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NUMERICAL SOLUTION OF FIFTH ORDER INITIAL VALUE PROBLEMS USING BLOCK METHOD WITH GENERALISED POINTS

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Abstract This article develops a block method of order six to solve fifth order initial value problems (IVPs) directly. This implies that the fifth order IVPs considered in this article were not reduced to a system of first order IVPs before solving. The schemes in the developed block method were derived using a linear block approach with the basic properties for convergence and stability satisfied. Since the method has generalised points, the steplength with efficient accuracy was investigated while also comparing with existing studies. From the numerical results obtained, it was observed that the developed method performs well as its results were quite close to the exact solution and better than existing methods.

1 Introduction

Numerical methods are constantly being introduced with the main aim of obtaining more accurate approximate solutions to differential equation models [1, 2, 3, 4]. Studies such as [5, 6, 7] have explored solving first, second, third, and fourth orders differential equations in various domains. The objective of this article is to develop a new method with good accuracy for solving fifth order initial value problems (IVPs) of the form

$$y^{(5)} = f\left(x, y, y', y'', y''', y^{(4)}\right);$$
$$y\left(x_0\right) = a, y'\left(x_0\right) = b, y''\left(x_0\right) = c, y'''\left(x_0\right) = d, y^{(4)}\left(x_0\right) = f;$$
$$x_0 \in \left[x_0, x_n\right]. \quad (1.1)$$

Although few articles consider the numerical solution of fifth order IVPs, the quest for approximate solutions with better accuracy than existing approaches is still being studied by recent literature. [8], and [9] developed a multistep block method with better accuracy than inbuilt MATLAB ODE45 software, conventional linear multistep methods (LMM), and analytical solutions. The one-step hybrid block method of order six by [10] performed better than the implicit LMM by [3] and order eight one-step hybrid method by [11]. Other studies that have shown improved accuracy over other numerical approaches for solving fifth order IVPs include [12] and [13] whose generalised Runge-Kutta method integrators and general implicit block method solved fifth order IVPs efficiently.

This article aims to develop a numerical approach with better accuracy than these existing studies and thus develops a block method with generalised points for solving fifth order IVPs. Linear block approach is adopted for the development of the method and its basic properties to ensure convergence and stability are investigated before implementing the method to solve problems in the form of Equation (1.1). The flow chart for the whole manuscript which summarizes the phases in the development and implementation of the proposed method in this article are displayed in Figure 1 below:

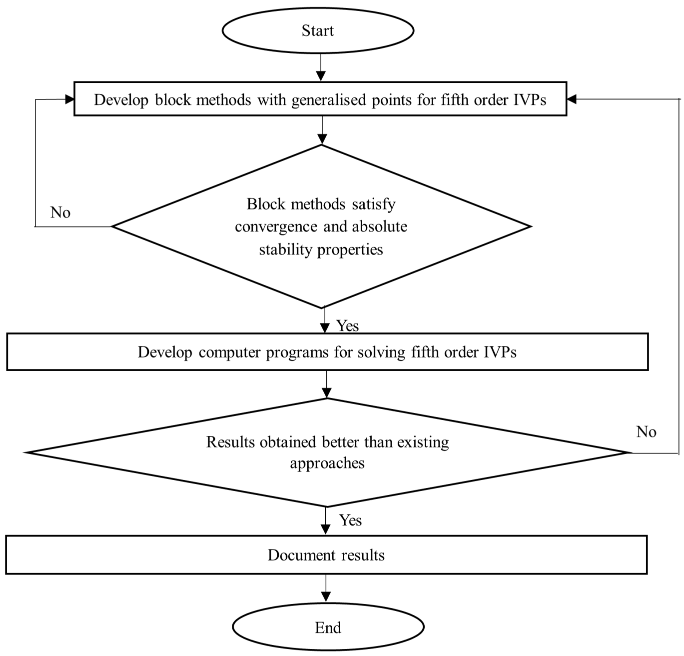

Figure 1. Flowchart of Development and Implementation of Developed Method

Section 2 discusses the phase of the development of the block method with generalised points for solving fifth order initial value problems while Section 3 tests if the developed method satisfies the convergence and absolute stability conditions of linear multistep methods. Having justified the properties in Section 3, sample numerical problems are considered and comparison is made with existing studies. The computation of results and conclusion of the study are documented in Sections 4 and 5 respectively.

