar Gupta, Sandeep Gupta and Sumit Goel

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 40A05; Secondary 46A45.

Keywords and phrases: Natural density, statistical dense set, statistical convergence, statistical boundedness.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *In this paper, we introduce and study the notions of statistical means and statistical ratios. We utilize these concepts to develop statistical analogs of certain well established classical results, including Cauchy's theorems on limits and Cauchy's formula on limits. Additionally, we present a statistical version of the squeeze principle for the convergence of real sequences.*

1 Introduction

The concept of statistical convergence has been widely generalized and extensively studied in functional analysis, leveraging statistical and number theory attributes. To develop the theory of convergence of sequences, statistical convergence of sequences was introduced by Fast [12] in a short note in 1951, exemplified as 'Convergence in density' by Buck [5] in 1953, and independently studied as a summability method by Schoenberg [25] in 1959. It is also contained in Zygmund [30] (see lemma in p. 181) as almost convergence. Later, the investigation of statistical convergence from the perspective of sequence space and its connection to summability was carried out by Connor [9], Fridy and Orhan [16], Kolk [18], Maddox [20], Rath and Tripathy [23], Šalàt [24], Tripathy [26], Tripathy and Sen [28], and numerous others. In the literature, statistical boundedness, statistical limit point, statistical cluster point, statistical limit superior, statistical limit inferior and many more notions were independently introduced and subsequently combined to enhance the field of summability theory over time. One can refer to ([1], [2], [4], [7], [8], [10], [11], [14], [15], [17], [19], [21], [27] and [29]) for associated works in the field of statistical convergence.

Statistical convergence relies on the pillar of natural density. The natural density of a subset A of natural numbers is a measure of how "dense" or abundant the subset is within the set of natural numbers. If the natural density of A is 1(unity), it means that A contains a "large" proportion of the natural numbers. Conversely, if the natural density is 0, it indicates that A contains a negligible proportion of the natural numbers. Thus, the density of a subset of natural numbers quantifies the relative abundance or "density" of the subset within the infinite set of natural numbers. The idea of asymptotic density (natural density), as discussed by Niven, Zuckerman, and Montgomery [22], plays a crucial role in understanding the concept of statistical convergence.

Definition 1.1. ([22]): The *asymptotic density* (or simply natural density $\delta(A)$) of a subset A of $\mathbb{N}$ is defined as

$$\lim_{n\to\infty} \frac{1}{n}\big|\{m \le n : m \in A\}\big|$$

provided that the limit exists finitely, where $\big|\{m \le n : m \in A\}\big|$ is the cardinality of the set of those elements of A which are less than or equal to n.

Clearly, all finite subsets of $\mathbb{N}$ *have zero natural density;* $\delta(\mathbb{N}) = 1$. *For a fixed natural number* k, $\delta(k\mathbb{N}) = \frac{1}{k}$. *Additionally,* $\delta(\mathbb{N}^k) = 0$, *where* $\mathbb{N}^k = \{1^k, 2^k, 3^k, 4^k, \dots\}$, *and* $\delta(k^{\mathbb{N}}) =$

0, *where* $k^{\mathbb{N}} = \{k^1, k^2, k^3, k^4, \dots\}$. *Furthermore, the density of the complement of A is given by* $\delta(A^c) = \delta(\mathbb{N} - A) = 1 - \delta(A)$. *It is to be noted that* $k \geq 2$ *is a fixed natural number.*

We also present some other important definitions and results which will be used in the sequel.

Definition 1.2. ([6]): A subset A of the set $\mathbb{N}$ is said to be *statistically dense* if $\delta(A) = 1$. Set A may also be referred to as a *unital density set*.

Lemma 1.3. *([6]):*

(i) A statistically dense subset of a statistically dense set is a statistically dense set.

(ii) The intersection and union of two statistically dense sets are statistically dense sets.

Definition 1.4. ([13]): A given number sequence $z = (z_m)$ is called *statistically convergent* to L if for every $\epsilon > 0$,

$$\delta\big(\{m \in \mathbb{N} : |z_m - L| \geq \epsilon\}\big) = 0.$$

We write $z_m \xrightarrow{\text{stat.}} L$ or $\text{stat-lim}_{m\to\infty} z_m = L$.

Definition 1.5. ([13]): A sequence $z = (z_m)$ is said to be *statistically Cauchy* if for every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$\delta\big(\{m \in \mathbb{N} : |z_m - z_{n_0}| \geq \epsilon\}\big) = 0.$$

Definition 1.6. ([3, 16]): A real number sequence $z = (z_m)$ is said to be *statistically bounded* if there is a number K such that

$$\delta\big(\{m \in \mathbb{N} : |z_m| > K\}\big) = 0.$$

Lemma 1.7. *([25]): If* $D\text{-}\lim z_m = z$ *and* $g(u)$, *defined for all real* u, *is continuous at* $u = z$, *then* $D\text{-}\lim g(z_m) = g(z)$.

