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Weak e-reversible rings

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Abstract We introduce the notion of weak e-reversible rings. It is proved that the classes of e-reversible and weak e-symmetric rings are properly contained in the class of weak e-reversible rings. Basic properties and some characterizations of the notion of weak e-reversible are provided. In particular, we show that R is a weak e-reversible ring if and only if $eR(1-e)Re = 0$ and eRe is a reversible ring. As an application, we show that a ring R is a left min-abel ring if and only if R is a weak e-reversible ring for any $e \in ME_l(R)$. Furthermore, we introduce the notion of a weak e-reduced ring and study some properties of it. Finally, we investigate the conditions under which weak e-reversibility holds in some ring extensions.

1 Introduction

Throughout this article, all rings are associative and noncommutative with identity unless otherwise stated. We denote the center, the set of all nilpotent elements, and the set of all idempotent elements of a ring R by $Z(R)$, $N(R)$ and $E(R)$, respectively. Let $M_n(R)$, $T_n(R)$ be the ring of all $n \times n$ matrices, and upper triangular matrices over the ring R, respectively.

An element r of a ring R is central if $ar = ra$ for all $a \in R$, and R is said to be abelian if every idempotent is central. An idempotent e of R is called right (resp., left) semicentral if for each $a \in R, ea = eae$ (resp., $ae = eae$). A ring R is said to be semiabelian if every idempotent of R is either left semicentral or right semicentral. A ring R is called reduced if it has no nonzero nilpotent elements. Cohn in [4] called a ring R reversible if $ab = 0$ implies $ba = 0$ for all $a, b \in R$. In fact, reversible property lies between "commutative" and "2-primal" properties. Cohn shows that the Köthe Conjecture is true for the class of reversible rings. Lambek in [18] introduced a stronger condition than "reversible" which he calls symmetric. A ring R is called symmetric if, for all $a, b, c \in R$, $abc = 0$ implies $acb = 0$. Equivalently, whenever a product of any number of elements is zero, any permutation of the factors still yields product zero. It is clear that symmetric rings are reversible but the converse is not true in general (see [20, Example 5]). Idempotent elements are important tools for studying the structure of a ring. In [21], the authors extended the notions of symmetric and reduced via idempotent elements of the rings, namely, e-symmetric and e-reduced, respectively. A ring R is called e-symmetric if $abc = 0$ implies $acbe = 0$ for all $a, b, c \in R$. A ring R is called right (resp., left) e-reduced if $N(R)e = 0$ (resp., $eN(R) = 0$). Clearly, reduced rings are left and right e-reduced. It is proved that right e-reduced rings are e-symmetric (see [21, Corollary 4.3]). Following this perspective, the authors in [13] studied a version of reversibility depending on idempotent elements, namely, right (resp., left) e-reversible rings. A ring R is called right e-reversible (resp., left e-reversible) if for any $a, b \in R, ab = 0$ implies $bae = 0$ (resp., $eba = 0$). The ring R is e-reversible if it is both left and right e-reversible. It has been shown that the class of e-reversible contains the classes of e-reduced rings and e-symmetric rings.

In [22], the authors introduced a ring R to be weak e-symmetric if $abc = 0$ implies $eacbe = 0$

for all $a, b, c \in R$. Obviously, R is a symmetric ring if and only if R is a weak 1-symmetric. It has been shown that e-symmetric ring is weak e-symmetric but the converse is not true in general (see [22, Corollary 2.4 and Remark 2.5]). In the light of aforementioned concepts and inspired by the work in [22], we introduce the notions of weak e-reversible and weak e-reduced rings. For $e \in E(R)$, we call a ring R is weak e-reversible if for $a, b \in R$, where $ab = 0$ then $ebae = 0$. In Section 2, we study the basic properties and give some characterizations of weak e-reversible. In particular, we show that the class of weak e-symmetric and the class of e-reversible rings are properly contained in the class of weak e-reversible rings (Proposition 2.2). Further, we provide examples of weak e-reversible which are not e-reversible (Example 2.3) and not weak e-symmetric (Example 2.4). In Section 3, we introduce the notion of weak e-reduced rings. A ring R is called weak e-reduced if $eN(R)e = 0$. Among other results, we show that a weak e-reduced ring is weak e-symmetric (Proposition 3.3). We prove that over a prime ring, the classes of weak e-reduced, weak e-symmetric, and e-reversible coincide (Proposition 3.7). In Section 4, we study the weak e-reversible ring property of several kinds of ring extensions, for instance upper triangular matrices $T_n(R)$, polynomial rings $R[X]$, power series rings $R[[x]]$, and the Laurent polynomial rings $R[[x, x^{-1}]]$ with an indeterminate x over a ring R.

2 Some properties of weak e-reversible rings

Motivated by [22], in this section, we introduce the notion of weak e-reversible rings and study its basic properties. Examples are provided to show that the class of weak e-reversible properly contains the classes of e-reversible and that of e-symmetric (Proposition 2.2). Furthermore, we give characterizations of weak e-reversible (Theorem 2.8 and Proposition 2.14). We discuss some properties of weak e-reversible rings that will be in use through our study (Proposition 2.15 and Proposition 2.21). It is known that if a ring is reversible, then every idempotent is central, while if a ring is right e-reversible, then e is left semicentral idempotent (see [13, Theorem 2.9]). Unlike the previous cases, in the case of weak e-reversible, we show that any idempotent isomorphic to left or right semicentral idempotent is left or right semicentral (Theorem 2.18). Finally, we investigate the relation of weak e-reversible and other classes of rings (Theorem 2.16 and Theorem 2.23).

Definition 2.1. Let R be a ring and $e \in E(R)$. A ring R is called weak e-reversible if for any $a, b \in R$, $ab = 0$ implies $ebae = 0$. Obviously, R is a reversible ring if and only if R is a weak 1-reversible ring.

The following result provides a source of examples for weak e-reversible rings.

Proposition 2.2. *For any ring R, the following conditions hold:*
(1) Every one-sided e-reversible ring is weak e-reversible.
(2) Every weak e-symmetric ring is weak e-reversible.

Proof. (1) It is obvious.
(2) It is clear by (1), since every e-symmetric ring is right e-reversible ring, noting that $ab = 1ab = 0$ implies $bae = 1bae = 0$. □

The following examples show that the converse of Proposition 2.2 is not generally true.