2 Methodology

A block method is developed in this article with generalised points using a linear block approach and taking the form

$$y_{n+\xi r}=\sum_{i=0}^{4}\frac{(\xi rh)^{i}}{i!}y_{n}^{(i)}+\sum_{i=0}^{5}\phi_{\xi i}f_{n+ri},\ \xi=1,\ 2,\ldots,5 \tag{2.1}$$

for the main scheme, while its required derivative schemes are obtained from

$$y_{n+\xi r}^{(\alpha)}=\sum_{i=0}^{5-(\alpha+1)}\frac{(\xi rh)^{i}}{i!}y_{n}^{(i+\alpha)}+\sum_{i=0}^{5}\omega_{\xi i\alpha}f_{n+ri}\ ;$$
$$\alpha=1_{(\xi=1,\,2,\,\cdots,\,5)},\ 2_{(\xi=1,\,2,\,\cdots,\,5)},\ldots,\ 5_{(\xi=1,\,2,\,\cdots,\,5)}. \tag{2.2}$$

In Equations (2.1) and (2.2), h is the step-size and r is the equal distance between consecutive points. The value of the steplength is selected as 5 to produce a block method of order 6 which will be of the same or lesser order than the existing methods used for comparison in the numerical results section. This follows the proposition by [14] which states that the order of a -step block method of the form $A^{0}Y_{n+\xi}=A^{i}Y_{n-\xi}+\sum_{i=1}^{m-1}B^{i}Y_{n-\xi}^{(i)}+h^{m}\left(C^{0}Y_{n+\xi}^{(m)}+C^{i}Y_{n-\xi}^{(m)}\right)$ is $\xi+1$. In that regard, the linear block approach is adopted to obtain the values of $\phi_{\xi i}$ and $\omega_{\xi i\alpha}$ in Equations

(2.1) and (2.2) respectively.
The resultant main scheme of the block method is thus obtained as:

$$y_{n+r} = y_n + (rh)\,y'_n + \tfrac{(rh)^2}{2!}y''_n + \tfrac{(rh)^3}{3!}y'''_n + \tfrac{(rh)^4}{4!}y^{(4)}_n + \tfrac{(rh)^5}{7257600}(42157f_n + 34845f_{n+1r} - 29530f_{n+2r} + 18830f_{n+3r} + 6915f_{n+4r} + 1093f_{n+5r}) \quad (2.3)$$

$$y_{n+2r} = y_n + (2rh)\,y'_n + \tfrac{(2rh)^2}{2!}y''_n + \tfrac{(2rh)^3}{3!}y'''_n + \tfrac{(2rh)^4}{4!}y^{(4)}_n + \tfrac{(rh)^5}{28350}(3853f_n + 6035f_{n+1r} - 4110f_{n+2r} + 2570f_{n+3r} - 935f_{n+4r} + 147f_{n+5r}) \quad (2.4)$$

$$y_{n+3r} = y_n + (3rh)\,y'_n + \tfrac{(3rh)^2}{2!}y''_n + \tfrac{(3rh)^3}{3!}y'''_n + \tfrac{(3rh)^4}{4!}y^{(4)}_n + \tfrac{(rh)^5}{89600}(71253f_n + 152685f_{n+1r} - 78570f_{n+2r} + 52110f_{n+3r} - 19035f_{n+4r} + 2997f_{n+5r}) \quad (2.5)$$

$$y_{n+4r} = y_n + (4rh)\,y'_n + \tfrac{(4rh)^2}{2!}y''_n + \tfrac{(4rh)^3}{3!}y'''_n + \tfrac{(4rh)^4}{4!}y^{(4)}_n + \tfrac{(rh)^5}{14175}(38336f_n + 97920f_{n+1r} - 35840f_{n+2r} + 29440f_{n+3r} - 10560f_{n+4r} + 1664f_{n+5r}) \quad (2.6)$$

$$y_{n+5r} = y_n + (5rh)\,y'_n + \tfrac{(5rh)^2}{2!}y''_n + \tfrac{(5rh)^3}{3!}y'''_n + \tfrac{(5rh)^4}{4!}y^{(4)}_n + \tfrac{(rh)^5}{290304}(2003125f_n + 5703125f_{n+1r} - 1406250f_{n+2r} + 1718750f_{n+3r} - 546875f_{n+4r} + 88125f_{n+5r}). \quad (2.7)$$