Using Lemma 1.7 and the definition of statistical convergence, we have the following:

Lemma 1.8. *If* $\text{stat-lim}_{m\to\infty} z_m = z$ *and* $g(u)$, *defined for all real* u, *is continuous at* $u = z$, *then*

$$\text{stat-lim}_{m\to\infty} g(z_m) = g(z),$$

i.e., continuity preserves statistical convergence.

Fridy [13] and Šalàt [24] established some relations between statistical convergence and convergence of sequences. We procure those results below.

Theorem 1.9. *[Fridy 13] For a sequence* $z = (z_m)$, *the following statements are equivalent:*

(i) (z_m) *is a statistically convergent sequence;*

(ii) (z_m) *is a statistically Cauchy sequence;*

(iii) (z_m) *is a sequence for which there is a convergent sequence* (y_m) *such that* $(y_m) = (z_m)$ *a.a.m.*

Theorem 1.10. *[Šalàt 24] A sequence* (z_m) *statistically converges to L if and only if there exists a set* $M = \{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$ *such that* $\delta(M) = 1$ *and* $\lim_{n\to\infty} z_{m_n} = L$.

In this paper, we introduce statistical arithmetic means, statistical geometric means, statistical ratios and statistical inequalities for real sequences. We explore these concepts and establish significant results, which generalize classical results such as Cauchy's first and second theorems on limits, Cauchy's formula on limits, and the squeeze principle, all of which are fundamental in analysis.

In the second section, we introduce sequences of statistical arithmetic means and statistical geometric means, generalizing Cauchy's first and second theorems on limits, as well as Cauchy's formula on limits. Moreover, we introduce statistical inequalities, extending the existing squeeze principle for real sequence convergence in the third section.

Throughout the paper, a sequence refers to a sequence of real numbers.

2 Sequence of Statistical Means

Fridy [13] introduced the notion of "almost all m" (a.a.m.) *as follows: If $z = (z_m)$ is any sequence of real numbers satisfying a property P for all indices m of its terms, excluding a set of indices having zero natural density, then one can say that $z = (z_m)$ satisfies property P for* "almost all m", *and abbreviate it as* a.a.m..

Using this notion, statistical convergence is redefined as follows: A real sequence $z = (z_m)$ is statistically convergent *to L if for a given $\epsilon > 0$,* $|z_m - L| < \epsilon$ a.a.m..

In this section of our work, we introduce the concept of a sequence of statistical means, *leveraging the notion of* "almost all m", *which proves fruitful in generalizing some important classical notions and results regarding the convergence of real sequences.*

2.1 Sequence of Statistical Arithmetic Means

Definition 2.1. (Sequence of Statistical Arithmetic Means): Let $z = (z_m)$ be any sequence of real numbers such that z_m satisfies some property P for *"almost all m"*. We construct a sequence $z' = (z'_m)$ as follows:

$$z'_m = \begin{cases} \frac{\sum_{k=1;k\in M}^{m} z_k}{|\{k\in M:1\leq k\leq m\}|} & m \in M \\ z_m & m \notin M \end{cases}$$

where $M = \{m \in \mathbb{N} : z_m$ satisfies property $P\}$ and $|\{k \in M : 1 \leq k \leq m\}|$ denotes the cardinality of the set $\{k \in M : 1 \leq k \leq m\}$. Then $z' = (z'_m)$ is called the *sequence of statistical arithmetic means* of the sequence $z = (z_m)$ corresponding to M.

Example 2.2. Consider the sequence

$$z_m = \begin{cases} (-1)^m & \text{if } m \neq 2^n \\ m & \text{if } m = 2^n \end{cases} \qquad n = 1, 2, 3, \ldots$$

No doubt, $|z_m| \leq 1$ a.a.m., *i.e., $z = (z_m)$ is statistically bounded. Here, the corresponding dense set is $M = \{m \in \mathbb{N} : m \neq 2^n\}$. The sequence of statistical arithmetic means of the sequence $z = (z_m)$ corresponding to M is given by*

$$z' = (z'_m) = (-1, 2, -1, 4, -1, -\frac{1}{2}, -\frac{3}{5}, 8, -\frac{2}{3}, -\frac{3}{7}, \ldots).$$

It is evident that $|z'_m| \leq 1$ for all $m \in M$. As $\delta(M) = 1$, $z' = (z'_m)$ is also statistically bounded. Thus, the present illustration demonstrates that the sequence of statistical arithmetic means follows the same property of boundedness as its original sequence for a.a.m..