Example 2.3. (i) Consider a ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Let $a, b \in R$ with $ab = 0$, then $ba = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ where $x \in \mathbb{Z}$. Assume $x \neq 0$, then $eba \neq 0$ and so R is not left e-reversible(see [13, Example 2.3]). However, $ebae = 0$, R is weak e-reversible.

(ii) Consider the ring $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$. Therefore R is weak e-reversible but not right e-reversible for $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Indeed, if $xy = 0$ for $x, y \in R$, then $yx = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$.

Assume $b \neq 0$, then $yxe \neq 0$ while $eyxe = 0$. Hence R is not right e-reversible but weak e-reversible.

Example 2.4. Let R be a reversible ring which is not symmetric. Let $x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2(R)$, then $xe = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = exe$ for $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore e is left semicentral, and so by [22, Remark 2.3] we have $eR(1-e)Re = 0$. Since $R \cong eT_2(R)e$, $eT_2(R)e$ is not symmetric. By [22, Theorem 2.2], $T_2(R)$ is not week e-symmetric. On the other hand, by [13, Example 2.6], for a reversible ring R, the ring $T_2(R)$ is right e-reversible but not reversible. Also, $T_2(R)$ is not left e-reversible but is weak e-reversible. Indeed, let $x, y \in T_2(R)$ such that $xy = 0$. Then $eyx = \begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix} \neq 0$ while $eyxe = 0$.

The following result provides a condition under which the class of weak e-reversible coincides with that of e-reversible.

Proposition 2.5. *Let R be a ring and $e \in E(R)$. A ring R is weak e-reversible and e is left (right) semicentral idempotent if and only if R is right (left) e-reversible.*

Proof. Assume that R is weak e-reversible and e is left semicentral idempotent. Let $ab = 0$ for $a, b \in R$, then $ebae = 0 = bae$. Therefore R is right e-reversible. The converse is true by Proposition 2.2 and [13, Theorem 2.9]. □

The following result shows that the notion of weak e-reversible inherits by its subrings.

Lemma 2.6. *Let S be any subring of a ring R and $e \in E(S)$. If R is weak e-reversible ring, then S is weak e-reversible.*

Lemma 2.7. *Let $(R_i)_{i \in I}$ be a family of rings and $(e_i)_{i \in I} \in E(\prod_{i \in I} R_i)$. Then $\prod_{i \in I} R_i$ is weak $(e_i)_{i \in I}$-reversible ring if and only if for each $i \in I$, R_i is weak e_i-reversible ring.*

Proof. Necessity: Let $i \in I$ and $a_i, b_i \in R_i$ with $a_i b_i = 0$. Consider $a = (0, 0, ..., a_i, ..., 0, 0), b = (0, 0, ..., b_i, ..., 0, 0) \in R = \prod_{i \in I} R_i$. Then $ab = 0$. Since R is weak $(e_i)_{i \in I}$-reversible ring, $ebae = 0$ for $e = (e_i)_{i \in I} \in E(\prod_{i \in I} R_i)$. Consequently, $e_i b_i a_i e_i = 0$. Therefore R_i is weak e_i-reversible.
Sufficiency: Let $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I} \in R$ such that $ab = 0$. Then we have $a_i b_i = 0$ for each $i \in I$. Since R_i is weak e_i-reversible, $e_i b_i a_i e_i = 0$ for each $i \in I$ and $e_i \in E(R_i)$. Consequently, $ebae = 0$ and therefore R is weak e-reversible. □

Lam in [16] defined a ring idempotent $e \in R$ to be quarter-central (or q-central for short) if $eR(1-e)Re = 0$. Also, he called a ring R to be quarter-abelian (or q-abelian for short) if all idempotents in R are q-central. In the following result, we show that e is q-central and the corner ring eRe inherits the abelianness property if the ring R is weak e-reversible.

Theorem 2.8. *Let R be a ring. Then R is a weak e-reversible ring if and only if e is q-central and eRe is a reversible ring.*

Proof. ($\Rightarrow$) Assume that R is a weak e-reversible ring. Let $x, y \in eRe$ such that $x = eae$, $y = ebe$, and $xy = 0$. Then $eaebe = 0 = ebeae$, i.e, $yx = 0$. So eRe is a reversible ring. Now let $h = ea - eae$, $he = 0, eh = h$. Noting that $rhe = 0$ for all $r \in R$. Since R is a weak e-reversible ring, then $ehre = 0 = hre$. Therefore, $ea(1-e)re = 0$ and so for all $a, r \in R$, we have $eR(1-e)Re = 0$ and so e is q-central.
($\Leftarrow$) Suppose that $ab = 0$ for $a, b \in R$. Then we have $ea(1-e)be = 0$, $eb(1-e)ae = 0$, and so $eaebe = 0$. Since eRe is reversible, $ebeae = 0 = ebae = 0$. Therefore, R is weak e-reversible. □

Corollary 2.9. *Let R be a ring. If eRe is a reversible ring and R is a q-abelian ring, then R is a weak e-reversible.*

The following example shows that there is a weak e-reversible ring that is not q-abelian.

Example 2.10. Let D be a division ring and $R = \begin{pmatrix} D & D & D \\ 0 & D & D \\ 0 & 0 & D \end{pmatrix}$. Consider the idempotent $e = e_{11} + e_{33}$ where $e_{ij} \in R$ denote the matrix unit whose (i, j)-th entry is 1 and the entries are zero. We have $eR(1-e)Re = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ (see [27, P. 1858]), then e is not q-central and so R is not q-abelian. Now, let $a, b \in R$ with $ab = 0$, then $ba = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$ for $x, y, z \in D$. Assume $x, y, z \neq 0$, then $e_{11}bae_{11} = 0$ for $e_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, R is weak e_{11}-reversible.

In [14], a ring R is said to be left idempotent reflexive if $eRa = 0$ implies $aRe = 0$ for all $a \in R$ and $e \in E(R)$. Clearly, abelian rings are left idempotent reflexive. The following result provides a condition under which a weak e-reversible ring is abelian.

Theorem 2.11. *Let R be a ring and $e \in E(R)$. If R is a weak e-reversible and left idempotent reflexive ring, then e is a central idempotent element.*

Proof. Since R is weak e-reversible and by Theorem 2.8, e is q-central i.e., $eR(1-e)Re = 0$. As R is left idempotent reflexive, $(1-e)Re = 0$. Again using the left idempotent reflexivity of R, we have $eR(1-e) = 0$. Therefore, e is a central idempotent element. □

As a consequence of Theorem 2.11, we conclude that if a ring R is weak e-reversible and left idempotent reflexive for all $e \in E(R)$, then R is abelian. The following example shows that the converse of the above result is not true, in general.