The first derivative schemes are obtained as

$$y'_{n+r} = y'_n + (rh)\,y''_n + \tfrac{(rh)^2}{2!}y'''_n + \tfrac{(rh)^3}{3!}y^{(4)}_n + \tfrac{(rh)^4}{1814400}(49126f_n + 49045f_{n+1r} - 40160f_{n+2r} + 25430f_{n+3r} - 9310f_{n+4r} + 1469f_{n+5r}) \quad (2.8)$$

$$y'_{n+2r} = y'_n + (2rh)\,y''_n + \tfrac{(2rh)^2}{2!}y'''_n + \tfrac{(2rh)^3}{3!}y^{(4)}_n + \tfrac{(rh)^4}{14175}(4264f_n + 7960f_{n+1r} - 4910f_{n+2r} + 3080f_{n+3r} - 1120f_{n+4r} + 176f_{n+5r}) \quad (2.9)$$

$$y'_{n+3r} = y'_n + (3rh)\,y''_n + \tfrac{(3rh)^2}{2!}y'''_n + \tfrac{(3rh)^3}{3!}y^{(4)}_n + \tfrac{(rh)^4}{22400}(25488f_n + 63315f_{n+1r} - 26460f_{n+2r} + 19170f_{n+3r} - 7020f_{n+4r} + 1107f_{n+5r}) \quad (2.10)$$

$$y'_{n+4r} = y'_n + (4rh)\,y''_n + \tfrac{(4rh)^2}{2!}y'''_n + \tfrac{(4rh)^3}{3!}y^{(4)}_n + \tfrac{(rh)^4}{14175}(40448f_n + 116480f_{n+1r} - 29440f_{n+2r} + 33280f_{n+3r} - 11360f_{n+4r} + 1792f_{n+5r}) \quad (2.11)$$

$$y'_{n+5r} = y'_n + (5rh)\,y''_n + \tfrac{(5rh)^2}{2!}y'''_n + \tfrac{(5rh)^3}{3!}y^{(4)}_n + \tfrac{(rh)^4}{72576}(418250f_n + 1315625f_{n+1r} - 175000f_{n+2r} + 418750f_{n+3r} - 106250f_{n+4r} + 18625f_{n+5r}). \quad (2.12)$$

Similarly, the second, third, and fourth derivative schemes are given below

$$y''_{n+r} = y''_n + (rh)\,y'''_n + \tfrac{(rh)^2}{2!}y^{(4)}_n + \tfrac{(rh)^3}{40320}(3929f_n + 4975f_{n+1r} - 3862f_{n+2r} + 2422f_{n+3r} - 883f_{n+4r} + 139f_{n+5r}) \quad (2.13)$$

$$y''_{n+2r} = y''_n + (2rh)\, y'''_n + \tfrac{(2rh)^2}{2!} y^{(4)}_n + \tfrac{(rh)^3}{630} (317 f_n + 734 f_{n+1r} - 380 f_{n+2r} + 244 f_{n+3r} - 89 f_{n+4r} + 14 f_{n+5r}) \quad (2.14)$$

$$y''_{n+3r} = y''_n + (3rh)\, y'''_n + \tfrac{(3rh)^2}{2!} y^{(4)}_n + \tfrac{(rh)^3}{4480} (5481 f_n + 16119 f_{n+1r} - 4374 f_{n+2r} + 4230 f_{n+3r} - 1539 f_{n+4r} + 243 f_{n+5r}) \quad (2.15)$$

$$y''_{n+4r} = y''_n + (4rh)\, y'''_n + \tfrac{(4rh)^2}{2!} y^{(4)}_n + \tfrac{(rh)^3}{315} (712 f_n + 2336 f_{n+1r} - 224 f_{n+2r} + 704 f_{n+3r} - 200 f_{n+4r} + 32 f_{n+5r}) \quad (2.16)$$

$$y''_{n+5r} = y''_n + (5rh)\, y'''_n + \tfrac{(5rh)^2}{2!} y^{(4)}_n + \tfrac{(rh)^3}{8064} (29125 f_n + 101875 f_{n+1r} + 1250 f_{n+2r} + 38750 f_{n+3r} - 4375 f_{n+4r} + 1375 f_{n+5r}) \quad (2.17)$$

$$y'''_{n+r} = y'''_n + (rh)\, y^{(4)}_n + \tfrac{(rh)^2}{10080} (2462 f_n + 4315 f_{n+1r} - 3044 f_{n+2r} + 1882 f_{n+3r} - 682 f_{n+4r} + 107 f_{n+5r}) \quad (2.18)$$

$$y'''_{n+2r} = y'''_n + (2rh)\, y^{(4)}_n + \tfrac{(rh)^2}{630} (355 f_n + 1088 f_{n+1r} - 370 f_{n+2r} + 272 f_{n+3r} - 101 f_{n+4r} + 16 f_{n+5r}) \quad (2.19)$$

$$y'''_{n+3r} = y'''_n + (3rh)\, y^{(4)}_n + \tfrac{(rh)^2}{1120} (984 f_n + 3501 f_{n+1r} - 72 f_{n+2r} + 870 f_{n+3r} - 288 f_{n+4r} + 45 f_{n+5r}) \quad (2.20)$$

