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On $\mathfrak{A}$-gr-n-ideals of graded commutative rings

Ala'a Abu Qayass and Khaldoun Al-Zoubi

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Corresponding Author: K. Al-Zoubi

Abstract *Let G be a group with identity e, $\mathfrak{D}$ be a commutative G-graded ring with unity, $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a multiplicatively closed subset of $\mathfrak{D}$ and $\mathfrak{P}$ be a graded ideal of $\mathfrak{D}$ such that $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. In this article, we introduce the concept of $\mathfrak{A}$-gr-n-ideals which is a generalization of gr-n-ideals. We say that $\mathfrak{P}$ is $\mathfrak{A}$-gr-n-ideal if there exists $s \in \mathfrak{A}$ such that for all $r, t \in h(\mathfrak{D})$, if $sr \in \mathfrak{P}$, then either $sr \in Gr(0)$ or $st \in \mathfrak{P}$. We investigate some basic properties of $\mathfrak{A}$-gr-n-ideals.*

1 Introduction

Throughout this article, we assume that G is a group with identity e, $\mathfrak{D}$ is a commutative G-graded ring with nonzero unity 1 and $\mathfrak{A} \subseteq h(\mathfrak{D})$ is a multiplicatively closed subset (briefly, m.c.s.) of $\mathfrak{D}$.

The concept of graded prime ideal of a graded ring was introduced in [12] and studied in [3]. The concept of graded primary ideal was introduced and studied by M. Refai and K. Al-Zoubi in [11]. The authors in [9, 10] used a new approach to generalize graded prime ideals by defining graded $\mathfrak{A}$-prime ($\mathfrak{A}$-gr-prime) ideals. Then analogously, Alshehry in [1] introduced the notion of graded $\mathfrak{A}$-primary ($\mathfrak{A}$-gr-primary) ideals. In [2] the notion of graded n-ideals (gr-n-ideals) was first introduced and studied by Al-Zoubi, Al-Turman and Celikel. Here,we introduce the concept of $\mathfrak{A}$-gr-n-ideals which is a generalization of gr-n-ideals. We investigate several properties of $\mathfrak{A}$-gr-n-ideals.

2 Preliminaries

The purpose of this section is to provide the definitions and results that will be needed in the next section.

Definition 2.1. (a) Let G be a group with identity e and $\mathfrak{D}$ be a commutative ring with identity $1_{\mathfrak{D}}$. Then $\mathfrak{D}$ is G-graded ring if there exist additive subgroups $\mathfrak{D}_g$ of $\mathfrak{D}$ indexed by the elements $g \in G$ such that $\mathfrak{D} = \bigoplus_{g\in G} \mathfrak{D}_g$ and $\mathfrak{D}_g\mathfrak{D}_h \subseteq \mathfrak{D}_{gh}$ for all $g, h \in G$. The elements of $\mathfrak{D}_g$ are called homogeneous of degree g. The set of all homogeneous elements of $\mathfrak{D}$ is denoted by $h(\mathfrak{D})$, i.e. $h(\mathfrak{D}) = \bigcup_{g\in G} \mathfrak{D}_g$, see [6].

(b) Let $\mathfrak{D} = \bigoplus_{g\in G} \mathfrak{D}_g$ be G-graded ring, an ideal K of $\mathfrak{D}$ is called a graded ideal if $K = \sum_{h\in G} K \cap \mathfrak{D}_h = \sum_{h\in G} K_h$. By $K \leq^{id}_G \mathfrak{D}$, we mean that K is a G-graded ideal of $\mathfrak{D}$, (see [6]).

(c) The *graded radical* of a graded ideal K, denoted by $Gr(K)$, is the set of all $r = \sum_{g\in G} r_g \in \mathfrak{D}$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $r_g^{n_g} \in K$. Note that, if t is a homogeneous element, then $t \in Gr(K)$ if and only if $t^n \in K$ for some $n \in \mathbb{N}$, see [11, 4, 5].

(d) A proper graded ideal K of $\mathfrak{D}$ is said to be a graded prime (briefly, gr-prime) if whenever $r_g, s_h \in h(\mathfrak{D})$ with $r_g s_h \in K$, then either $r_g \in K$ or $s_h \in K$, see [11].

(e) A proper graded ideal K of $\mathfrak{D}$ is called a graded primary(briefly, gr-primary) ideal if whenever $r_g, s_h \in h(\mathfrak{D})$ and $r_g s_h \in K$, then either $r_g \in K$ or $s_h \in Gr(K)$, see [11].

(f) A proper graded ideal K of $\mathfrak{D}$ is said to be a graded n-ideal (briefly, gr-n-ideal) of $\mathfrak{D}$ if whenever $r_g, s_h \in h(\mathfrak{D})$ with $r_g s_h \in K$ and $r_g \notin Gr(0)$, then $s_h \in K$, see [2].

(g) Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. We say that $\mathfrak{P}$ is a graded $\mathfrak{A}$-prime (briefly, $\mathfrak{A}$-gr-prime) ideal of $\mathfrak{D}$ If there exists an $s \in \mathfrak{A}$ such that for all $r, t \in h(\mathfrak{D})$, if $rt \in \mathfrak{P}$, then either $sr \in \mathfrak{P}$ or $st \in \mathfrak{P}$., see [9].

(h) Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. We say that $\mathfrak{P}$ is a graded $\mathfrak{A}$-primary (briefly, $\mathfrak{A}$-gr-primary) ideal of $\mathfrak{D}$ If there exists an $s \in \mathfrak{A}$ such that for all $r, t \in h(\mathfrak{D})$, if $rt \in \mathfrak{P}$, then either $sr \in \mathfrak{P}$ or $st \in Gr(\mathfrak{P})$, see[1].

(i) For G-graded rings $\mathfrak{D}$ and $\mathfrak{D}'$, a G-graded ring homomorphism $f : \mathfrak{D} \to \mathfrak{D}'$ is a ring homomorphism with $f(\mathfrak{D}_g) \subseteq \mathfrak{D}'_g$ for every $g \in G$, see [6].

(j) A G-graded ring $\mathfrak{D}$ is called graded reduced ring if whenever $a \in h(\mathfrak{D})$ with $a^2 = 0$, then $a = 0$, i.e $Grad(0) = \{0\}$.