Example 2.12. Consider the following ring

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d(\text{mod}2); b \equiv c \equiv 0(\text{mod}2); a, b, c, d \in \mathbb{Z} \right\}$$

Then R is an abelian ring because $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotent elements in R (see [6, Example 2.14]). Hence R is left idempotent reflexive, while R is not weak e-reversible ring for $e = I_2$. Indeed, take $x = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $y = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in R$, then $xy = 0$ but $eyxe = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \neq 0$.

A ring R is called von Neumann regular, if for every a in R there exists x in R such that $a = axa$. The following result provides a condition for which abelian ring is weak e-reversible.

Proposition 2.13. *Let R be a von Neumann regular abelian ring. Then R is a weak e-reversible ring for all $e \in E(R)$.*

Proof. Since an abelian von Neumann regular ring R is reduced, so R is reversible. Hence R is weak e-reversible for each idempotent $e \in R$. □

The following result provides a characterization for a weak e-reversible ring in terms of subsets.

Proposition 2.14. *Let R be a ring, then the following are equivalent:*
(1) R is a weak e-reversible ring.
(2) $AB = 0$ implies $eBAe = 0$ for any nonempty subsets A and B of R.

Proof. $(1) \Rightarrow (2)$ Assume that R is weak e-reversible and $AB = 0$ for any nonempty subsets A and B of R. Consequently, for any $a \in A$ and $b \in B$ we have $ab = 0$. Since R is weak e-reversible, $ebae = 0$. Therefore we get $eBAe = \sum_{b\in B, a\in A} ebae = 0$.
$(2) \Rightarrow (1)$ It is clear. □

Recall that, an idempotent is called full idempotent if $ReR = R$.

Proposition 2.15. *Let R be a ring, then the following statements hold:*
(1) If R is weak e-reversible, then $eabe = eaebe$ for $a, b \in R$.
(2) Assume that R is a weak e-reversible ring. If e is a full idempotent element in R, then $e = 1$.
(3) If R is a weak e-reversible ring and M is a maximal left ideal or maximal right ideal in R, then we have either $e \in M$ or $1 - e \in M$.

Proof. (1) From Theorem 2.8, we have e is q-central and so $eR(1 - e)Re = 0$. Here, the expression $eR(1 - e)Re$ is intended to denote the set of all finite sums $\sum_i er_i(1 - e)s_ie$ where $r_i, s_i \in R$. For any $r, s \in R$, then $er(1 - e)se = 0$, Hence $erse = erese$.
(2) Since R is weak e-reversible, by Theorem 2.8, $eR(1 - e)Re = 0$. Since $ReR = R, R(1 - e)R = ReR(1 - e)ReR = 0$. Therefore $e = 1$.
(3) Let M be a maximal right ideal in R and assume that $e, (1-e) \notin M$, then there exist $r, r' \in R$ and $m, m' \in M$ such that $1 = er + m$ also $1 = (1 - e)r' + m'$. Hence by using (1), we have $e = (1 - e)r'e + m'e = (er + m)(1 - e)r'e + m'e = err'e - erer'e + mr'e - mer'e + m'e = mr'e - mer'e + m'e \in M$, a contradiction. A similar proof can be provided if M is a maximal left ideal in R. □

Recall that, a ring R is directly finite if for $a, b \in R$ with $ab = 1$, then $ba = 1$. It is known that every reversible ring is directly finite. In the following result, we generalize this statement and show that every weak e-reversible ring is directly finite.

Theorem 2.16. *For a ring R. If R is a weak e-reversible ring for all $e \in E(R)$, then R is directly finite.*

Proof. Since R is weak e-reversible for all $e \in E(R)$, then by Theorem 2.8, we have e is q-central for all $e \in E(R)$. Consequently, R is q-abelian. Therefore R is directly finite by [16, Propsition 2.6(2)]. □

The following example shows that the converse of Theorem 2.16 is not true, in general.

Example 2.17. Consider $R = M_2(\mathbb{Z}_2)$ and let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then we have $AB = I$ and $BA = I$. Therefore, R is directly finite, while R is not weak e-reversible for $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Recall that, two idempotent elements e and f are said to be isomorphic if $eR \cong fR$ as right R-modules. Equivalently, two idempotent elements e and f are isomorphic if and only if there are elements $a, b \in R$ such that $e = ab$ and $f = ba$ (for more information see [15, Page 292]). In [13, Theorem 2.9], it has been shown that: “A ring R is right e-reversible if and only if eRe is reversible and e is left semicentral idempotent". In the case of a weak e-reversible ring, we get the following result.

Theorem 2.18. *For a ring R. If R is weak e-reversible for all $e \in E(R)$, then any idempotent isomorphic to left or right semicentral idempotent is left or right semicentral.*

Proof. By using Theorem 2.16 we have, R is directly finite. Now apply [19, Theorem 6]. □

Proposition 2.19. *Let $e, f \in E(R)$ such that $e = ef, f = fe$ and e is a right semicentral idempotent element. Then R is a weak f-reversible ring if and only if R is a weak e-reversible ring.*

Proof. ($\Rightarrow$) Let $a, b \in R$ such that $ab = 0$, then $abe = 0$. Since R is weak f-reversible, $fbeaf = 0$. As e is right semicentral we have $fbeaef = 0$. Multiplying e from the left, $efbeaef = 0$. Since $ef = e$, we have $ebeae = 0$. Again using e as being right semicentral, $ebae = 0$. Therefore, R is weak e-reversible.
($\Leftarrow$) Let $a, b \in R$ such that $ab = 0$. Since R is weak e-reversible, $ebae = 0$. Multiplying both sides by f, $febaef = 0$. Since e is right semicentral and $f = fe$, we get $fbaf = 0$. Therefore, R is weak f-reversible. □

In the following result, we present an alternative condition for the previously stated result to hold.