$$y'''_{n+4r} = y'''_n + (4rh)\, y^{(4)}_n + \tfrac{(rh)^2}{315} (376 f_n + 1424 f_{n+1r} + 176 f_{n+2r} + 608 f_{n+3r} - 80 f_{n+4r} + 16 f_{n+5r}) \quad (2.21)$$

$$y'''_{n+5r} = y'''_n + (5rh)\, y^{(4)}_n + \tfrac{(rh)^2}{2016} (3050 f_n + 11875 f_{n+1r} + 2500 f_{n+2r} + 6250 f_{n+3r} + 1250 f_{n+4r} + 275 f_{n+5r}) \quad (2.22)$$

$$y^{(4)}_{n+r} = y^{(4)}_n + \tfrac{(rh)}{1440} (475 f_n + 1427 f_{n+1r} - 798 f_{n+2r} + 482 f_{n+3r} - 173 f_{n+4r} + 27 f_{n+5r}) \quad (2.23)$$

$$y^{(4)}_{n+2r} = y^{(4)}_n + \tfrac{(rh)}{90} (28 f_n + 129 f_{n+1r} + 14 f_{n+2r} + 14 f_{n+3r} - 6 f_{n+4r} + f_{n+5r}) \quad (2.24)$$

$$y^{(4)}_{n+3r} = y^{(4)}_n + \tfrac{(rh)}{160} (51 f_n + 219 f_{n+1r} + 114 f_{n+2r} + 114 f_{n+3r} - 21 f_{n+4r} + 3 f_{n+5r}) \quad (2.25)$$

$$y^{(4)}_{n+4r} = y^{(4)}_n + \tfrac{(rh)}{45} (14 f_n + 64 f_{n+1r} + 24 f_{n+2r} + 64 f_{n+3r} + 14 f_{n+4r}) \quad (2.26)$$

$$y^{(4)}_{n+5r} = y^{(4)}_n + \tfrac{(rh)}{288} (95 f_n + 375 f_{n+1r} + 250 f_{n+2r} + 250 f_{n+3r} + 375 f_{n+4r} + 95 f_{n+5r}). \quad (2.27)$$

The next section discusses the basic properties to ensure convergence and stability of the method.

3 Properties of the Developed Method

The properties examined for the derived block method in this section will include those required to assure convergence criterion and absolute stability. For the properties that assure convergence, a linear multistep method is said to be convergent if and only if it is consistent and zero-stable. Zero-stability guarantees that the block method's solution converges in the limit as h tends to zero. This indicates that the numerical solution converges to the exact/precise solution as h approaches zero. Additionally, the method's consistency shows that the local truncation error goes to zero faster than the step length, h. The local truncation error denoted by LTE is defined as the difference between the method's result y_{n+s} and the precise solution of the equation at a given mesh point x_{n+s} provided that all prior numbers up to and including $y_{n+s-1}, \ldots, y_n$ are exact.

In addition, a linear multistep method is said to be absolutely stable for a given $\overline{h}$ if, for that $\overline{h}$, all the roots r_s of the stability polynomial $\pi(r, \overline{h}) = \rho(r) - \overline{h}\sigma(r) = 0$, satisfy $|r_s| < 1, s = 1, 2, \ldots, k$ [1]. Furthermore, an interval (α, β) of the real line is said to be an interval of stability if the method is absolutely stable for all $\overline{h} \in (\alpha, \beta)$. A region R_A of the complex plane is said to be a region of absolute stability if the method is stable for all $\overline{h}$ in R_A.

This section will demonstrate how the derived technique satisfies these convergence and absolute stability properties.

3.1 Zero-Stability Property of the Developed Block Method

According to [1], a linear multi-step method (LMM) is said to be zero-stable if it satisfies the root condition. In other words, LMM is said to be zero-stable if no root of the 1st characteristic polynomial has modulus greater than one, and if every root with modulus one is simple (not repeated). Therefore, to analyse the zero-stability property of the derived block method, it will be shown that the root of the first characteristic polynomial

$$\delta(\sigma) = \left| \sigma I_5 - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| \tag{3.1}$$

must be less than unity and simple.
Thus,

$$\delta(\sigma) = \left| \sigma \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = \sigma^4 (\sigma - 1) \tag{3.2}$$

Solving Equation (3.2), one obtains

$$\delta(\sigma) = \sigma^4 (\sigma - 1) = 0 \tag{3.3}$$

Therefore, $\sigma = 0, 0, 0, 0, 1$. This shows that the derived block method is zero stable, since $\delta(\sigma)$ has roots satisfying $|\sigma_i| \leq 1$.