The question arises whether the sequence of statistical arithmetic means exhibits all the properties of its original sequence for a.a.m.. *The answer is no, in light of Example 2.2., where the sequence $z = (z_m)$ is not statistically convergent, but the sequence of its statistical arithmetic means statistically converges to 0.*

Another natural question arises: Is the sequence of statistical arithmetic means of a statistically convergent sequence statistically convergent, similar to how the sequence of arithmetic means of a convergent sequence converges in the ordinary sense? The answer is positive, and it can be presented in the following form:

Theorem 2.3. *The sequence $z' = (z'_m)$ of statistical arithmetic means of a statistically convergent sequence $z = (z_m)$ is also statistically convergent with the same limit L, i.e., if*

$$\textit{stat-lim}_{m\to\infty} z_m = L,$$

then

$$\textit{stat-lim}_{m\to\infty} z'_m = L.$$

Proof. Let $z = (z_m)$ be any real number sequence which is statistically convergent to L. Theorem 1.10. assures the existence of a statistically dense set M such that $\lim_{m\in M} z_m = L$.
Let $z' = (z'_m)$ be the sequence of statistical arithmetic means of the sequence $z = (z_m)$, corresponding to this M, defined as

$$z'_m = \begin{cases} \frac{\sum_{k=1;k\in M}^{m} z_k}{|\{k\in M:1\leq k\leq m\}|} & m\in M \\ z_m & m\notin M \end{cases}$$

In order to prove

$$\text{stat-lim}_{m\to\infty} z'_m = L,$$

construct a sequence $y = (y_m)$ as follows:

$$y_m = \begin{cases} z_m - L & m\in M \\ z_m & m\notin M. \end{cases}$$

As $\lim_{m\in M} z_m = L$, therefore, $\lim_{m\in M} y_m = 0$. It implies

$$\text{stat-lim}_{m\to\infty} y_m = 0$$

due to $\delta(M) = 1$. Now $z = (z_m)$ can be written as:

$$z_m = \begin{cases} y_m + L & m\in M \\ y_m & m\notin M. \end{cases}$$

Building upon it, $z' = (z'_m)$ may be rewritten as:

$$z'_m = \begin{cases} \frac{\sum_{k=1;k\in M}^{m} (y_k+L)}{|\{k\in M:1\leq k\leq m\}|} & m\in M \\ y_m & m\notin M \end{cases}$$

$$= \begin{cases} \frac{|\{k\in M:1\leq k\leq m\}|L}{|\{k\in M:1\leq k\leq m\}|} + \frac{\sum_{k=1;k\in M}^{m} y_k}{|\{k\in M:1\leq k\leq m\}|} & m\in M \\ y_m & m\notin M \end{cases}$$

$$= \begin{cases} L + \frac{\sum_{k=1;k\in M}^{m} y_k}{|\{k\in M:1\leq k\leq m\}|} & m\in M \\ y_m & m\notin M. \end{cases}$$

In reference to it, proving the given statement amounts to proving y'_m statistically converges to zero when

$$\text{stat-lim}_{m\to\infty} y_m = 0,$$

where

$$y'_m = \begin{cases} \frac{\sum_{k=1;k\in M}^{m} y_k}{|\{k\in M:1\leq k\leq m\}|} & m\in M \\ y_m & m\notin M. \end{cases}$$

As $\lim_{m\in M} y_m = 0$, so, for a given $\epsilon > 0$, there exists a non-negative integer p_0 such that:

$$|y_m| < \frac{\epsilon}{2} \quad \text{for all } m(\in M) \geq p_0. \tag{2.1}$$

As (y_m) is statistically convergent, it is necessarily statistically bounded. As such, there exists a positive number t such that:

$$|y_m| < t \quad \text{for all } m\in M. \tag{2.2}$$

By the use of the triangle inequality, we have

$$\begin{aligned}
\left|\frac{\sum_{k=1;k\in M}^{m} y_k}{|\{k\in M: 1\le k\le m\}|}\right| &\le \frac{1}{|\{k\in M: 1\le k\le m\}|}\sum_{k=1;k\in M}^{m}|y_k| \\
&\le \frac{1}{|\{k\in M: 1\le k\le m\}|}\left[\sum_{k=1;k\in M}^{p_0}|y_k| + \sum_{k=p_0+1;k\in M}^{m}|y_k|\right] \\
&\le \frac{1}{|\{k\in M: 1\le k\le m\}|}\Big[|\{k\in M: 1\le k\le p_0\}|t \\
&\quad +|\{k\in M: p_0+1\le k\le m\}|\frac{\epsilon}{2}\Big] \quad \text{for all } m(\in M)\ge p_0 \text{ [By (2.1) and (2.2)]} \\
&= \frac{|\{k\in M: 1\le k\le p_0\}|}{|\{k\in M: 1\le k\le m\}|}t + \frac{|\{k\in M: p_0+1\le k\le m\}|}{|\{k\in M: 1\le k\le m\}|}\frac{\epsilon}{2} \\
&\le \frac{|\{k\in M: 1\le k\le p_0\}|}{|\{k\in M: 1\le k\le m\}|}t + \frac{\epsilon}{2}. \qquad (2.3)
\end{aligned}$$

Choose a positive integer p_1 such that

$$\frac{|\{k\in M: 1\le k\le p_0\}|}{|\{k\in M: 1\le k\le m\}|}t < \frac{\epsilon}{2}, \quad \text{for all } m(\in M)\ge p_1.$$

Taking $p = \max\{p_0, p_1\}$, we have

$$\left|\frac{\sum_{k=1;k\in M}^{m} y_k}{|\{k\in M: 1\le k\le m\}|}\right| < \epsilon \quad \text{for all } m(\in M)\ge p \quad \text{[By (2.3)]}.$$

This implies

$$\lim_{m\in M} y'_m = 0.$$

As $\delta(M) = 1$, therefore, y'_m statistically converges to 0. Hence, z'_m statistically converges to L. This amounts to the stated result. □

Corollary 2.4. (Cauchy's First Theorem on Limits): *If any sequence $z = (z_m)$ of real numbers converges to L, then the sequence of arithmetic means of terms of the sequence $z = (z_m)$ also converges to L, i.e., if*

$$\lim_{m\to\infty} z_m = L,$$

then

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m} z_k = L.$$

Proof. Taking $M = \mathbb{N}$ in Theorem 2.3., the result follows. □

2.2 Sequence of Statistical Geometric Means

Definition 2.5. (Sequence of Statistical Geometric Means): Let $z = (z_m)$ be any sequence of positive real numbers such that z_m satisfies some property P for "almost all m". We construct a sequence $z'' = (z''_m)$ as follows:

$$z''_m = \begin{cases} \left(\prod_{\substack{k=1;\\ k\in M}}^{m} z_k\right)^{1/q} & m\in M \\ z_m & m\notin M \end{cases}$$

where $M = \{m\in\mathbb{N} : z_m \text{ satisfies property } P\}$ and $q = |\{k\in M: 1\le k\le m\}|$.

Then, the sequence $z'' = (z''_m)$ is called the sequence of statistical geometric means of the sequence $z = (z_m)$ corresponding to M.

Theorem 2.6. *If a sequence $z = (z_m)$ of positive real numbers converges to a positive statistical limit L, then so does the sequence $z'' = (z''_m)$ of its geometric statistical means, i.e., if*

$$\textit{stat-lim}_{m\to\infty} z_m = L,$$

then

$$\textit{stat-lim}_{m\to\infty} z''_m = L.$$

Proof. Let $z = (z_m)$ be any sequence of positive real numbers which is statistically convergent to L ($L > 0$). In view of Theorem 1.10., there exists a statistically dense set M such that

$$\lim_{m\in M} z_m = L.$$

Let $z'' = (z''_m)$ be the sequence of statistical geometric means of the sequence $z = (z_m)$, corresponding to this M, defined as

$$z''_m = \begin{cases} \left(\prod_{k=1;\, k\in M}^{m} z_k\right)^{1/q} & \text{if } m \in M \\ z_m & \text{if } m \notin M. \end{cases}$$

Let $t = (t_m) = (\log z_m)$. In view of Lemma 1.8.,

$$\text{stat-lim}_{m\to\infty} t_m = \text{stat-lim}_{m\to\infty} \log z_m = \log\left(\text{stat-lim}_{m\to\infty} z_m\right) = \log L.$$

Let $t' = (t'_m)$ be the sequence of statistical arithmetic means of the sequence $t = (t_m)$, corresponding to this M, i.e.,

$$t'_m = \begin{cases} \frac{\sum_{k=1;\, k\in M}^{m} t_k}{|\{k\in M: 1\le k\le m\}|} & \text{if } m \in M \\ t_m & \text{if } m \notin M \end{cases}$$

which can also be expressed as

$$t'_m = \begin{cases} \frac{\sum_{k=1;\, k\in M}^{m} \log z_k}{|\{k\in M: 1\le k\le m\}|} & \text{if } m \in M \\ \log z_m & \text{if } m \notin M \end{cases}$$

and hence

$$t'_m = \begin{cases} \log\left(\prod_{k=1;\, k\in M}^{m} z_k\right)^{1/q} & \text{if } m \in M \\ \log z_m & \text{if } m \notin M \end{cases} = \log z''_m.$$