Proposition 2.20. *Let R be a ring and $e, f \in E(R)$ such that $eR = fR$. Then R is weak e-reversible if and only if R is weak f-reversible.*

Proof. Let R be weak e-reversible, then by Theorem 2.8 we have e is q-central and eRe is a reversible ring. Hence $eR(1-e)Re = 0$. By using [15, Exercise 21.4] and right multiplying by f, we have $fR(1-f)Ref = 0$. Thus $fR(1-f)Rf = 0$ and so f is q-central. Therefore, R is weak f-reversible. Conversely, let R be weak f-reversible, hence $fR(1-f)Rf = 0$. Then $eR(1-e)Rf = 0$, right multiplying by e we have $eR(1-e)Rfe = 0$. Since $e = fe$, this leads to $eR(1-e)Re = 0$. Therefore, R is weak e-reversible. □

Proposition 2.21. *For a ring R. If R is weak e-reversible for $e \in E(R)$. Then for any $a, b \in R$ and $ab \in E(R)$, implies $ebae \in E(R)$.*

Proof. Assume that $a, b \in R$ with $ab \in E(R)$. Then we have $a(1-ba)b = 0$. Since R is weak e-reversible, $eba(1-ba)e = 0$. Now by Proposition 2.15 (1), we get $ebae = ebabae = ebaebae$. Therefore, $ebae \in E(R)$. □

The following example shows that the converse of the aforementioned result does not necessarily hold true in general.

Example 2.22. Consider $R = M_2(\mathbb{Z}_2)$. Let $a, b \in R$ such that $a = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $e = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in E(R)$. Then we have $ab = 0 \in E(R)$, and $ebae = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in E(R)$. But R is not week e-reversible since $ebae \neq 0$.

Recall from [3], a ring R is called strongly e-reversible if for any $a, b \in R, ab = 0$ implies $bea = 0$. In the following result, we provide a condition under which a strongly e-reversible ring is weak e-reversible.

Theorem 2.23. *A ring R is strongly e-reversible if and only if R is weak e-reversible and an idempotent reflexive ring.*

Proof. ($\Rightarrow$) Assume that $ab = 0$ for any $a, b \in R$. Since R is strongly e-reversible ring, $bea = 0$. Now by [3, Theorem 2.5], e is a central idempotent. Therefore, R is weak e-reversible and left and right idempotent reflexive, i.e., R is idempotent reflexive.
($\Leftarrow$) Let R be a weak e-reversible and an idempotent reflexive ring. By Theorem 2.8, we have $eR(1-e)Re = 0$, and eRe is a reversible ring. Suppose that $eR(1-e) \neq 0 \neq (1-e)Re$. Then there exists $0 \neq x \in eR(1-e)$ and $0 \neq y \in (1-e)Re$. Since $xRe = 0 = eRy$ and R is idempotent reflexive, $eRx = 0 = yRe$. Then $ex = ye = 0$. But $x = ex$ and $y = ye$, we have $x = 0 = y$, a contradiction. Hence $eR(1-e) = 0 = (1-e)Re$, and so e is a central element. Therefore, R is a strongly e-reversible ring. □

Recall that, a ring R is called clean if every element in R is the sum of a unit and an idempotent. In [7], the authors asked the following question: "If R is a clean ring and $e \in E(R)$, is the ring eRe clean?". In general, if R is a clean ring, then eRe need not be clean (see [24, Example 3.4]). In the following results, we show if R is weak e-reversible, the corner ring is clean.

Proposition 2.24. *Let R be a weak e-reversible ring for $e \in E(R)$. If $a \in R$ is clean, then eae is clean.*

Proof. Since a is clean, there exist $f \in E(R)$ and $u \in U(R)$ such that $a = f + u$. Then $eae = efe + eue$. Since R is a weak e-reversible ring, by Proposition 2.15 (1) we have $(efe)^2 = efe$. On the other hand, $(eue - (1-e))(eu^{-1}e - (1-e)) = 1$. Consequently, we have $eae = (efe + (1-e)) + (eue - (1-e))$. Therefore eae is clean. □

Theorem 2.25. *Let R be a weak e-reversible ring for $e \in E(R)$. If R is clean, then eRe is clean.*

Proof. It follows directly from Proposition 2.24. □

Let R be a ring, an idempotent element $e \in E(R)$ is called a left minimal idempotent if the left ideal Re is minimal. Assume that $ME_l(R) = \{e \in E(R)$ such that Re is minimal left ideal of $R\}$. A ring R is called left min-abel if either $ME_l(R) = \phi$ or every element of $ME_l(R)$ is left semicentral (for more information see [25]).

Theorem 2.26. *For a ring R, the following conditions are equivalent:*
(1) R is a left min-abel ring.
(2) R is weak e-symmetric for each $e \in ME_l(R)$.
(3) R is weak e-reversible for each $e \in ME_l(R)$.

Proof. (1) $\Longleftrightarrow$ (2) It follows from [22, Theorem 2.15].
(2) $\Longrightarrow$ (3) It follows from Proposition 2.2.
(3) $\Longrightarrow$ (1) Let $e \in ME_l(R)$ and $h = (1-e)re$ for some $r \in R$. If $h \neq 0$, then $eh = 0$ and $he = h$. Since R is weak e-reversible and by Theorem 2.8, $eR(1-e)Re = 0$. By hypotheses Re is minimal left ideal of R, $Rh = Re$ and $eRh = eRe$. Hence $er'(1-e)re = ere \neq 0$ for some $r, r' \in R$, a contradiction. Hence $h = 0 = (1-e)re$ and so e is left semicentral. Therefore, R is a left min-abel ring. □

According to [17], a ring R is called a left (or a right) quasi-duo if every maximal left (or right) ideal of R is an ideal, respectively. A ring R is called MELT if every essential maximal left ideal of R is an ideal. Clearly, every left quasi-duo ring is MELT ring. In [25, Theorem 1.2], it has been shown that R is a left quasi-duo ring if and only if R is a left min-abel MELT ring. We get the following result.

Theorem 2.27. *For a ring R, the following are equivalent:*
(1) R is a left quasi-duo ring.
(2) R is weak e-reversible MELT ring for each $e \in ME_l(R)$.

Proof. (1) $\Rightarrow$ (2) Let R be a left quasi-duo ring. By [25, Theorem 1.2], R is a left min-abel MELT ring. Now apply Theorem 2.26, we have R is weak e-reversible.
(2) $\Rightarrow$ (1) Suppose that R is weak e-reversible MELT ring for each $e \in ME_l(R)$. By Theorem 2.26, R is a left min-abel ring. Therefore R is a left quasi-duo ring by [25, Theorem 1.2]. □

Recall from [25], a ring R is called strongly left min-abel if for every left minimal idempotent element $e \in R$, $Re = eR$.