3.2 Order of the Developed Block Method

To obtain the order of accuracy of the derived method, the values and values in the equation

$$y_{n+\xi r} - \sum_{i=0}^{4} \frac{(\xi r h)^i}{i!} y_n^{(i)} - \sum_{i=0}^{5} \phi_{\xi i} f_{n+ri} = 0,\ \xi = 1,\ 2, \ldots, 5 \tag{3.4}$$

are evaluated using Taylor series expansions about the point $x = x_n$ to obtain the order of the integrators of the block method to be of order $[6, 6, 6, 6, 6]^T$. Therefore, the block method is consistent since each integrator has order greater than 1.

3.3 Convergence Property of the Developed Block Method

The derived block method is consistent and zero stable, hence it is convergent.

3.4 Absolute Stability Property of the Developed Block Method

Considering the correctors of the block method in Equation(2.1), the stability region is obtained by plotting the loci of the roots of the stability polynomial

$$\pi(q;z) = -(q^{\xi}) + \sum_{i=0}^{m-1} \frac{(z\xi)^i}{i!} + z^m \left(\sum_{i=0}^{k} \phi_{i\xi} q^i \right), \xi = 1, 2, \ldots, k \tag{3.5}$$

for $z = e^{i\theta}$ as θ ranges from 0 to 2π. The resultant stability polynomial of the developed block method is obtained as

$$\begin{aligned}\pi(q;z) = {} & \tfrac{1}{224\,042\,112} q^{15} r^{25} z^{25} + \tfrac{2108\,233}{1234\,517\,760\,000} q^{15} r^{20} z^{20} + \tfrac{447\,985}{1170\,505\,728} q^{15} r^{15} z^{15} + \tfrac{38\,293}{2177\,280} q^{15} r^{10} z^{10} \\ & - q^{15} + \tfrac{18\,245}{5377\,010\,688} q^{10} r^{25} z^{25} + \tfrac{377\,479}{9217\,732\,608} q^{10} r^{24} z^{24} + \tfrac{34\,270\,373}{138\,265\,989\,120} q^{10} r^{23} z^{23} + \tfrac{7425\,157}{8062\,156\,800} q^{10} r^{22} z^{22} \\ & + \tfrac{24\,446\,851\,691}{11\,851\,370\,496\,000} q^{10} r^{21} z^{21} + \tfrac{193\,211\,143\,529}{138\,265\,989\,120\,000} q^{10} r^{20} z^{20} - \tfrac{1345\,765\,625}{147\,483\,721\,728} q^{10} r^{19} z^{19} - \tfrac{16\,705\,625}{390\,168\,576} q^{10} r^{18} z^{18} \\ & - \tfrac{2147\,840\,375}{21\,069\,103\,104} q^{10} r^{17} z^{17} - \tfrac{518\,068\,525}{3511\,517\,184} q^{10} r^{16} z^{16} - \tfrac{318\,943\,705}{3511\,517\,184} q^{10} r^{15} z^{15} + \tfrac{1051\,375}{8128\,512} q^{10} r^{14} z^{14} \\ & + \tfrac{65\,217\,925}{146\,313\,216} q^{10} r^{13} z^{13} + \tfrac{157\,105}{193\,536} q^{10} r^{12} z^{12} + \tfrac{265\,967}{193\,536} q^{10} r^{11} z^{11} + \tfrac{2910\,541}{1088\,640} q^{10} r^{10} z^{10} + \tfrac{390\,625}{72\,576} q^{10} r^{9} z^{9} \\ & + \tfrac{78\,125}{8064} q^{10} r^{8} z^{8} + \tfrac{15\,625}{1008} q^{10} r^{7} z^{7} + \tfrac{3125}{144} q^{10} r^{6} z^{6} + \tfrac{625}{24} q^{10} r^{5} z^{5} + \tfrac{625}{24} q^{10} r^{4} z^{4} + \tfrac{125}{6} q^{10} r^{3} z^{3} + \tfrac{25}{2} q^{10} r^{2} z^{2} \\ & + 5 q^{10} r z + q^{10}\end{aligned} \tag{3.6}$$

The region of absolute stability is determined by plotting the roots of the polynomial as shown below.