(k) An element a of $h(\mathfrak{D})$ is called regular if $Ann(a) = 0$. Then we denote the set of all regular elements of $h(\mathfrak{D})$ by $reg(\mathfrak{D})$. For a graded ring $\mathfrak{D}$, we will denote by $U(\mathfrak{D})$, $reg(\mathfrak{D})$ and $Z(\mathfrak{D})$, the set of unit homogeneous elements, regular homogeneous elements and zero-divisor homogeneous elements of $\mathfrak{D}$, respectively.

3 Results

Definition 3.1. Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. Then $\mathfrak{P}$ is said to be $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ if there exists a fixed $s_\alpha \in \mathfrak{A}$ such that for all $r_g, t_h \in h(\mathfrak{D})$ if $r_g t_h \in \mathfrak{P}$ and $s_\alpha r_g \notin Gr(0)$, then $s_\alpha t_h \in \mathfrak{P}$. This fixed homogenous element $s_\alpha \in \mathfrak{A}$ is called an $\mathfrak{A}$-homogenous element of $\mathfrak{P}$.

Let $\mathfrak{P}$ be a gr-n-ideal of a graded ring $\mathfrak{D}$ and $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a multiplicatively closed subset of $\mathfrak{D}$ disjoint with $\mathfrak{P}$. Then clearly $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal. However, it is clear that the classes of gr-n-ideals and $\mathfrak{A}$-gr-n-ideals coincide if $\mathfrak{A} \subseteq U(\mathfrak{D})$.

Moreover, obviously any $\mathfrak{A}$-gr-n-ideal is a $\mathfrak{A}$-gr-$primary$ ideal and the two concepts coincide if the graded ideal is contained in $Gr(0)$. However, the converses of these implications are not true in general as we can see in the following examples.

Example 3.2. Let $\mathfrak{D} = \mathbb{Z}_{12}$ and $G = \mathbb{Z}_2$. Then $\mathfrak{D}$ is a G-graded ring with $\mathfrak{D}_0 = \mathbb{Z}_{12}$ and $\mathfrak{D}_1 = \{0\}$. Let $\mathfrak{A} = \left\{\bar{1}, \bar{3}, \bar{9}\right\} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and consider the graded ideal $\mathfrak{P} = \left\langle \bar{4} \right\rangle$ of $\mathfrak{D}$. Choose $s_\alpha = \bar{3} \in \mathfrak{A}$ and let $r_g, t_h \in h(\mathfrak{D})$ with $r_g t_h \in \mathfrak{P}$ but $\bar{3} t_h \notin \mathfrak{P}$. Now, $r_g t_h \in \langle \bar{2} \rangle$ implies $r_g \in \langle \bar{2} \rangle$ or $t_h \in \langle \bar{2} \rangle$. Assume that $r_g \notin \langle \bar{2} \rangle$ and $t_h \in \langle \bar{2} \rangle$. Since $r_g \notin \langle \bar{2} \rangle$, then $r_g \in \left\{\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \bar{11}\right\}$ and since $\bar{3} t_h \notin \mathfrak{P}$, we have $t_h \in \{\bar{2}, \bar{6}, \bar{10}\}$. Thus in each case $r_g t_h \notin \mathfrak{P}$, a contradiction. Hence, we must have $r_g \in \langle \bar{2} \rangle$ and so $\bar{3} r_g \in \left\langle \bar{6} \right\rangle = Gr\left(\bar{0}\right)$. On the other hand, $\mathfrak{P}$ is not a gr-n-ideal as $\bar{2}.\bar{2} \in \mathfrak{P}$ but neither $\bar{2} \in Gr\left(\bar{0}\right)$ nor $\bar{2} \in \mathfrak{P}$.

A (gr-$prime$) gr-$primary$ ideal of a graded ring $\mathfrak{D}$ that is not a gr-n-ideal is a direct example of a ($\mathfrak{A}$-gr-$prime$) $\mathfrak{A}$-gr-$primary$ that is not an $\mathfrak{A}$-gr-n-ideal where $\mathfrak{A} = \{1\}$. For a less trivial example, we have the following.

Example 3.3. Consider $\mathfrak{D} = \mathbb{Z}[x]$ and $G = \mathbb{Z}$. Then $\mathfrak{D}$ is G-graded ring by $\mathfrak{D}_j = \mathbb{Z}x^j$ for $j \geq 0$ and $\mathfrak{D}_j = \{0\}$ otherwise. Consider the graded ideal $\mathfrak{P} = \langle 4x \rangle$ of $\mathfrak{D}$ and the multiplicative subset $\mathfrak{A} = \{4^m : m \in \mathbb{N} \cup \{0\}\}$ of $\mathfrak{D}$. We show that $\mathfrak{P}$ is a $\mathfrak{A}$-gr-$prime$ ideal of $\mathfrak{D}$. Note that $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. Let $f(x) g(x) \in \mathfrak{P}$ for some $f(x), g(x) \in h(\mathfrak{D})$. Then x divides $f(x) g(x)$, and then x divides $f(x)$ or x divides $g(x)$, which implies that $4f(x) \in \mathfrak{P}$ or $4g(x) \in \mathfrak{P}$. Therefore, $\mathfrak{P}$ is a gr-$\mathfrak{A}$-$prime$ (gr-$\mathfrak{A}$-$primary$) ideal of $\mathfrak{D}$. However, $\mathfrak{P}$ is not an $\mathfrak{A}$-gr-n-ideal since for all $s = 4^m \in \mathfrak{A}$, we have $(2x)(2) \in \mathfrak{P}$ but $s(2x) \notin Gr\left(0_{\mathbb{Z}[x]}\right)$ and $s(2) \notin \mathfrak{P}$.

Theorem 3.4. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$.*

(i) *If $\mathfrak{P}$ be an $\mathfrak{A}$-gr-n-ideal, then $s_\alpha \mathfrak{P} \subseteq Gr(0)$ for some $s_\alpha \in \mathfrak{A}$. If moreover, $\mathfrak{A} \subseteq reg(\mathfrak{D})$, then $\mathfrak{P} \subseteq Gr(0)$.*

(ii) *$Gr(0)$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ if and only if $Gr(0)$ is a $\mathfrak{A}$-gr-$prime$ ideal of $\mathfrak{D}$.*

(iii) *Let $\mathfrak{A} \subseteq reg(\mathfrak{D})$. Then $\{0\}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ if and only if $\{0\}$ is a gr-n-ideal.*

Proof. (i) Let $r_g \in \mathfrak{P}$. Since $\mathfrak{P} \cap \mathfrak{A} = \emptyset$, $s_\alpha.1 \notin \mathfrak{P}$ for all $s_\alpha \in \mathfrak{A}$. So, $r_g.1 \in \mathfrak{P}$ and hence there exists an $s_\alpha \in \mathfrak{A}$ with $s_\alpha r_g \in Gr(0)$. Thus, $s_\alpha \mathfrak{P} \subseteq Gr(0)$. Moreover, if $\mathfrak{A} \subseteq reg(\mathfrak{D})$, then clearly $\mathfrak{P} \subseteq Gr(0)$.