As a consequence of Theorem 2.3.,

$$\text{stat-lim}_{m\to\infty} t'_m = \text{stat-lim}_{m\to\infty} t_m = \log L,$$

which implies that

$$\text{stat-lim}_{m\to\infty} \log z''_m = \log L.$$

Now, using Lemma 1.8. and the continuity of the exponential function, we have

$$\text{stat-lim}_{m\to\infty} z''_m = L.$$

□

Corollary 2.7. (Cauchy's Second Theorem on Limits): *If any sequence $z = (z_m)$ of positive real numbers converges to L, then the sequence of geometric means of the terms of the sequence $z = (z_m)$ also converges to L.*

Proof. Taking $M = \mathbb{N}$ in Theorem 2.6., the result follows. □

2.3 Sequence of Statistical Ratios and Sequence of Statistical q-th Roots

Definition 2.8. (Sequence of Statistical Ratios):
Let $z = (z_m)$ be any sequence of real numbers possessing positive terms for "almost all m". Take $M = \{m \in \mathbb{N} : z_m > 0\}$, then $\delta(M) = 1$. Construct a sequence $\alpha = (\alpha_m)$ as follows:

$$\alpha_m = \begin{cases} \frac{z_m}{z_{\lambda_m}} & \text{if } m \in M \\ |z_m| & \text{if } m \notin M \end{cases}$$

where for any $m \in M$, $\lambda_m = \max\{k : k \in M \text{ and } k < m\} \cup \{0\}$ and $z_0 = 1$. We call $\alpha = (\alpha_m)$ the sequence of statistical ratios for $z = (z_m)$.

Definition 2.9. (Sequence of Statistical q-th Roots): Let $z = (z_m)$ be any sequence of real numbers possessing positive terms for "almost all m". Let $M = \{m \in \mathbb{N} : z_m > 0\}$, then $\delta(M) = 1$. Construct a sequence $\beta = (\beta_m)$ as follows:

$$\beta_m = \begin{cases} z_m^{1/q} & \text{if } m \in M \\ |z_m| & \text{if } m \notin M \end{cases}$$

where $q = |\{k \in M : 1 \leq k \leq m\}|$. We refer to $\beta = (\beta_m)$ as the sequence of statistical q-th roots for $z = (z_m)$.

Example 2.10. Consider the sequence $z = (z_m)$, where

$$z_m = \begin{cases} m & \text{if } m \neq 7^n \\ 0 & \text{if } m = 7^n \end{cases} \quad n = 1, 2, 3, \ldots$$

Both the sequence $\alpha = (\alpha_m)$, the sequence of statistical ratios, and the sequence $\beta = (\beta_m)$, the sequence of statistical q-th roots for $z = (z_m)$, have the same statistical limit of 1.

Example 2.11. Consider the sequence $z = (z_m)$, where

$$z_m = \begin{cases} 3^{(-m+(-1)^m)} & \text{if } m \neq n^2 \\ -m^3 & \text{if } m = n^2 \end{cases} \quad n = 1, 2, 3, \ldots$$

The sequence $\alpha = (\alpha_m)$ of statistical ratios for $z = (z_m)$ does not converge statistically, but the sequence $\beta = (\beta_m)$, the sequence of statistical q-th roots for $z = (z_m)$, statistically converges to $\frac{1}{3}$.

From Example 2.10. and Example 2.11., a natural question arises: Is there any connection between the statistical convergence of sequence of statistical q-th roots and sequence of statistical ratios of a given sequence. The answer to this question is affirmative in form of following:

Theorem 2.12. *For a sequence $z = (z_m)$ possessing positive terms for "almost all m", the statistical limits of the sequence $\beta = (\beta_m)$ of statistical q-th roots and the sequence $\alpha = (\alpha_m)$ of statistical ratios of the sequence $z = (z_m)$ are equal, provided the latter statistical limit exists.*

Proof. We have

$$\beta_m = \begin{cases} \left(\prod_{k=1;k\in M}^{m} \frac{z_k}{z_{\lambda_k}}\right)^{1/q} & \text{if } m \in M \\ |z_m| & \text{if } m \notin M. \end{cases}$$