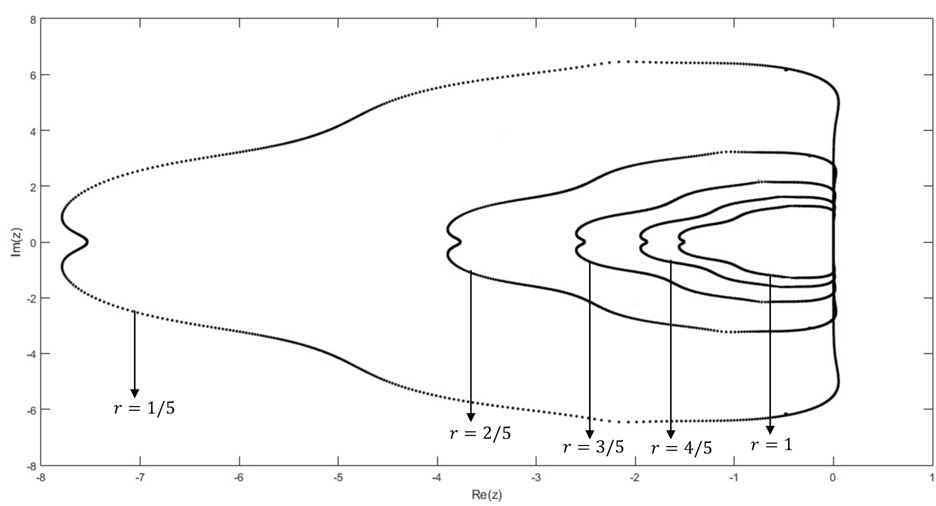

Figure 2. Region of Absolute Stability for Two-Step Block Method for Second Order ODEs

Figure 2 shows the absolute stability region of the developed method for the values of r where $r = [r_1, r_2, r_3, r_4, r_5] = \left[\frac{(^1/_5)}{5} : \frac{(^1/_5)}{5} : \frac{1}{5} \right]$ as likewise adopted in the numerical examples. According to [1], the interval of absolute stability cannot include the positive real line in the neighbourhood of the origin, hence Figure 2 shows that the developed method is absolutely stable for all values of r.

4 Numerical Examples and Results

In the section, two objectives are considered. The first objective is to compare the efficiency of the values of r where $r = [r_1, r_2, r_3, r_4, r_5] = \left[\frac{(1/5)}{5} : \frac{(1/5)}{5} : \frac{1}{5}\right]$ for solving various linear and nonlinear fifth order IVPs, and the second objective is to compare solutions to existing studies with developed methods of either equal or higher order. The following problems were solved.
Problem 1 [8]

$$y^{(5)} = -\left(\cos x + \sin x\right);$$
$$y(0) = y'(0) = 1, y''(0) = -2, y'''(0) = 1, y^{(4)}(0) = 2. \quad (4.1)$$

Exact Solution: $y(x) = 2x - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cos x - \sin x$.
Problem 2 [10]

$$y^{(5)} = 2y'y'' - yy^{(4)} - y'y''' - 8x + \left(x^2 - 2x - 3\right)e^x;$$
$$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, y^{(4)}(0) = 1. \quad (4.2)$$

Exact Solution: $y(x) = e^x + x^2$.
Problem 3 [8]

$$y^{(5)} = y^{(4)} + y' - y;$$
$$y(0) = 0, y'(0) = 0, y''(0) = 1, y'''(0) = 0, y^{(4)}(0) = 0. \quad (4.3)$$

Exact Solution: $y(x) = \frac{1}{2}\left(\cosh x - \cos x\right)$.
Problem 4 [13]

$$y^{(5)} = \cos x;$$
$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = -1, y^{(4)}(0) = 0. \quad (4.4)$$

Exact Solution: $y(x) = \sin x$.

Tables 1 and 3 show comparison of the order 6 hybrid method developed in this article with the order 10 method by [8] (denoted by JM), while Table 2 shows a comparison of the order 6 hybrid method developed in this article with the method by [10] (denoted by DM) of same order and the problem whose results were displayed in Table 4 was not compared with any study, although the problem was obtained from [13] but the results were presented graphically and not numerically in [13]. Figures

Table 1. Comparison of Results for Problem 1 with $h = 0.1$

x	Exact Solution	Absolute Error (JM)	Absolute Error $(r = r_1)$	Absolute Error $(r = r_2)$	Absolute Error $(r = r_3)$	Absolute Error $(r = r_4)$	Absolute Error $(r = r_5)$
0.06	1.0564370734557595	-	0.000000e+00	-	0.000000e+00	-	0.000000e+00
0.08	1.0736887190001134	-	0.000000e+00	-	-	0.000000e+00	0.000000e+00
0.10	1.0901749152978641	4.884981e-15	-	-	-	-	2.220446e-16
0.20	1.1614639137128471	9.103829e-15	2.220446e-16	-	-	-	0.000000e+00
0.30	1.2151537824642666	1.776357e-15	0.000000e+00	-	0.000000e+00	-	0.000000e+00
0.40	1.2527093183609015	7.704948e-14	0.000000e+00	0.000000e+00	-	0.000000e+00	0.000000e+00
0.50	1.2757611899528365	9.992007e-15	4.440892e-16	-	-	-	2.220446e-16
0.60	1.2860931415146433	1.199041e-14	2.220446e-16	2.220446e-16	4.440892e-16	-	0.000000e+00
0.70	1.2856286667134640	1.909584e-14	4.440892e-16	-	-	-	6.661338e-16
0.80	1.2764172851143094	1.110223e-15	0.000000e+00	2.220446e-16	-	0.000000e+00	1.110223e-15
0.90	1.2606205586431809	5.995204e-15	4.440892e-16	-	6.661338e-16	-	1.332268e-15
1.00	1.2404979877269100	6.106227e-14	0.000000e+00	2.220446e-16	-	-	2.664535e-15
1.02	1.2361592693022867	-	0.000000e+00	-	8.881784e-16	-	3.108624e-15
1.04	1.2317601366561062	-	4.440892e-16	8.881784e-16	-	4.440892e-16	3.552714e-15
MAXE $x = (0:1)$		7.704948e-14	8.881784e-16	8.881784e-16	8.881784e-16	6.661338e-16	2.886580e-15