(ii) It follows from [2, Corollary 1].

$(iii$ Let s_α be an $\mathfrak{A}$-homogenous element of $\{0\}$ and $r_g t_h = 0$ for some $r_g, t_h \in h(\mathfrak{D})$. So $s_\alpha r_g \in Gr(0)$ or $s_\alpha t_h = 0$ and hence $s_\alpha^n r_g^n = 0$ for some positive integer n or $s_\alpha t_h = 0$. Since $\mathfrak{A} \subseteq reg(\mathfrak{D})$, $r_g^n = 0$ or $t_h^n = 0$. □

Theorem 3.5. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. Then the following statements are equivalent.*

(i) *$\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$.*

(ii) *There exists an $s_\alpha \in \mathfrak{A}$ such that for any two graded ideals I, J of $\mathfrak{D}$, if $IJ \subseteq \mathfrak{P}$, then $s_\alpha I \subseteq Gr(0)$ or $s_\alpha J \subseteq \mathfrak{P}$.*

Proof. $(i) \Rightarrow (ii)$. Assume that $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. Assume on the contrary that for each $s_\alpha \in \mathfrak{A}$, there exist two graded ideals I', J' of $\mathfrak{D}$ with $I'J' \subseteq \mathfrak{P}$ but $s_\alpha I' \nsubseteq Gr(0)$ and $s_\alpha J' \nsubseteq \mathfrak{P}$. Then for each $s_\alpha \in \mathfrak{A}$, we can find two elements $r_g \in I'$ and $t_h \in J'$ with $r_g t_h \in \mathfrak{P}$ such that $s_\alpha r_g \notin Gr(0)$ and $s_\alpha t_h \notin \mathfrak{P}$. By this contradiction, we are done.

$(ii) \Rightarrow (i)$. Let $r_g, t_h \in h(\mathfrak{D})$ with $r_g t_h \in \mathfrak{P}$. Let $I = Rr_g$ and $J = Rt_h$ be two graded ideals of $\mathfrak{D}$ generated by r_g and t_h, respectively. Then we get the result. □

Theorem 3.6. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. Then*

(i) *If $(\mathfrak{P} :_{\mathfrak{D}} w)$ is a gr-n-ideal of $\mathfrak{D}$ for some $w \in \mathfrak{A}$, then $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal.*

(ii) *If $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal and $(Gr(0) :_{\mathfrak{D}} w)$ is a gr-n-ideal of $\mathfrak{D}$ where $w \in \mathfrak{A}$ is an $\mathfrak{A}$-homogenous element of $\mathfrak{P}$, then $(\mathfrak{P} :_{\mathfrak{D}} w)$ is a gr-n-ideal of $\mathfrak{D}$.*

(iii) *If $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal and $\mathfrak{A} \subseteq reg(\mathfrak{D})$, then $(\mathfrak{P} :_{\mathfrak{D}} w)$ is gr-n-ideal of $\mathfrak{D}$ for any $\mathfrak{A}$-element w of $\mathfrak{P}$.*

Proof. (i) Assume that $(\mathfrak{P} :_{\mathfrak{D}} w)$ is a gr-n-ideal of $\mathfrak{D}$ for some $w \in \mathfrak{A}$. We show that w is an $\mathfrak{A}$-homogenous element of $\mathfrak{P}$. Let $r, t \in h(\mathfrak{D})$ with $rt \in \mathfrak{P}$ and $wr \notin Gr(0)$. Hence $rt \in (\mathfrak{P} :_{\mathfrak{D}} w)$ and $r \notin Gr(0)$ so $t \in (\mathfrak{P} :_{\mathfrak{D}} w)$. Thus, $wt \in \mathfrak{P}$ and $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal.

(ii) Assume that $rt \in (\mathfrak{P} :_{\mathfrak{D}} w)$ for some $r, t \in h(\mathfrak{D})$. Hence $r(wt) \in \mathfrak{P}$ and thus $wr \in Gr(0)$ or $w^2 t \in \mathfrak{P}$. Suppose $wr \in Gr(0)$. Since $(Gr(0) :_{\mathfrak{D}} w)$ is a gr-n-ideal, by [2, Lemma 1], we have $(Gr(0) :_{\mathfrak{D}} w) = Gr(0)$. So $r \in Gr(0)$. Now, suppose $w^2 t \in \mathfrak{P}$. If $wt \notin \mathfrak{P}$, then $w^3 \in Gr(0)$ as $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal. So $w \in Gr(0)$ which contradicts the assumption that $(Gr(0) :_{\mathfrak{D}} w)$ is proper. Hence, $wt \in \mathfrak{P}$ and $t \in (\mathfrak{P} :_{\mathfrak{D}} w)$ as needed.