Taking the logarithm:

$$\log \beta_m = \begin{cases} \frac{1}{q} \sum_{k=1;k\in M}^{m} \log \frac{z_k}{z_{\lambda_k}} & \text{if } m \in M \\ \log |z_m| & \text{if } m \notin M. \end{cases}$$

Let

$$A_k = \log \frac{z_k}{z_{\lambda_k}} \text{ for all } k \ (1 \leq k \leq m;\ m \in M) \in M, \quad A_m = \log |z_m| \text{ for } m \notin M.$$

Thus

$$\log \beta_m = \begin{cases} \frac{\left(\sum_{k=1;\, k\in M}^{m} A_k\right)}{|\{k\in M: 1\le k\le m\}|} & \text{if } m \in M \\ A_m & \text{if } m \notin M \end{cases} = A'_m.$$

Where (A'_m) is the sequence of statistical arithmetic means of the sequence (A_m) corresponding to the statistical dense set M. It implies that

$$\text{stat-lim}_{m\to\infty} \log \beta_m = \text{stat-lim}_{m\to\infty} A'_m.$$

Using Theorem 2.3. on statistical arithmetic means, we have

$$\text{stat-lim}_{m\to\infty} A'_m = \text{stat-lim}_{m\to\infty} A_m.$$

Thus,

$$\text{stat-lim}_{m\to\infty} \log \beta_m = \text{stat-lim}_{m\to\infty} A_m.$$

Since $A_m = \log \alpha_m$, it follows that

$$\text{stat-lim}_{m\to\infty} \log \beta_m = \text{stat-lim}_{m\to\infty} \log \alpha_m.$$

By the continuity of the exponential function and Lemma 1.8., we have

$$\text{stat-lim}_{m\to\infty} \beta_m = \text{stat-lim}_{m\to\infty} \alpha_m.$$

□

Corollary 2.13. (Cauchy's Formula on Limits): *Let $z = (z_m)$ be a sequence such that $z_m > 0$ for all m. Then,*

$$\lim_{m\to\infty} z_m^{1/m} = \lim_{m\to\infty} \frac{z_m}{z_{m-1}},$$

provided the latter limit exists.

Proof. Taking $M = \mathbb{N}$ in Theorem 2.12., the result follows. □

3 Generalization of Squeeze Principle

In this section of our work, we initialize with the statistical inequalities as follows:

Definition 3.1. Let $z = (z_m)$ and $y = (y_m)$ be two sequences of real numbers. We say that $z_m \le y_m$ for almost all m if $\delta(M) = 1$, where $M = \{m \in \mathbb{N} : z_m \le y_m\}$.

Example 3.2. Consider

$$z_m = \begin{cases} 1 & \text{if } m \text{ is non-square} \\ m+1 & \text{if } m \text{ is a perfect square} \end{cases}$$

and

$$y_m = \begin{cases} 3 & \text{if } m \text{ is non-square} \\ m & \text{if } m \text{ is a perfect square} \end{cases}$$

Clearly, $M = \{m \in \mathbb{N} : z_m \le y_m\} = \{m \in \mathbb{N} : m \text{ is non-square}\}$ and $\delta(M) = 1$. Therefore, $z_m \le y_m$ a.a.m.

Definition 3.1 may also be generalized for three sequences of reals as follows:

Definition 3.3. Let $z = (z_m)$, $y = (y_m)$, and $t = (t_m)$ be three sequences of real numbers. We say that $z_m \le y_m \le t_m$ for almost all m if $\delta(M) = 1$, where $M = \{m \in \mathbb{N} : z_m \le y_m \le t_m\}$.

Example 3.4. Consider

$$z_m = \begin{cases} \frac{1}{m+2} & \text{if } m \neq n^3 \\ m+2 & \text{if } m = n^3 \end{cases} \quad n = 1, 2, 3, \ldots$$

$$y_m = \begin{cases} \frac{1}{m+1} & \text{if } m \neq n^3 \\ m+1 & \text{if } m = n^3 \end{cases} \quad n = 1, 2, 3, \ldots$$

and

$$t_m = \begin{cases} \frac{1}{m} & \text{if } m \neq n^3 \\ m & \text{if } m = n^3 \end{cases} \quad n = 1, 2, 3, \ldots$$

Clearly, $z_m \leq y_m \leq t_m$ a.a.m., as $\delta(M) = \delta(\{m \in \mathbb{N} : m \neq n^3\}) = 1$. From this example, we observe that all three sequences converge to the same statistical limit, implying a result similar to the squeeze principle.