Table 2. Comparison of Results for Problem 2 with $h = 0.01$

x	Exact Solution	Absolute Error (DM)	Absolute Error $(r = r_1)$	Absolute Error $(r = r_2)$	Absolute Error $(r = r_3)$	Absolute Error $(r = r_4)$	Absolute Error $(r = r_5)$
0.10	1.1151709180756477	1.459721e-12	4.440892e-16	2.220446e-16	-	-	2.220446e-16
0.20	1.2614027581601699	4.187584e-11	2.220446e-16	0.000000e+00	-	2.220446e-16	2.220446e-16
0.30	1.4398588075760033	3.221776e-10	0.000000e+00	2.220446e-16	4.440892e-16	-	0.000000e+00
0.40	1.6518246976412709	1.365175e-09	1.110223e-15	6.661338e-16	-	4.440892e-16	4.440892e-16
0.50	1.8987212707001291	4.166737e-09	8.881784e-16	2.220446e-16	-	-	8.881784e-16
0.60	2.1821188003905103	1.033164e-08	8.881784e-16	0.000000e+00	1.332268e-15	1.332268e-15	1.332268e-15
0.70	2.5037527074704782	2.218906e-08	0.000000e+00	0.000000e+00	-	-	1.776357e-15
0.80	2.8655409284924698	4.288554e-08	4.440892e-16	4.440892e-16	-	1.776357e-15	2.664535e-15
0.90	3.2696031111569526	7.645583e-08	0.000000e+00	1.332268e-15	2.664535e-15	-	3.108624e-15
1.00	3.7182818284590486	1.278750e-07	1.776357e-15	2.220446e-15	-	3.552714e-15	4.884981e-15
MAXE $x =$ $(0:1)$		1.278750e-07	2.220446e-15	2.664535e-15	3.996803e-15	4.440892e-15	4.884981e-15

Table 3. Comparison of Results for Problem 1 with $h = 0.1$

x	Exact Solution	Absolute Error (JM)	Absolute Error $(r = r_1)$	Absolute Error $(r = r_2)$	Absolute Error $(r = r_3)$	Absolute Error $(r = r_4)$	Absolute Error $(r = r_5)$
0.06	0.001800000064800023	-	2.190088e-17	-	2.233456e-17	-	2.168404e-17
0.08	0.003200000364088962	-	6.938894e-17	6.852158e-17	-	6.852158e-17	6.852158e-17
0.10	0.005000001388888897	1.099988e-14	1.994932e-17	-	-	-	1.994932e-17
0.20	0.020000088888917134	2.997602e-15	2.081668e-17	2.081668e-17	-	-	2.081668e-17
0.30	0.045001012501627247	1.399575e-14	0.000000e+00	-	1.387779e-17	-	6.938894e-18
0.40	0.080005688917784978	1.799949e-14	9.714451e-17	9.714451e-17	-	2.775558e-17	5.551115e-17
0.50	0.125021701658004190	1.099121e-14	1.110223e-16	-	-	-	8.326673e-17
0.60	0.180064801666294960	1.900036e-14	2.220446e-16	2.498002e-16	5.551115e-17	-	1.110223e-16
0.70	0.245163409173227710	1.498801e-14	3.330669e-16	-	-	-	8.326673e-17
0.80	0.320364118478840050	1.898481e-14	4.440892e-16	4.440892e-16	-	1.110223e-16	1.665335e-16
0.90	0.405738208589055520	1.199041e-14	4.996004e-16	-	3.330669e-16	-	1.665335e-16
1.00	0.501389164473552640	4.996004e-15	4.440892e-16	4.440892e-16	-	-	2.220446e-16
1.02	0.521764450408519930	-	4.440892e-16	-	4.440892e-16	-	4.440892e-16
1.04	0.542557795461196420	-	4.440892e-16	5.551115e-16	-	2.220446e-16	3.330669e-16
MAXE x = (0 : 1)		1.900036e-14	7.21645e-16	6.106227e-16	4.996004e-16	2.220446e-16	2.220446e-16