(iii) Assume that $\mathfrak{A} \subseteq reg(\mathfrak{D})$ and $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal with $\mathfrak{A}$-homogeneous element w of $\mathfrak{P}$. Let $r, t \in h(\mathfrak{D})$ with $rt \in (\mathfrak{P} :_{\mathfrak{D}} w)$ so that $r(wt) \in \mathfrak{P}$. If $wr \in Gr(0)$, then $w^m r^m = 0$ for some integer m. Since $\mathfrak{A} \subseteq reg(\mathfrak{D})$, we have $r^m = 0$ and hence $r \in Gr(0)$. If $w^2 t \in \mathfrak{P}$, then similar to the proof of (ii) we get $t \in (\mathfrak{P} :_{\mathfrak{D}} w)$. Thus $(\mathfrak{P} :_{\mathfrak{D}} w)$ is a gr-n-ideal of $\mathfrak{D}$. □

Theorem 3.7. *Let $\mathfrak{A} \subseteq reg(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P}$ be a $\mathfrak{A}$-gr-prime ideal of $\mathfrak{D}$. Then the following statements are equivalent.*

(i) $\mathfrak{P}$ *is an* $\mathfrak{A}$-gr-n-*ideal*

(ii) $(\mathfrak{P} :_{\mathfrak{D}} w) = Gr(0)$ *for some* $w \in \mathfrak{A}$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ and w_1 be an $\mathfrak{A}$-homogenous element of $\mathfrak{P}$. By Theorem 3.6 (i), we have $(\mathfrak{P} :_{\mathfrak{D}} w_1)$ is a gr-n-ideal of $\mathfrak{D}$. Moreover, $(\mathfrak{P} :_{\mathfrak{D}} ww_1)$ is a gr-n-ideal for all $w \in \mathfrak{A}$. Indeed, if $rt \in (\mathfrak{P} :_{\mathfrak{D}} ww_1)$ for $r, t \in h(\mathfrak{D})$, then $rtww_1 \in \mathfrak{P}$ and hence either $w_1^2 r \in Gr(0)$ or $w_1 wt \in \mathfrak{P}$. If $w_1^2 r \in Gr(0)$, then $r \in Gr(0)$ as $\mathfrak{A} \subseteq reg(\mathfrak{D})$. Otherwise, we get $t \in (\mathfrak{P} :_{\mathfrak{D}} ww_1)$ as needed. Since $\mathfrak{P}$ is a $\mathfrak{A}$-gr-$prime$ ideal of $\mathfrak{D}$, we have $(\mathfrak{P} :_{\mathfrak{D}} w_2)$ is a gr-$prime$ ideal of $\mathfrak{D}$ where $w_2 \in \mathfrak{A}$ such that whenever $rt \in \mathfrak{P}$ for $r, t \in h(\mathfrak{D})$, either $w_2 r \in \mathfrak{P}$ or $w_2 t \in \mathfrak{P}$ by [10, Proposition 2.4]. Similar to the above argument, we can also conclude that $(\mathfrak{P} :_{\mathfrak{D}} ww_2)$ is a gr-$prime$ ideal for all $w \in \mathfrak{A}$. Now, choose $s = w_1 w_2$. Then $(\mathfrak{P} :_{\mathfrak{D}} s)$ is both a gr-$prime$ and a gr-n-ideal of $\mathfrak{D}$ and hence $(\mathfrak{P} :_{\mathfrak{D}} s) = Gr(0)$ by [2, Theorem 1].

$(ii) \Rightarrow (i)$ Suppose $(\mathfrak{P} :_{\mathfrak{D}} w) = Gr(0)$ for some $w \in \mathfrak{A}$. Then $(\mathfrak{P} :_{\mathfrak{D}} w')$ is a gr-$prime$ ideal of $\mathfrak{D}$ for some $w' \in \mathfrak{A}$ as $\mathfrak{P}$ is a $\mathfrak{A}$-gr-$prime$ ideal. Now, if $r \in (\mathfrak{P} :_{\mathfrak{D}} w') \cap h(\mathfrak{D})$, then $rw' \in \mathfrak{P} \subseteq (\mathfrak{P} :_{\mathfrak{D}} w) \subseteq Gr(0)$ and so $r \in Gr(0)$ as $\mathfrak{A} \subseteq reg(\mathfrak{D})$. Hence $(\mathfrak{P} :_{\mathfrak{D}} w') = Gr(0)$ is a gr-$prime$ ideal. By [2, Theorem 1], we have $(\mathfrak{P} :_{\mathfrak{D}} w')$ is gr-n-ideal. Thus $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal by Theorem 3.6 (i). □

Theorem 3.8. *Let $\mathfrak{D}$ be a G-graded reduced ring and let $\mathfrak{A} \subseteq reg(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$. Then $\mathfrak{D}$ is a graded integral domain if and only if there exists a gr-$\mathfrak{A}$-prime ideal of $\mathfrak{D}$ which is also an $\mathfrak{A}$-gr-n-ideal.*

Proof. Assume that $\mathfrak{D}$ is a graded integral domain. Since $0 = Gr(0)$ is a gr-prime, by [2, Corollary 1], we have $Gr(0)$ is a gr-n-ideal. Thus $Gr(0)$ is both $\mathfrak{A}$-gr-prime ideal and $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$, as required. Conversely, Assume that $\mathfrak{P}$ is both $\mathfrak{A}$-gr-prime and $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. Then $(\mathfrak{P} :_{\mathfrak{D}} s) = Gr(0)$ for some $s \in \mathfrak{A}$ by Theorem 3.7 and hence $\mathfrak{P}$ is gr-n-ideal by Theorem 3.6 (i). Hence $Gr(0) = 0$ is also a gr-prime ideal by [2, Corollary 1]. Thus $\mathfrak{D}$ is a graded integral domain. □

A graded ideal $\mathfrak{P}$ of a graded ring $\mathfrak{D}$ is called a maximal $\mathfrak{A}$-gr-n-ideal if there is no $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ that contains I properly. In the following Theorem, we observe the relationship between maximal $\mathfrak{A}$-gr-n-ideals and gr-$\mathfrak{A}$-prime ideals.

Theorem 3.9. *Let $\mathfrak{A} \subseteq reg(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$. If $\mathfrak{P}$ is a maximal $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$, then $\mathfrak{P}$ is a $\mathfrak{A}$-gr-$prime$ (and so $(\mathfrak{P} :_{\mathfrak{D}} s) = Gr(0)$ for some $s \in \mathfrak{A}$).*

Proof. Assume that $\mathfrak{P}$ is a maximal $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ and $w \in \mathfrak{A}$ is an $\mathfrak{A}$-homogenous element of $\mathfrak{P}$. By Theorem 3.6 (iii), we have $(\mathfrak{P} :_{\mathfrak{D}} w)$ is a gr-n-ideal of $\mathfrak{D}$. Moreover, $(\mathfrak{P} :_{\mathfrak{D}} w)$ is a maximal gr-n-ideal of $\mathfrak{D}$. Indeed, if $(\mathfrak{P} :_{\mathfrak{D}} w) \subsetneq I$ for some gr-n-ideal (and so $\mathfrak{A}$-gr-n-ideal) I of $\mathfrak{D}$, then $\mathfrak{P} \subseteq (\mathfrak{P} :_{\mathfrak{D}} w) \subsetneq I$ which is a contradiction. By [2, Theorem 7] , $(\mathfrak{P} :_{\mathfrak{D}} w) = Gr(0)$ is a gr-$prime$ ideal of $\mathfrak{D}$. Hence $\mathfrak{P}$ is a $\mathfrak{A}$-gr-$prime$ ideal by [10, Proposition 2.4]. □