Proposition 2.28. *A ring R is strongly left min-abel if and only if R is e-reversible for any $e \in ME_l(R)$.*

Proof. Let R be a strongly left min-abel ring and $ab = 0$ for $a, b \in R$. Then R is a left min-abel by [25, Theorem 1.8]. From Theorem 2.26, R is weak e-reversible ring for $e \in ME_l(R)$. Then we have $ebae = 0$. Since e is a central idempotent, we get $eba = 0$, therefore R is e-reversible. Conversely, let R is e-reversible for any $e \in ME_l(R)$, e is semicentral. By Theorem 2.26, R is a left min-abel ring. Now we show that e is a central idempotent element. For any $r \in R$, let $h = er(1-e)$, then $h = eh$ and $he = 0$. By hypothesis, $ehe = 0 = he$. Since R is a left min-abel ring and $e \neq 0$, then $h = 0$, so $er = ere = re$. Hence e is a central idempotent element and so R is a strongly left min-abel ring. □

Theorem 2.29. *Let R be a ring with an ideal I and $e \in E(R)$. If R/I is weak $\bar{e}$-reversible and I is a reduced ring, then R is weak e-reversible.*

Proof. Let $a, b \in R$ with $ab = 0$. Then $\bar{a}\bar{b} = \bar{0}$ in R/I. Since R/I is weak $\bar{e}$-reversible, then $\bar{e}\bar{b}\bar{a}\bar{e} = \bar{0}$. So $ebae \in I$. By Proposition 2.15 (1), $(ebae)^2 = ebaebae = ebabae = 0$. Consequently, $ebae = 0$ as I is reduced. Therefore, R is weak e-reversible. □

Let $S(R)$ be the nonempty set of all the proper ideals of R generated by central idempotent elements. Recall that, if P is a maximal element of the set $S(R)$, then the factor ring R/P is called a Pierce stalk of R (for more information see [10]).

Proposition 2.30. *Let R be a ring and $e \in E(R)$. Then the following conditions are equivalent:*
(1) R is a weak e-reversible ring.
(2) R/I is a weak $\bar{e}$-reversible ring for any ideal I which is generated by central idempotent elements of R.
(3) R/P is a weak $\bar{e}$-reversible ring for any pierce stalk ideal P of R.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$ such that $\bar{a}\bar{b} = \bar{0}$ and I is an ideal generated by central idempotent elements. Then $ab \in I$ and so there exists a central idempotent element f of R such that $ab \in Rf = I$. Then $ab(1-f) = 0$. Since R is a weak e-reversible ring, $eb(1-f)ae = 0$. Since f is a central idempotent element, $ebae(1-f) = 0$. This implies $ebae = ebaef \in Rf = I$ and so $\bar{e}\bar{b}\bar{a}\bar{e} = \bar{0}$. Therefore, R/I is a weak $\bar{e}$-reversible ring.
(2) $\Rightarrow$ (3) It is clear.
(3) $\Rightarrow$ (1) Assume that R is not a weak e-reversible ring, then there exists $a, b \in R$ such that $ab = 0$ but $ebae \neq 0$. Define $S(R) = \{J | J$ is a proper ideal of R generated by central idempotent elements and $ebae \neq J\}$. Obviously, $0 \in S(R)$, $S(R)$ is a nonempty set. Clearly, $(S(R), \geq)$ is a partially ordered set defined by $J_1 \geq J_2$ if and only if $J_1 \supseteq J_2$ for any $J_1, J_2 \in S(R)$. Noting that the partially ordered set $(S(R), \geq)$ is inductive. Then by Zorn's Lemma, $S(R)$ contains a maximal element. Let P be a maximal element, then P is a Pierce stalk ideal of R. By (3), R/P is a weak $\bar{e}$-reversible ring. Since $\bar{a}\bar{b} = \bar{0}, \bar{e}\bar{b}\bar{a}\bar{e} = \bar{0}$. Hence $ebae \in P$, a contradiction. Therefore R is a weak e-reversible ring. □

Proposition 2.31. *Let R be a weak e-reversible ring, then the following conditions hold:*
(1) $\alpha : R \to R$ defined by $\alpha(r) = ere$ where $r \in R$ is an endomorphism,
(2) If $ab = 0$, then $\alpha(b)\alpha(a) = 0$ for any $a, b \in R$.

Proof. (1) Assume that R is weak e-reversible. For $a, b \in R$, we have $\alpha(a+b) = \alpha(a) + \alpha(b)$ and $\alpha(ab) = eabe = eaebe = \alpha(a)\alpha(b)$. Therefore, α is an endomorphism.
(2) Let $ab = 0$ for $a, b \in R$. Since R is weak e-reversible, then $0 = ebae = ebeae = \alpha(b)\alpha(a)$. □

Let R be a ring and $a, d \in R$. If there exists $y \in R$ such that $y \in dR \cap Rd$ and $yad = d = day$, then a is called invertible along d, and y is called the inverse of a along d. It is well known that such y is unique and written usually by $a^{||d}$. A ring R is called weakly left idempotent reflexive if $ae = 0$ implies $ea = 0$ for all $a \in R$ and left semi-central idempotent e of R (for more information see [26]). Clearly, abel rings are weakly left idempotent reflexive.

Theorem 2.32. *Let R be a weak e-reversible ring. Then R is weakly left idempotent reflexive if and only if for any $a \in R$, $a^{||e}$ exists to imply $e = a^{||e}a$.*

Proof. ($\Longrightarrow$) Let $a^{||e} = y$. Then $y = ey = ye$ and $yae = e = eay$, this gives $(ay-1)e = 0$. Since R is a weak e-reversible ring, $e(ay-1)e = 0$, it follows that $(eya - e)e = 0$. Hence, $(ya - e)e = 0$. Noting that R is a weakly idempotent reflexive ring. Then $e(ya - e) = 0$, this gives $e = ya = a^{||e}a$.
($\Longleftarrow$) Assume that $a \in R$ with $ae = 0$. If $ea \neq 0$. Set $g = e + ea$. Then $eg = g, ge = e$ and $g^2 = g$. Clearly, $g^{||e} = e$. Then, by hypothesis, one has $e = ge = g$, that is, $ea = 0$, which is a contradiction. Hence, $ea = 0$. □

3 Weak e-reduced rings

Recall that, a ring R is called reduced if it has no nonzero nilpotent elements. A ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. In [21], the notions of e-reduced and e-symmetric rings are introduced as a generalization of reduced and symmetric rings, respectively. A ring R is called e-symmetric if $abc = 0$ implies $acbe = 0$ for all $a, b, c \in R$, and a ring R is called right (resp., left) e-reduced if $N(R)e = 0$ (resp., $eN(R) = 0$) for $e \in E(R)$. It has been proved that right e-reduced rings are e-symmetric. In this section, we introduce the notion of weak e-reduced which contains the class of e-reduced rings. Some properties and examples of it are provided.