Table 4. Comparison of Results for Problem 4 with $h = 0.01$

x	Exact Solution	Absolute Error $(r = r_1)$	Absolute Error $(r = r_2)$	Absolute Error $(r = r_3)$	Absolute Error $(r = r_4)$	Absolute Error $(r = r_5)$
0.10	0.099833416646828224	6.938894e-17	5.551115e-17	-	-	1.387779e-17
0.20	0.198669330795061360	1.387779e-16	1.942890e-16	-	8.326673e-17	0.000000e+00
0.30	0.295520206661339770	1.665335e-16	2.775558e-16	1.110223e-16	-	0.000000e+00
0.40	0.389418342308650740	2.220446e-16	2.775558e-16	-	3.330669e-16	1.665335e-16
0.50	0.479425538604203280	2.220446e-16	2.775558e-16	-	-	5.551115e-17
0.60	0.564642473395035700	2.220446e-16	4.440892e-16	6.661338e-16	4.440892e-16	1.110223e-16
0.70	0.644217687237691460	4.440892e-16	6.661338e-16	-	-	1.110223e-16
0.80	0.717356090899523120	7.771561e-16	7.771561e-16	-	4.440892e-16	2.220446e-16
0.90	0.783326909627483860	6.661338e-16	8.881784e-16	5.551115e-16	-	1.110223e-16
1.00	0.841470984807896840	9.992007e-16	6.661338e-16	-	3.330669e-16	1.110223e-16
MAXE $x = (0:1)$		6.938894e-17	9.992007e-16	7.771561e-16	6.661338e-16	3.330669e-16

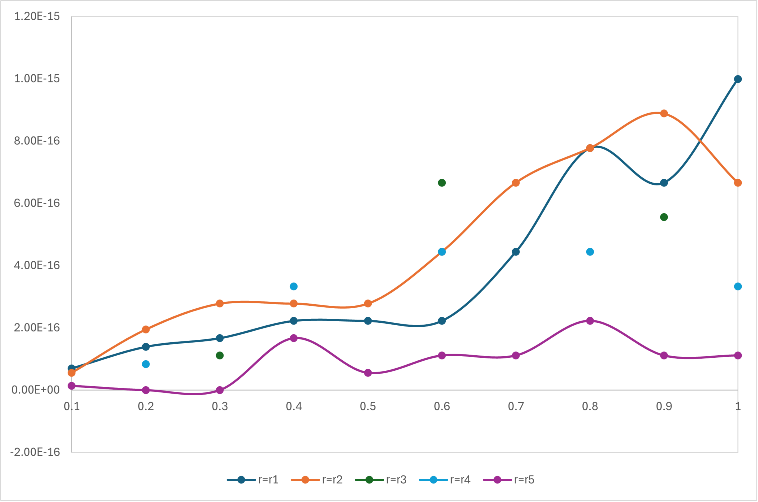

Figure 3. Solution of Problem 4 for $r = [r_1, r_2, r_3, r_4, r_5]$

In Tables 1–4 and Figure 3, the newly developed block method performed better than the existing methods compared with. Furthermore, regarding the different values of r, $r = r_4$ had the least maximum error for Problem 1, closely followed by $r = r_1$, $r = r_2$, and $r = r_3$. For Problem 2, $r = r_1$ had the least maximum error, followed by the other values, while in Problem 3 $r = r_4$ and $r = r_5$ had the least maximum error followed by r_3, and finally $r = r_1$ had the least maximum error in Problem 4 followed $r = r_4$. Overall, $r = r_1$ is the better performing among the considered $r = [r_1, r_2, r_3, r_4, r_5] = \left[\frac{\left({}^1/_5\right)}{5} : \frac{\left({}^1/_5\right)}{5} : \frac{1}{5}\right]$. This accuracy is also affirmed in Figure 3.

5 Conclusion

A new method for solving fifth order IVPs is developed in this article. The IVPs considered included both linear and nonlinear fifth order IVPs and the results displayed the new approach being of better accuracy than existing studies. In addition, information is given about the best value of in the interval considered. Moreover, it is worth mentioning that models describing real-life scenarios may not have exact solutions and this article has shown that the developed method has impressive accuracy compared to the exact solutions. Thus, the method will give accurate results when adopted to solve fifth order IVP models void of exact solutions. In a nutshell, the new hybrid block method is suitable method for solving fifth order IVPs with and without exact solutions.

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