Theorem 3.10. *Let $\mathfrak{A}$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. If $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal, and $K \leq_g^{id} \mathfrak{D}$ with $K \cap \mathfrak{A} \neq \emptyset$, then PK and $\mathfrak{P} \cap K$ are $\mathfrak{A}$-gr-n-ideals of $\mathfrak{D}$.*

Proof. Let $r, t \in h(\mathfrak{D})$ with $rt \in PK$. Since $rt \in \mathfrak{P}$ and $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal, we have $wr \in Gr(0)$ or $wt \in \mathfrak{P}$ where w is an $\mathfrak{A}$-homogenous element of $\mathfrak{P}$. Since $K \cap \mathfrak{A} \neq \emptyset$, there exists $w' \in K \cap \mathfrak{A}$. Thus $(w'w)r \in KGr(0) \subseteq Gr(0)$ or $(w'w)t \in PK$. Therefore, PK is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. The proof that $\mathfrak{P} \cap K$ is an $\mathfrak{A}$-gr-n-ideal is similar. □

Let $\mathfrak{D}$ be a G-graded ring and let $\mathfrak{A}_1, \mathfrak{A}_2 \subseteq h(\mathfrak{D})$ be two multiplicatively closed subsets of a graded ring $\mathfrak{D}$ with $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$. Let $\mathfrak{P}$ be a graded ideal such that $\mathfrak{P} \cap \mathfrak{A}_2 = \emptyset$. It is clear that if $\mathfrak{P}$ is a $\mathfrak{A}_1$-gr-n-ideal, then it is $\mathfrak{A}_2$-gr-n-ideal. The converse is not true, in Example 3.2, the graded ideal $\mathfrak{P} = \langle 4 \rangle$ is a $\mathfrak{A}_2$-gr-n-ideal for $\mathfrak{A}_2 = \{1, 3, 9\}$, but it is not an $\mathfrak{A}_1$-gr-n-ideal for $\mathfrak{A}_1 = \{1\} \subseteq \mathfrak{A}_2$.

Theorem 3.11. *Let $\mathfrak{A}_1$, $\mathfrak{A}_2 \subseteq h(\mathfrak{D})$ be two multiplicatively closed subsets of $\mathfrak{D}$ with $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq h(\mathfrak{D})$ such that for each $w \in \mathfrak{A}_2$, there is a homogenous element $w' \in \mathfrak{A}_2$ with $ww' \in \mathfrak{A}_1$. If $\mathfrak{P}$ is a $\mathfrak{A}_2$-gr-n-ideal of $\mathfrak{D}$, then $\mathfrak{P}$ is an $\mathfrak{A}_1$-gr-n-ideal of $\mathfrak{D}$.*

Proof. Let $r, t \in h(\mathfrak{D})$ such that $rt \in \mathfrak{P}$. Then there is an $\mathfrak{A}_2$-homogenous element $w \in \mathfrak{A}_2$ of $\mathfrak{P}$ satisfying $wr \in Gr(0)$ or $wt \in \mathfrak{P}$. Hence there exists $w' \in \mathfrak{A}_2$ with $s = ww' \in \mathfrak{A}_1$ and thus $wr \in Gr(0)$ or $wt \in \mathfrak{P}$. Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a multiplicatively closed subset of a graded ring $\mathfrak{D}$. □

The gr-saturation of $\mathfrak{A}$ is the set $\mathfrak{A}^* = \{w \in h(\mathfrak{D}) : \frac{w}{1}$ is unit in $\mathfrak{A}^{-1}\mathfrak{D}\}$. It is clear that $\mathfrak{A}^*$ is a mutiplicatively closed subset of $\mathfrak{D}$ and that $\mathfrak{A} \subseteq \mathfrak{A}^*$. Moreover, it is well known that $\mathfrak{A}^* = \{w \in h(\mathfrak{D}) : wt \in \mathfrak{A}$ for some $t \in h(\mathfrak{D})\}$. The set $\mathfrak{A}$ is called gr-saturated if $\mathfrak{A}^* = \mathfrak{A}$.

Theorem 3.12. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. Then $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ if and only if $\mathfrak{P}$ is an $\mathfrak{A}^*$-gr-n-ideal of $\mathfrak{D}$.*

Proof. Assume that $\mathfrak{P}$ is an $\mathfrak{A}^*$-gr-n-ideal of $\mathfrak{D}$. Let $w \in \mathfrak{A}^*$ and choose $w' \in h(\mathfrak{D})$ such that $ww' \in \mathfrak{A}$. Then $w' \in \mathfrak{A}^*$ and $ww' \in \mathfrak{A}$. By Theorem 3.11, $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. The converse is obvious. □

Let $\mathfrak{D}$ be a G-graded ring and let $\mathfrak{A}$, $T \subseteq h(\mathfrak{D})$ be two m.c.s. of $\mathfrak{D}$ with $\mathfrak{A} \subseteq T \subseteq h(\mathfrak{D})$. Then clearly, $T^{-1}\mathfrak{A} = \{\frac{s}{t} : t \in T , s \in \mathfrak{A}\} \subseteq h(T^{-1}\mathfrak{D})$ is a multiplicatively closed subset of $T^{-1}\mathfrak{D}$.