Theorem 3.5. *If sequences $z = (z_m)$ and $t = (t_m)$ statistically converge to the same limit L and $z_m \leq y_m \leq t_m$ a.a.m., then $y = (y_m)$ also statistically converges to L.*

Proof. Let stat-$\lim_{m\to\infty} z_m = L$, so there exists $M_1 \subseteq \mathbb{N}$ such that $\delta(M_1) = 1$ and $\lim_{m\in M_1} z_m = L$ (refer to Theorem 1.10.). Similarly, there exists $M_2 \subseteq \mathbb{N}$ such that $\delta(M_2) = 1$ and $\lim_{m\in M_2} t_m = L$. Let $M' = M_1 \cap M_2$. Since the intersection of two statistically dense sets is again statistically dense, we have $\delta(M') = 1$, and

$$\lim_{m\in M'} z_m = \lim_{m\in M'} t_m = L.$$

On the other hand, $z_m \leq y_m \leq t_m$ a.a.m. suggests a statistically dense set $M'' \subseteq \mathbb{N}$ where $M'' = \{m \in \mathbb{N} : z_m \leq y_m \leq t_m\}$. Construct a common unital density set $M = M' \cap M''$ such that

$$\lim_{m\in M} z_m = \lim_{m\in M} t_m = L \quad \text{and} \quad z_m \leq y_m \leq t_m \text{ for all } m \in M.$$

Let $\epsilon > 0$ be given. There exists a positive integer m_0 such that

$$|z_m - L| < \epsilon \quad \text{and} \quad |t_m - L| < \epsilon \quad \text{for all } m \in M \text{ and } m \geq m_0.$$

Since $z_m \leq y_m \leq t_m$ for all $m \in M$, we have for any $m \in M$ with $m \geq m_0$,

$$L - \epsilon < z_m \leq y_m \leq t_m < L + \epsilon$$

which implies

$$L - \epsilon < y_m < L + \epsilon \quad \text{for all } m \in M \text{ and } m \geq m_0.$$

Thus, $|y_m - L| < \epsilon$ for all $m \in M$ and $m \geq m_0$. Since $\delta(M) = 1$, it follows that stat-$\lim_{m\to\infty} y_m = L$ by Theorem 1.10. □

Corollary 3.6. (Squeeze Principle): *If sequences $z = (z_m)$ and $t = (t_m)$ converge to the same limit L and $z_m \leq y_m \leq t_m$ for all $m \in \mathbb{N}$, then $y = (y_m)$ also converges to L.*

Proof. Taking $M = \mathbb{N}$ in Theorem 3.5., the result follows. □

References

[1] *A. Aizpuru, M. C. Lipton-Garela, and F. Rambla-Barreno,* Density by moduli and statistical convergence, *Quaest. Math.,* **37(4)***(2014), 525–530.* https://doi.org/10.2989/16073606.2014.981683

[2] *M. Altınok, Z. Kurtdişi, and M. Küçükaslan,* On Generalized Statistical Convergence, *Palest. J. Math.,* **5(1)***(2016), 50–58.*

[3] *V. K. Bhardwaj and S. Gupta,* On some generalizations of statistical boundedness, *J. Inequal. Appl.,* **12***(2014), 1–11.* https://doi.org/10.1186/1029-242X-2014-12

[4] *V. K. Bhardwaj, S. Gupta, S. A. Mohiuddine, and A. Kilicman,* Lacunary statistical boundedness, *J. Inequal. Appl.,* **311***(2014), 1–11.* https://doi.org/10.1186/1029-242X-2014-311

[5] *R. C. Buck,* Generalized asymptotic density, *Amer. J. Math.,* **75***(1953), 335–346.*https://doi.org/10.2307/2372456

[6] *M. Burgin and O. Duman,* Statistical Convergence and Convergence in Statistics, *arXiv:math/0612179 [math.GM](2007).* https://doi.org/10.48550/arXiv.math/0612179

[7] *R. Çolak,* Statistical convergence of order α, *Modern Methods in Analysis and Its Applications, Anamaya Publishers, New Delhi, India,(2010), 121–129.*

[8] *R. Çolak and Ç. A. Bektas,* λ-Statistical convergence of order α, *Acta Math. Sci. Ser. B (Engl. Ed.),* **31(3)***(2011), 953–959.* https://doi.org/10.1016/S0252-9602(11)60288-9.

[9] *J. S. Connor,* The statistical and strong p-Cesàro convergence of sequences, *Anly.,* **8***(1988), 47–63.* https://doi.org/10.1524/anly.1988.8.12.47.

[10] *S. Das and A. Ghosh,* A study on statistical versions of convergence of sequences of functions, *Math. Slovaca,* **72(2)***(2022), 443–458.* https://doi.org/10.1015/ms-2022-0030.