Definition 3.1. Let R be a ring, R is called weak e-reduced if $eN(R)e = 0$ for $e \in E(R)$.

Clearly, every one-sided e-reduced ring is weak e-reduced but the converse is not true in the following example.

Example 3.2. Let F be a field and $R = T_3(F)$. Then $N(R) = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ and $N(R)e = 0$ while $eN(R) \neq 0$ for $e = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (See [21, Example 4.1]). Consequently, R is right e-reduced but not left e-reduced. Therefore R is weak e-reduced as $eN(R)e = 0$.

Proposition 3.3. *For a ring R. If R is a weak e-reduced ring, then R is weak e-symmetric.*

Proof. Let $abc = 0$, then $(cab)^2 = cabcab = 0$. Since R is weak e-reduced ring, $ecabe = 0$. By [22, Corollary 2.9], R is weak e-symmetric ring. □

The following example shows that the converse of the above result is not true in, general.

Example 3.4. Let R be a symmetric ring. Then $T_2(R)$ is e-symmetric for $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ (see [21, Example 3.6]) and so R is weak e-symmetric. On the other hand, R is not weak e-reduced. Indeed, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$, where $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a nilpotent element in $T_2(R)$.

The following diagram summarizes the relationships between the concepts of weak e-reversible, weak e-reduced, and other related classes of rings.

$$\begin{array}{ccccc} \text{Reduced} & \Longrightarrow & e\text{-reduced} & \Longrightarrow & \text{weak } e\text{-reduced} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Symmetric} & \Longrightarrow & e\text{-symmetric} & \Longrightarrow & \text{weak } e\text{-symmetric} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Reversible} & \Longrightarrow & e\text{-reversible} & \Longrightarrow & \text{weak } e\text{-reversible.} \end{array}$$

Proposition 3.5. *Let R be a ring. If R is a weak e-reduced ring, then eRe is a reduced ring and e is q-central.*

Proof. Suppose that R is a weak e-reduced ring. Since $(1-e)ae \in N(R)$ for any $a \in R$, then $eN(R)e = 0$. Since $N(eRe) \subseteq eN(R)e = 0$. Hence $N(eRe) = 0$. Therefore eRe is a reduced ring. Now let $h = er(1-e)$ which is a nilpotent element in R since $h^2 = 0 = er(1-e)er(1-e)$. As R is a weak e-reduced ring, we have $er(1-e)er(1-e)e = 0$, hence $er(1-e)he = 0$ for any $r, h \in R$. Then $eR(1-e)Re = 0$ and so e is q-central. □

Theorem 3.6. *Let R be a ring. If R is a left min-abel ring, then R is a weak e-reduced ring for each $e \in ME_l(R)$.*

Proof. Assume that R is left min-abel and $e \in ME_l(R)$. If $eN(R)e \neq 0$, then there exists $a \in N(R)$ such that $eae \neq 0$, say $h = eae$. Since R is left min-abel $Re = Rh$ (i.e., $Re = Reae$). Hence there exists $b \in R$ such that $e = beae$. Since e is left semicentral, we have $e = bae$. By [21, Proposition 2.4], $e = abe$. Consequently, $e = bae = ba(abe) = ba^2be = bea^2be = b^2a^2e = ... = b^na^neb^na^ne = ...$ for each $n \geq 1$. Since $a \in N(R), e = 0$, a contradiction. Hence $eN(R)e = 0$ and therefore R is weak e-reduced. □

Recall that, a ring R is called prime if for any $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$. It is natural to ask when the classes of weak e-reversible, weak e-symmetric, and weak e-reduced rings coincide. The following result provides an answer to that question.

Proposition 3.7. *For a prime ring R, the following are equivalent:*
(1) R is weak e-reversible.
(2) R is weak e-symmetric.
(3) R is weak e-reduced.

Proof. (1) $\iff$ (3) Let $a^n = 0$ for $a \in R$. We may assume that n is even and $n = 2t$. Since $a^n = a^ta^t = 0$, then $a^ta^tr = 0$. Since R is weak e-reversible $(ea^t)R(a^te) = 0$. By primness of R, $a^te = 0$. Again we may assume that $t = 2k$. Similarly, $a^ke = 0$. Continuing this way, we may reach $ea^2e = 0 = eaaeR$. Hence $(eae)R(eae) = 0$. As R is prime again, we have $eae = 0$. Therefore, R is weak e-reduced. The converse follows directly by using Proposition 3.3 and Proposition 2.2.
(1) $\iff$ (2) Let $abc = 0 = abcrec$ for any $r \in R$. Since R is weak e-reversible, we have $ecrecabe = 0$. By hypothesis $ecabe = 0$, and so R is weak e-symmetric. The converse is clear by Proposition 2.2. □

4 Some extensions of weak e-reversible rings

In the following section, we study the transfer of weak e-reversible notion to some ring extensions. In particular, for a ring R and $e \in E(R)$, we show that if R is a weak e-reversible ring, then the upper triangular matrices over the ring R, $T_n(R)$ is weak $\mathcal{E}$-reversible for $\mathcal{E} \in E(T_n(R))$ (Theorem 4.2). Among other results, we prove that over the Armendariz ring, R is weak e-reversible if and only if the polynomial ring $R[X]$ is weak e-reversible (Proposition 4.7).

In the following example, we show that for a reduced ring R, $M_n(R)$ is not weak e-reversible for some $e \in E(M_n(R))$.

Example 4.1. Let R be a reduced ring, and e_{ij} denote the matrix unit in $M_n(R)$ whose (i, j)-th entry is 1 and the others are zero. Then $M_n(R)$ is neither right e-reversible nor left e-reversible for some $e \in E(M_n(R))$ (see [13, Example 2.4]). In fact, $M_n(R)$ is not also weak e-reversible. Indeed, let $a = e_{23}$, $b = e_{12}$ and $e = e_{11} + e_{33} \in E(M_n(R))$. Then $ab = 0$ and $ebae \neq 0$.

We have the following result.