Theorem 3.13. *Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two m.c.s. of $\mathfrak{D}$ with $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq h(\mathfrak{D})$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A}_2 = \emptyset$. If $\mathfrak{P}$ is an $\mathfrak{A}_1$-gr-n-ideal of $\mathfrak{D}$, then $\mathfrak{A}_2^{-1}\mathfrak{P}$ is an $\mathfrak{A}_2^{-1}\mathfrak{A}_1$-$gr$-$n$-ideal of $\mathfrak{A}_2^{-1}\mathfrak{D}$. Moreover, we have $\mathfrak{A}_2^{-1}\mathfrak{P} \cap \mathfrak{D} = (\mathfrak{P} :_{\mathfrak{D}} u)$ for some $\mathfrak{A}$-homogenous element u of $\mathfrak{P}$.*

Proof. Assume that $\mathfrak{P}$ is an $\mathfrak{A}_1$-gr-n-ideal. If $\mathfrak{A}_2^{-1}\mathfrak{P} \cap \mathfrak{A}_2^{-1}\mathfrak{A}_1 \neq \emptyset$, then there exists $\frac{r}{l} \in h(\mathfrak{A}_2^{-1}\mathfrak{D})$ with $\frac{r}{l} \in \mathfrak{A}_2^{-1}\mathfrak{P} \cap \mathfrak{A}_2^{-1}\mathfrak{A}_1$. Hence $r \in \mathfrak{A}_1$ and $lr \in \mathfrak{P}$ for some $l \in \mathfrak{A}_2$. Since $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$, we have $lr \in \mathfrak{A}_2 \cap \mathfrak{P}$, a contradiction. So $\mathfrak{A}_2^{-1}\mathfrak{P}$ is proper in $\mathfrak{A}_2^{-1}\mathfrak{D}$ and $\mathfrak{A}_2^{-1}\mathfrak{P} \cap \mathfrak{A}_2^{-1}\mathfrak{A}_1 = \emptyset$. Let $w \in \mathfrak{A}_1$ be an $\mathfrak{A}_1$-homogenous element of $\mathfrak{P}$. Then $\frac{w}{1} \in \mathfrak{A}_2^{-1}\mathfrak{A}_1$. Now, let $r, t \in h(\mathfrak{D})$ and $w_1, w_2 \in \mathfrak{A}_2$ with $\frac{r}{w_1}\frac{t}{w_2} \in \mathfrak{A}_2^{-1}\mathfrak{P}$ and $\frac{w}{1}\frac{r}{w_1} \notin Gr(0_{\mathfrak{A}_2^{-1}\mathfrak{D}})$. So $urt \in \mathfrak{P}$ for some $u \in \mathfrak{A}_2$ and $wr \notin Gr(0)$. Then $wut \in \mathfrak{P}$ as $\mathfrak{P}$ is a $\mathfrak{A}_1$-gr-n-ideal. Hence $\frac{w}{1}\frac{t}{w_2} = \frac{wut}{uw_2} \in \mathfrak{A}_2^{-1}\mathfrak{P}$ as needed. Let $r \in \mathfrak{A}_2^{-1}\mathfrak{P} \cap h(\mathfrak{D})$ and choose $a \in \mathfrak{P} \cap h(\mathfrak{D})$, $t \in \mathfrak{A}_2$ with $\frac{r}{1} = \frac{a}{t}$. So $ur \in \mathfrak{P}$ for some $u \in \mathfrak{A}_2$. Then there exists $w \in \mathfrak{A}_1 \subseteq \mathfrak{A}_2$ with $wu \in Gr(0)$ or $wr \in \mathfrak{P}$ as $\mathfrak{P}$ is an $\mathfrak{A}_1$-gr-n-ideal. Since $\mathfrak{A}_2 \cap Gr(0) = \emptyset$, $wu \notin Gr(0)$ and hence $wr \in \mathfrak{P}$. Thus $r \in (\mathfrak{P} :_{\mathfrak{D}} w)$ for some $\mathfrak{A}_1$-homogenous element w of $\mathfrak{P}$. Since clearly $(\mathfrak{P} :_{\mathfrak{D}} w') \subseteq \mathfrak{A}_2^{-1}\mathfrak{P} \cap \mathfrak{D}$ for all $w' \in \mathfrak{A}_2$, the proof is completed. □

In particular, if $\mathfrak{A}_1 = \mathfrak{A}_2$, then all homogeneous element of $\mathfrak{A}_2^{-1}\mathfrak{A}_1$ are units in $\mathfrak{A}_2^{-1}\mathfrak{D}$. As a special case of a Theorem 3.13, we have the following.

Corollary 3.14. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. If $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$, then $\mathfrak{A}^{-1}\mathfrak{P}$ is a gr-n-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$. Moreover, we have $\mathfrak{A}^{-1}\mathfrak{P} \cap \mathfrak{D} = (\mathfrak{P} :_{\mathfrak{D}} w)$ for some $\mathfrak{A}$-homogenous element w of $\mathfrak{P}$.*

Proof. Assume that $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal. Then $\mathfrak{A}^{-1}\mathfrak{P}$ is an $\mathfrak{A}^{-1}\mathfrak{A}$-$gr$-$n$-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$ by Theorem 3.13. Let $\frac{r}{w_1}\frac{t}{w_2} \in \mathfrak{A}^{-1}\mathfrak{P}$ for some $r, t \in h(\mathfrak{D})$, $w_1, w_2 \in \mathfrak{A}$. Then $\frac{w}{u}\frac{a}{s_1} \in Gr\,(0_{\mathfrak{A}^{-1}\mathfrak{D}})$ or $\frac{w}{u}\frac{b}{s_2} \in \mathfrak{A}^{-1}\mathfrak{P}$ for some $\mathfrak{A}^{-1}\mathfrak{A}$-homogenous element $\frac{w}{u}$ of $\mathfrak{A}^{-1}\mathfrak{P}$. Since $\frac{w}{u}$ is a unit in $\mathfrak{A}^{-1}\mathfrak{D}$, then $\mathfrak{A}^{-1}\mathfrak{P}$ is a gr-n-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$ as required. A second part follows directly from Theorem 3.13. □

Corollary 3.15. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$ and and $\mathfrak{P} \leq_g^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$. Then $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ if and only if $\mathfrak{A}^{-1}\mathfrak{P}$ is a gr-n-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$, $\mathfrak{A}^{-1}\mathfrak{P} \cap \mathfrak{D} = (\mathfrak{P} :_{\mathfrak{D}} w)$ and $\mathfrak{A}^{-1}Gr\,(0) \cap \mathfrak{D} = (Gr\,(0) :_{\mathfrak{D}} w')$ for some $w, w' \in \mathfrak{A}$.*