[11] *M. Et, V. K. Bhardwaj and S. Gupta,* On deferred statistical boundedness of order α, *Comm. Stat. Theory Methods,* **51(3)***(2021), 1–13.* https://doi.org/10.1080/03610926.2021.1906434.

[12] *H. Fast,* Sur la convergence statistique, *Colloq. Math.,* **2(3-4)***(1951), 241–244.* https://doi.org/10.4064/cm-2-3-4-241-244.

[13] *J. A. Fridy,* On statistical convergence, *Anly.,* **5***(1985), 301–313, .* https://doi.org/10.1524/anly.1985.5.4.301.

[14] *J. A. Fridy,* Statistical limit points, *Proc. Amer. Math. Soc.,* **118***(1993), 1187–1192.* https://doi.org/10.1090/S0002-9939-1993-1181163-6.

[15] *J. A. Fridy and C. Orhan,* Lacunary statistical convergence, *Pacific J. Math.,* **160(1)***(1993), 43–51.*

[16] *J. A. Fridy and C. Orhan,* Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.,* **125(12)***(1997), 3625–3631.* https://doi.org/10.1090/S0002-9939-97-04000-8.

[17] *S. Gupta and V. K. Bhardwaj,* On deferred f-statistical convergence, *Kyungpook Math. J.,* **50***(2018), 91–103.* https://doi.org/10.5666/KMJ.2018.58.1.91.

[18] *E. Kolk,* Matrix summability of statistical convergent sequences, *Anly.,* **13***(1993), 77–83.* https://doi.org/10.1524/anly.1993.13.12.77.

[19] *M. Kucukaslan and M. Yilmazturk,* On deferred statistical convergence of sequences, *Kyungpook Math. J.,* **56(2)***(2016), 357–366.* http://dx.doi.org/10.5666/KMJ.2016.56.2.357

[20] *I. J. Maddox,* A tauberian theorem for statistical convergence, *Math. Proc. Cambridge Philos. Soc.,* **106(2)***(1989), 277–280.* https://doi.org/10.1017/S0305004100078105

[21] *Mursaleen,* λ-statistical convergence, *Math. Slovaca,* **50(1)***(2000), 111–115.*

[22] *I. Niven, H. S. Zuckerman and H. L. Montgomery,* An introduction to the theory of numbers, *John Wiley and Sons Inc., (1991). ISBN-13: 978-0471625469.*

[23] *D. Rath and B. C. Tripathy,* Matrix maps on sequence spaces associated with sets of integers, *Indian J. Pure Appl. Math.,* **27(2)***(1996), 197–206.*

[24] *T. Šalat,* On statistically convergent sequences of real numbers, *Math. Slovaca,* **30***(1980), 139–149.*

[25] *I. J. Schoenberg,* The integrability of certain functions and related summability methods, *Amer. Math. Monthly,* **66***(1959), 361–375.* http://doi.org/10.2307/2308747

[26] *B. C. Tripathy,* On statistical convergence, *Proc. Est. Acad. Sci. Phy. Math.,* **47(4)***(1998), 299–303.* https://doi.org/10.3176/phys.math.1998.4.06

[27] *B. C. Tripathy and M. Et,* On generalized difference lacunary statistical convergence, *Stud. Univ. Babes-Bolyai Math.,* **L(1)***(2005), 119–130.*

[28] *B. C. Tripathy and M. Sen,* On generalized statistically convergent sequences, *Indian J. Pure Appl. Math.,* **32(11)***(2001), 1689–1694.*

[29] *A. K. Verma and L. K. Singh,* $(\Delta^k_{(v,)} f)$-Lacunary statistically convergent sequences of order α, *Proyecciones,* **41(3)***(2022), 791–804.* http://dx.doi.org/10.22199/issn.0717-6279-4757

[30] *A. Zygmund,* Trigonometric Series, *Cambridge University Press, (1993).*

Author information

Mridul Kumar Gupta, Department of Mathematics, Chaudhary Charan Singh University, Meerut - 250001, Uttar Pradesh, India.
E-mail: mkgupta2k2@gmail.com

Sandeep Gupta, Department of Mathematics, Arya (P.G.) College, Panipat - 132103, Haryana, India.
E-mail: sandeep80.gupta@rediffmail.com

Sumit Goel, Department of Mathematics, Chaudhary Charan Singh University, Meerut - 250001, Uttar Pradesh, India.
E-mail: sumitgoelmath01@gmail.com

Received: 2024-02-05
Accepted: 2024-12-26