Theorem 4.2. *Let R be a ring. Then the following conditions are equivalent:*
(1) R is a weak e-reversible ring.
(2) $T_n(R)$ is a weak $\mathcal{E}$-reversible ring where $\mathcal{E} = \begin{pmatrix} e & er_1 & ... & er_{n-1} \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}$ *is an idempotent matrix in $T_n(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $A, B \in T_n(R)$ for $A = [a_{ij}], B = [b_{ij}]$ such that $AB = 0$. Then $a_{ii}b_{ii} = 0$ for any i. Since R is weak e-reversible, we have $eb_{ii}a_{ii}e = 0$. Consequently, $\mathcal{E}BA\mathcal{E} = \begin{pmatrix} ebae & ebaer_1 & ... & ebaer_{n-1} \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix} = 0$ where $\mathcal{E} = \begin{pmatrix} e & er_1 & ... & er_{n-1} \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}$. Therefore, $T_n(R)$

is weak $\mathcal{E}$-reversible.
(2) $\Rightarrow$ (1) Let $a, b \in R$ such that $ab = 0$. Assume that $A = aE_{11}, B = bE_{11} \in T_n(R)$, then $AB = 0$ and so $\mathcal{E}BA\mathcal{E} = 0$ since $T_n(R)$ is weak $\mathcal{E}$-reversible. This implies that $ebae = 0$ and so R is weak e-reversible. □

Let S be an (R, R)-bimodule. The trivial extension of R by S is the ring $T(R, S) = R \oplus S$, where the addition is usual and the multiplication is defined as follows:
$(r_1, s_1)(r_2, s_2) = (r_1r_2, r_1s_2 + s_1r_2), s_i \in S, r_i \in R$ for $i = 1, 2$. $T(R, S)$ is isomorphic to the ring $\left\{\begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \mid t \in R, s \in S\right\}$, where the operations are usual matrix operations.

Proposition 4.3. *If $T(R, R)$ is a weak $\mathcal{E}$-reversible ring where $\mathcal{E} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$, then R is weak e-reversible.*

Proof. Let $w, h \in R$ such that $wh = 0$. Then $\begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $T(R, R)$ is weak $\mathcal{E}$-reversible, $\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0$. Hence $ehwe = 0$. Therefore, R is weak e-reversible. □

Proposition 4.4. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are rings, and ${}_RM_S$ an (R, S)-bimodule. If T is weak $\mathcal{E}$-reversible where $\mathcal{E} = \begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in E(T)$, then the following hold:*
(1) R is a weak e-reversible ring.
(2) S is a weak g-reversible ring.

Proof. If T is weak $\mathcal{E}$-reversible where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in E(T)$. Then by easy computations we can check that $e \in E(R), g \in E(S)$ and $ek + kg = k$.
(1) Assume that $ab = 0$ for $a, b \in R$. Consider the following elements, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then we have $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = 0$. Since T is weak $\mathcal{E}$-reversible, we get

$$\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} = 0$$

Consequently, $ebae = 0$. Therefore, R is a weak e-reversible ring.
(2) Assume that $\alpha\beta = 0$ for $\alpha, \beta \in S$. Consider the following elements $\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \in T$. Then we have $\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} = 0$. Since T is weak $\mathcal{E}$-reversible, we get

$$\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} = 0$$

Hence, $g\alpha\beta g = 0$ and so S is a weak g-reversible ring. □

Proposition 4.5. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are rings, and ${}_RM_S$ an (R, S)-bimodule. If R is weak e-reversible ring where $e \in E(R)$, then T is weak $\mathcal{E}$-reversible where $\mathcal{E} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$.*

Proof. Assume that R is a weak e-reversible ring and $e \in E(R)$. Let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}, \begin{pmatrix} p & n \\ 0 & q \end{pmatrix} \in T$ such that $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} p & n \\ 0 & q \end{pmatrix} = 0$. Hence, $ap = 0 \in R$. Since R is weak e-reversible, we have $epae = 0$. Consequently, we have $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & n \\ 0 & q \end{pmatrix} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} epae & 0 \\ 0 & 0 \end{pmatrix} = 0$. Therefore, T is a weak $\mathcal{E}$-reversible where $\mathcal{E} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$. □

Let R be a ring and S be a subring of R and $T[R,S] = \{(r_1, r_2, ..., r_n, s, s, ...) \mid r_i \in R, s \in S, n \geq 1, 1 \leq i \leq n\}$. Then $T[R,S]$ is a ring under the componentwise addition and multiplication. In the following result, we provide a necessary and sufficient condition for $T[R,S]$ to be weak e-reversible.

Proposition 4.6. *Let R be a ring and S a subring of R with the same identity as that of R. Let $e \in E(S)$ and $\mathcal{E} = (e, e, e, ...) \in E(T[R,S])$. Then the following are equivalent:*
(1) $T[R,S]$ is a weak $\mathcal{E}$-reversible ring.
(2) R and S are weak e-reversible rings.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$ such that $ab = 0$. Consider $A = (a, 0, 0, 0, ...), B = (b, 0, 0, 0, ...)$. Then A and B are in $T[R,S]$ and $AB = 0$. By (1), $\mathcal{E}BA\mathcal{E} = 0$ which implies that $ebae = 0$. Therefore R is weak e-reversible. Now let $s, t \in S$ such that $st = 0$. Set $X = (0, s, s, s, ...)$ and $Y = (0, t, t, t, ...) \in T[R,S]$, $XY = 0$. By (1), we have $\mathcal{E}YX\mathcal{E} = 0$ which implies $etse = 0$. Therefore, S is weak e-reversible.
(2) $\Rightarrow$ (1) Let $A = (a_1, a_2, ..., a_n, b, b, b, ...)$ and $B = (c_1, c_2, ..., c_m, d, d, ...) \in T[R,S]$ with $AB = 0$. Then $a_i c_i = 0$ and $bd = 0$ where $1 \leq i \leq n$. Since R and S are weak e-reversible rings we have $ec_i a_i e = 0$ and $edbe = 0$. If $n + 1 \leq i$, then $bc_i = 0$. Hence $ec_i be = 0$, it follows that $\mathcal{E}BA\mathcal{E} = 0$. Similarly, if $m > n$, then we have $\mathcal{E}BA\mathcal{E} = 0$. Therefore, $T[R,S]$ is weak $\mathcal{E}$-reversible. □

A ring R is called Armendariz if for any two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ such that $f(x)g(x) = 0$, then $a_i b_j = 0$ for all i, j. Since any Armendariz ring is abelian [12, Lemma 7], we have the following result.