Proof. Assume that $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. By Corollary 3.14, we have $\mathfrak{A}^{-1}\mathfrak{P}$ is a gr-n-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$. As for the other part of the implication, the approach used in the proof of Theorem 3.13 can be applied. Conversely, assume that $\mathfrak{A}^{-1}\mathfrak{P}$ is a gr-n-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$, $\mathfrak{A}^{-1}\mathfrak{P}\cap\mathfrak{D} = (\mathfrak{P} :_{\mathfrak{D}} w)$ and $\mathfrak{A}^{-1}Gr\,(0)\cap\mathfrak{D} = (Gr\,(0) :_{\mathfrak{D}} w')$ for some for some $w, w' \in \mathfrak{A}$. Choose $l = ww' \in \mathfrak{A}$. Let $rt \in \mathfrak{P}$ for some $r, t \in h\,(\mathfrak{D})$. So $\frac{r}{1}\frac{t}{1} \in \mathfrak{A}^{-1}\mathfrak{P}$ and hence $\frac{r}{1} \in Gr\left(\mathfrak{A}^{-1}0\right) = \mathfrak{A}^{-1}Gr\,(0)$ or $\frac{t}{1} \in \mathfrak{A}^{-1}\mathfrak{P}$ as $\mathfrak{A}^{-1}\mathfrak{P}$ is a gr-n-ideal of $\mathfrak{A}^{-1}\mathfrak{D}$. If $\frac{r}{1} \in Gr\left(\mathfrak{A}^{-1}0\right)$, then there is $u \in \mathfrak{A}$ such that $ur \in Gr\,(0)$. Hence, $r = \frac{ur}{u} \in \mathfrak{A}^{-1}Gr\,(0)\cap\mathfrak{D} = (Gr\,(0) :_{\mathfrak{D}} w')$. Thus, $w'r \in Gr\,(0)$ and so $lr = ww'r \in Gr\,(0)$. If $\frac{t}{1} \in \mathfrak{A}^{-1}\mathfrak{P}$, then there is $v \in \mathfrak{A}$ such that $vt \in \mathfrak{P}$ and so $t = \frac{vt}{v} \in \mathfrak{A}^{-1}\mathfrak{P}\cap\mathfrak{D} = (\mathfrak{P} :_{\mathfrak{D}} w)$ and so $lt = ww't \in \mathfrak{P}$. Therefore, $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. □

Theorem 3.16. *Let $\varphi : \mathfrak{D} \to T$ be a graded ring homomorphism and $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. $\mathfrak{D}$. Then*

(i) *If φ is a graded epimorphism and $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ containing $Ker\,(\varphi)$, then $\varphi\,(\mathfrak{P})$ is an $\varphi\,(\mathfrak{A})$-gr-n-ideal of T.*

(ii) *If $Ker\,(\varphi) \subseteq Gr\,(0_{\mathfrak{D}})$ and J is an $\varphi\,(\mathfrak{A})$-gr-n-ideal of T, then $\varphi^{-1}\,(J)$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$.*

Proof. (i) By [8, Lemma 3.11(2)], $\varphi(\mathfrak{P})$ is a graded ideal of T. Let $d \in \varphi\,(\mathfrak{P})\cap\varphi\,(\mathfrak{A})$. Then $d = \varphi\,(a) = \varphi\,(s)$ for some $a \in \mathfrak{P}\cap h(\mathfrak{D})$ and $s \in \mathfrak{A}$. So $a - s \in Ker\,(\varphi) \subseteq \mathfrak{P}$, which implies that $s \in \mathfrak{P}$, a contradiction. Thus $\varphi\,(\mathfrak{P})\cap\varphi\,(\mathfrak{A}) = \emptyset$. Now, let $tt' \in \varphi\,(\mathfrak{P})$ for some $t, t' \in h\,(T)$. Then there exist $a, b \in h\,(\mathfrak{D})$ such that $t = \varphi\,(a)$ and $t' = \varphi\,(b)$. Since $tt' = \varphi\,(a)\,\varphi\,(b) \in \varphi\,(\mathfrak{P})$ and $Ker\,(\varphi) \subseteq \mathfrak{P}$, we have $ab \in \mathfrak{P}$. Hence there exists an $s \in \mathfrak{A}$ such that $sa \in Gr\,(0_{\mathfrak{D}})$ or $sb \in \mathfrak{P}$. Hence, $\varphi\,(s)\,t \in Gr\,(0_T)$ or $\varphi\,(s)\,t' \in \varphi\,(\mathfrak{P})$. Thus, $\varphi\,(\mathfrak{P})$ is an $\varphi\,(\mathfrak{A})$-gr-n-ideal of T.

(ii) By [8, Lemma 3.11(1)], $\varphi^{-1}(J)$ is a graded ideal of $\mathfrak{D}$. Let $rr' \in \varphi^{-1}\,(J)$ for some $r, r' \in h\,(\mathfrak{D})$. So $\varphi\,(rr') = \varphi\,(r)\,\varphi\,(r') \in J$. Then there exists $\varphi\,(s) \in \varphi\,(\mathfrak{A})$ such that $\varphi\,(s)\,\varphi\,(r) \in Gr\,(0_T)$ or $\varphi\,(s)\,\varphi\,(r') \in J$ as J is an $\varphi\,(\mathfrak{A})$-gr-n-ideal of T. Hence, $sr \in Gr\,(0_{\mathfrak{D}})$ (as $Ker\,(\varphi) \subseteq Gr\,(0_{\mathfrak{D}})$) or $sr' \in \varphi^{-1}\,(J)$. □

In view of Theorem 3.16, we conclude the following result for $\bar{\mathfrak{A}}$-gr-n-ideals of $\mathfrak{D}/\mathfrak{P}$.

Corollary 3.17. *Let $\mathfrak{D}$ be a G-graded ring and let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$.*

(i) *If $\mathfrak{P} \subseteq J$ are two graded ideals of $\mathfrak{D}$ and J is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$, then $J/\mathfrak{P}$ is an $\bar{\mathfrak{A}}$-gr-n-ideal of $\mathfrak{D}/\mathfrak{P}$, where $\bar{\mathfrak{A}} = \{s + \mathfrak{P} : s \in \mathfrak{A}\}$.*

(ii) *If $\mathfrak{D}$ is a graded subring of $\mathfrak{D}'$ and $\mathfrak{P}'$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}'$, then $\mathfrak{P}'\cap\mathfrak{D}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$.*

Proof. (i) By [9, Lemma 3.2], $J/\mathfrak{P}$ is a graded ideal of $\mathfrak{D}/\mathfrak{P}$. Since $J\cap\mathfrak{A} = \emptyset$, we have $(J/\mathfrak{P})\cap\bar{\mathfrak{A}} = \emptyset$. Define $f : \mathfrak{D} \to \mathfrak{D}/\mathfrak{P}$ by $f(r) = r + \mathfrak{P}$. Then f is a graded epimorphism, and then the result follows by Theorem 3.16(i).

(ii) Apply the graded natural injection $i : \mathfrak{D} \to \mathfrak{D}'$ in Theorem 3.16(ii). □

Let $\mathfrak{D}$ be a commutative ring and M be an $\mathfrak{D}$-module. Then the idealization $\mathfrak{D}(+)M = \{(r, m) : r \in \mathfrak{D}$ and $m \in M\}$ is the ring whose elements are those of $\mathfrak{D}\times M$ equipped with addition and multiplication defined by $(r, m) + (r', m') = (r + r', m + m')$ and $(r, m)(r', m') = (rr', rm' + r'm)$ respectively. Let G be an abelian group, $\mathfrak{D} = \bigoplus_{g\in G}\mathfrak{D}_g$ be a G-graded ring and $M = \bigoplus_{g\in G} M_g$ be a G-grded $\mathfrak{D}$-module. Then $\mathfrak{D}(+)M$ is a G-graded ring with$(\mathfrak{D}(+)M)_g = \mathfrak{D}_g(+)M_g$, see [7, Proposition 3.1]. If $\mathfrak{P}$ is an ideal of $\mathfrak{D}$ and U is a submodule of M with $PM \subseteq U$. Then $\mathfrak{P}(+)U$ is a graded ideal of $\mathfrak{D}(+)M$ if and only if $\mathfrak{P}$ is a graded ideal of $\mathfrak{D}$ and U is a graded submodule of M, see [7, Proposition 3.3]. It is well known that if $\mathfrak{P}(+)U$ is a graded ideal of $\mathfrak{D}(+)M$, then $Gr(\mathfrak{P}(+)U) = Gr(\mathfrak{P})(+)M$ and in particular, $Gr(0_{\mathfrak{D}(+)M}) = Gr(0)(+)M$. If $\mathfrak{A} \subseteq h(\mathfrak{D})$ is a multiplicatively closed subset of $\mathfrak{D}$, then clearly the sets $\mathfrak{A}(+)M = \{(s, m) : s \in \mathfrak{A}, m \in M\}$and $\mathfrak{A}(+)0 = \{(s, 0) : s \in \mathfrak{A}\}$ are multiplicatively closed subsets of the ring $\mathfrak{D}(+)M$.

Theorem 3.18. *Let $\mathfrak{A} \subseteq$ be a m.c.s. of $\mathfrak{D}$, $\mathfrak{P} \leq_G^{id} \mathfrak{D}$ and U be a graded $\mathfrak{D}$-module of M such that $PM \subseteq U$. If $\mathfrak{P}(+)U$ is $\mathfrak{A}(+)M$-gr-n-ideal of $\mathfrak{D}(+)M$, then $\mathfrak{P}$ is a $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$.*

Proof. Assume that $\mathfrak{P}(+)U$ is a $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}(+)M$ and (s_g, m_g) is an $\mathfrak{A}$-homogenous element of $\mathfrak{P}(+)U$. Clearly, $\mathfrak{A} \cap \mathfrak{P} = \emptyset$. Let $rt \in \mathfrak{P}$ where $r, t \in h(\mathfrak{D})$. Then $(r, 0)(t, 0) \in \mathfrak{P}(+)U$. Since $\mathfrak{P}(+)U$ is a $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}(+)M$, we have either $(s_g, m_g)(r, 0) \in Gr(0)(+)M$ or $(s_g, m_g)(t, 0) \in \mathfrak{P}(+)N$. Hence, $s_g r \in Gr(0)$ or $s_g t \in \mathfrak{P}$ and $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$. □

Theorem 3.19. *Let $\mathfrak{A} \subseteq h(\mathfrak{D})$ be a m.c.s. of $\mathfrak{D}$, $\mathfrak{P} \leq_G^{id} \mathfrak{D}$ with $\mathfrak{P} \cap \mathfrak{A} = \emptyset$ and M be graded $\mathfrak{D}$-module. The following are equivalent.*

(i) $\mathfrak{P}$ *is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$.*

(ii) $\mathfrak{P}(+)M$ *is an $\mathfrak{A}(+)0$-gr-n-ideal of $\mathfrak{D}(+)M$.*

(iii) $\mathfrak{P}(+)M$ *is an $\mathfrak{A}(+)M$-gr-n-ideal of $\mathfrak{D}(+)M$.*

Proof. $(i) \Rightarrow (ii)$. Assume that $\mathfrak{P}$ is an $\mathfrak{A}$-gr-n-ideal of $\mathfrak{D}$ and w is $\mathfrak{A}$-homogenous element of $\mathfrak{P}$. Then $\mathfrak{P}(+)U$ is a graded ideal of $\mathfrak{D}(+)M$ by [7, Proposition 3.3]. Clearly, $\mathfrak{A}(+)0 \cap \mathfrak{P}(+)M = \emptyset$. Let $(r_g, m_g)(t_h, m'_h) \in \mathfrak{P}(+)M$ where $(r_g, m_g), (t_h, m'_h) \in h(\mathfrak{D}(+)M)$. So $r_g t_h \in \mathfrak{P}$ and hence either $wr_g \in Gr(0)$ or $wt_h \in \mathfrak{P}$. Thus $(w, 0)(r_g, m_g) \in Gr(0)(+)M = Gr(0_{\mathfrak{D}(+)M})$ or $(w, 0)(t_h, m'_h) \in \mathfrak{P}(+)M$. Therefore, $\mathfrak{P}(+)M$ is an $\mathfrak{A}(+)0$-gr-n-ideal of $\mathfrak{D}(+)M$ and $(w, 0)$ is a $\mathfrak{A}(+)0$-homogenous element of $\mathfrak{P}(+)M$.

$(ii) \Rightarrow (iii)$. It is clear.

$(iii) \Rightarrow (i)$. By Theorem 3.18. □

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Author information

Ala'a Abu Qayass, Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, Jordan.
E-mail: `awabuqayass20@sci.just.edu.jo`

Khaldoun Al-Zoubi, Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, Jordan.
E-mail: `kfzoubi@just.edu.jo`

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