Proposition 4.7. *Let R be an Armendariz ring, then R is weak e-reversible with $e \in E(R)$ if and only if $R[x]$ is weak e-reversible with $e \in E(R[x])$.*

Proof. It is enough to show that $R[x]$ is weak e-reversible. Assume that R is weak e-reversible and $f(x)g(x) = 0$, for $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$. Since R is an Armendariz ring, we have $a_i b_j = 0$, for all i and j. As R is weak e-reversible, we have $eb_j a_i e = 0$ for $0 \leq i \leq n, 0 \leq j \leq m$. Consequently, $eg(x)f(x)e = 0$. Therefore $R[x]$ is weak e-reversible for $e \in E(R[x])$. □

Note that, $E(R) = E(R[x]) = E(R[[x]])$ by [12, Lemma 8].

A ring R is called power-serieswise Armendariz if for every $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ such that $fg = 0$, then $a_i b_j = 0$ for every i and j (see [5], and [23]). It is clear that power-serieswise Armendariz rings are Armendariz, while the converse need not be true by [11, Example 2.1].

Proposition 4.8. *If R is a power-serieswise Armendariz ring, then the following conditions are equivalent:*
(i) R is weak e-reversible.
(ii) $R[x]$ is weak e-reversible.
(iii) $R[[x]]$ is weak e-reversible .

Proof. Let R be a power-serieswise Armendariz ring. Then it is sufficient to prove that $R[[x]]$ is weak e-reversible. Assume that R is weak e-reversible with $e \in E(R)$. Let $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$. Since R is a power-serieswise Armendariz ring we have $a_i b_j = 0$, for all i and j. As R is weak e-reversible, we have $eb_j a_i e = 0$ for all i, j. Consequently, $eg(x)f(x)e = 0$. Therefore $R[[x]]$ is weak e-reversible. □

For a ring R with an endomorphism α, we denote $R[x, \alpha]$ a skew polynomial ring (also called an Ore extension of endomorphism type) whose elements are the polynomials $f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in R$, where the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. Recall from [9], a ring R is called α-skew Armendariz for an endomorphism α of R if for any $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} a_j x^j \in R[x, \alpha]$ whenever $f(x)g(x) = 0$ then $a_i \alpha^i(b_j) = 0$ for all i and j. Following [1], a ring R is said to be α-compatible if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$.

The following lemma, which has been proved in [8, Lemma 2.1], will be helpful in our next result.

Lemma 4.9. *Let R be α-compatible ring. Then $ab = 0 \Leftrightarrow a\alpha^i(b) = 0 \Leftrightarrow \alpha^i(a)b = 0$ for any positive integer i and $a, b \in R$.*

Theorem 4.10. *Let R be a ring satisfying α-compatible for an endomorphism α of R. If R is α-skew Armendariz, then R is weak e-reversible if and only if $R[x, \alpha]$ is weak $e(x)$-reversible.*

Proof. We prove the necessary part only while the other part follows from the closedness of weak e-reversible rings under subrings. Let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ and $g(x) = b_0 + b_1 x + b_2 x^2 + ... + b_m x^m \in R[x, \alpha]$ such that $f(x)g(x) = 0$. Since R is α-skew Armendariz, $a_i \alpha^i(b_j) = 0$ for all i, j. By Lemma 4.9, $a_i b_j = 0$ for all i, j. Let $e(x) = e_0 + e_1 x + e_2 x^2 + ... + e_p x^p \in R[x, \alpha]$. Since R is weak e-reversible we have $e_l b_j a_i e_l = 0$ for all $l = 0, 1, ..., p$. It follows from Lemma 4.9 that $e_l \alpha^l(b_j) \alpha^{l+j}(a_i) \alpha^{l+j+i} e_l = 0$. Hence $e(x)g(x)f(x)e(x) = 0$. Therefore $R[x, \alpha]$ is weak $e(x)$-reversible. □

The set $\{x^j\}_{j \geq 0}$ is easily seen to be a left Ore subset of $R[x, \alpha]$, so that one can localize $R[x, \alpha]$ and form the skew Laurent polynomial ring $R[x, x^{-1}, \alpha]$. Elements of $R[x, x^{-1}, \alpha]$ are finite sums of elements of the form $x^{-j} a x^i$ where $a \in R$ and i and j are nonnegative integers. The skew power series ring is denoted by $R[[x, \alpha]]$, whose elements are the series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ for some $a_i \in R$ and nonnegative integers i. The skew Laurent power series ring $R[x, x^{-1}\alpha]$ which contains $R[[x, \alpha]]$ as a subring, arises as the localization of $R[[x, \alpha]]$ with respect to Ore set $\{x^j\}_{j \geq 0}$, and when is an automorphism of R, it consists elements of the form $x^s a_s + x^{s+1} a_{s+1} + ... + a_0 + a_1 x + ...$, for some $a_i \in R$, a negative integer s and integers i, j, where the addition is defined as usual and the multiplication is defined by the rule $xa = \alpha(a)x$ for any $a \in R$. Recall that a ring R with an endomorphism α is called skew power-serieswise Armendariz (or SPA for short) (see [23, Definition 2.1]), if for every skew power series $p(x) = \sum_{i=0}^{\infty} a_i x^i, q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x, \alpha]], p(x)q(x) = 0, a_i b_j = 0$ for all i, j.

Theorem 4.11. *Let R be an SPA ring and α an automorphism of R. Then the following are equivalent:*
(1) R is weak e-weak reversible, for each $e \in E(R)$.
(2) $R[x, \alpha]$ is e-weak reversible, for each $e \in E(R[x, \alpha])$.
(3) $R[x, x^{-1}, \alpha]$ is e-weak reversible, for each $e \in E(R[x, x^{-1}, \alpha])$.
(4) $R[[x, \alpha]]$ is e-weak reversible, for each $e \in E(R[[x, \alpha]])$.
(5) $R[[x, x^{-1}, \alpha]]$ is e-weak reversible, for each $e \in E(R[[x, x^{-1}, \alpha]])$.

Proof. It is enough to show that (1) $\Rightarrow$ (5) while the other parts follows by Lemma 2.6. Assume that (1) holds. Let $f(x)g(x) = 0$ for $f(x) = x^s a_s + x^{s+1} a_{s+1} + ... + a_0 + a_1 x + ..., g(x) = x^t b_t + x^{t+1} b_{t+1} + ... + b_0 + b_1 x + ...$ where a and b are integers with $s, t \leq 0$. Then $a_i b_j = 0$. Since R is weak e-weak reversible, we have that $e_l b_j a_i e_l = 0$ and so $e_l b_j \alpha^n(a_i) e_l = 0$ for any nonnegative integer n. Thus $e(x)g(x)f(x)e(x) = 0$ and therefore $R[[x, x^{-1}, \alpha]]$ is weak e-weak reversible. □

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