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Characterizations and Partial Orderings of s-k-unitary Matrices

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Abstract In this paper,we provide some basic characterizations of s-k unitary matrices and introduce the concept of s-k invariant partial orderings on matrices. A set of results related to Loewner partial ordering,star partial ordering, and minus partial ordering (rank subtractivity) of s-k-unitary matrices are presented. The relationship between left-star and right-star partial orderings is discussed. Furthermore, it is demonstrated that Loewner partial ordering is preserved under s-k-unitary similarity.Some equivalent conditions on s-k-unitary matrices are also derived.

1 Introduction

In 1976, A. Lee. [13] initiated the study of secondary symmetric and secondary skew-symmetric matrices as a generalization of symmetric and skew-symmetric matrices and also explained that the usual (primary) transpose A^T and secondary transpose A^S are related , as $A^S = VA^TV$, where $'V'$ is a permutation matrix with units at the secondary diagonal and all other elements are zero. Hill and Waters [8] have developed the theory of the k-real and k-hermitian matrix, where $'k'$ is the fixed product of disjoint transpositions in S_n, the set of all permutations on $\{1,2,3\dots,n\}$. Let 'K' be the associated permutation matrix, which follows that K is involutory, i.e.,$K^2 = I$. The concept of unitary matrices was introduced as a special case of normal matrices.Unitary matrices have significant importance in quantum mechanics because they preserve norm. In this way, some generalizations and modifications came time by time to the concept of matrices. Ekhad and Zeilberger describe the invariance properties of matrix powers, where they generalized Peter's property up to higher dimension [5]. Some work related to the invariance properties of matrices is also given in the literature (see, for instance, [11],[12]). The concept of partial ordering of matrices is well-known to us. The star partial ordering was introduced by Drazin [4]. A Matrix B is said to be a section of matrix A whenever two conditions of star partial ordering hold. Further, it is pointed out by Drazin that $A \underset{*}{\leq} B \Leftrightarrow A^\dagger A = B^\dagger A$ and $AA^\dagger = AB^\dagger$ as well as $A \underset{*}{\leq} B \Leftrightarrow A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$[2]. Bakasalary and Mitra introduced the concept of the left-star and right-star partial ordering [1]. Some properties of matrix partial ordering were discussed by [2] Bakasalary, Pukelshein and Styan. The Minus partial ordering and characterizations of these orderings were given by Hartwig and Styan [6]. Some additional conditions are discussed by them, which must be added so that rank subtractivity becomes star partial order. And also described a canonical form for rank subtractivity. Here, we described the invariance concept related to these partial orders under s-k-type generalization.The concept of s-k-normal matrices as a generalization of normal matrices is given by S.Krishnmoorthy and G.Bhuvaneswari. As we know, unitary matrices are a special type of normal matrices. So, in a similar way, they also introduced s-k-unitary matrices for describing some relationships between s-k-normal and s-k-unitary matrices [9]. In this paper, we extend the concept of s-k-unitary matrices by introducing some of their properties and partial ordering theories.

1.1 Preliminaries and notations

Let $\mathbb{C}_{n\times n}$ be the space of $n\times n$ complex matrices of order n. For a matrix $A \in \mathbb{C}_{n\times n}$, let $\overline{A}$, A^T, A^*, A^S, A^θ, $R(A)$, $N(A)$ denotes the conjugate, transpose, transpose conjugate (primary), secondary transpose, and secondary conjugate transpose, range space and null space, respectively. The Associated permutation matrix [8],$'K'$ satisfies properties $\overline{K} = K^T = K^S = K^* = K^\theta = K$, $K^2 = I$ and let 'V' be a permutation matrix [13], satisfies the properties $\overline{V} = V^T = V^S = V^* = V^\theta = V$, $V^2 = I$.The symbols $\underset{L}{\leq}$, $\underset{*}{\leq}$, $\underset{rs}{\leq}$, $* \leq$ and $\leq *$ denote the Loewner, star, minus, left-star, and right-star partial ordering, respectively. The symbol † stands for the Moore-Penrose inverse [14] The unique matrix satisfying the given four conditions of penrose $(i) AXA = A$ (ii) $XAX = X$ (iii) $(AX)^* = AX$ (iv)$(XA)^* = XA$ is called the penrose inverse of A. In the literature, three types of matrix partial orderings are considered in the set $\mathbb{C}_{n\times n}$. For matrices $A, B \in \mathbb{C}_{n\times n}$, the Loewner partial ordering is defined as $A \underset{L}{\leq} B \Leftrightarrow (B-A) \geq 0$. There are various ways of characterizing this ordering. One of these is shown below, which is in accordance with Baksalary, Liski, and Trenkler [3],[7]

$$A \underset{L}{\leq} B \Leftrightarrow \rho(B^\dagger A) \leq 1 \text{ and } R(A) \subseteq R(B) \tag{1.1}$$

where $\rho(B)$ = max $\{|\lambda|$: λ -an eigen value of $B\}$ is the spectral radius of B. This result explains the relationship between Loewner's partial order and the spectral radius of the transformation. According to Rao and Mitra [15]

$$\text{i) } N(A) \subseteq N(B) \Leftrightarrow R(B^*) \subseteq R(A^*)$$
$$\Leftrightarrow B = BA^-A \text{ for all } A^- \in A\{1\}$$

$$\text{ii) } N(A^*) \subseteq N(B^*) \Leftrightarrow R(B) \subseteq R(A)$$

$$\Leftrightarrow B = AA^-B \; for \; all \; A^- \in A\{1\} \tag{1.2}$$

The star partial ordering, which is defined by a binary relation

$$A \underset{*}{\leq} B \Leftrightarrow A^*A = A^*B \text{ and } AA^* = BA^*, \text{ for } A, B \in \mathbb{C}_{n\times n}$$

Modifying the binary relation introduced by Drazin [4], Baksalary and Mitra [1] put forward the left-star and right-star orderings characterized as; for $A, B \in \mathbb{C}_{n\times n}$,

$$A* \leq B \Leftrightarrow A^*A = A^*B \text{ and } R(A) \subseteq R(B)$$
$$A \leq *B \Leftrightarrow AA^* = BA^* \text{ and } R(A^*) \subseteq R(B^*)$$

Hartwig [6], defined the minus partial (rank subtractivity) ordering as

$$A \underset{rs}{\leq} B \Leftrightarrow rank(B-A) = rank(B) - rank(A), \text{ for } A, B \in \mathbb{C}_{n\times n}$$

Definition 1.1. [9] A matrix $A \in \mathbb{C}_{n\times n}$ is said to be s-k-unitary matrix, if $KA^{\theta}K = A^{-1}$ i.e. $KVA^*VK = A^{-1}$

Example: The matrix $\begin{bmatrix} 0 & \iota & 0 \\ 0 & 0 & \iota \\ \iota & 0 & 0 \end{bmatrix}$ is a s-k-unitary matrix, for k=(1) (2,3), the associated permutation matrix $K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

2 Characterizations of s-k-unitary matrices

In this section, some basic characterizations of s-k-unitary matrices analogous to those of unitary matrices are obtained. The necessary and sufficient conditions for the sum and difference of two s-k-unitary matrices to be s-k-unitary are determined. It is shown that the product of two s-k-unitary matrices is a s-k-unitary matrix, and we generalize this result up to nth product. Some equivalent conditions on s-k-unitary matrices are also given.

Theorem 2.1. *Let A be a s-k-unitary matrix. Then the following statements are true:.*

(i). A^T, $\overline{A}$ *and* A^{-1} *are s-k-unitary matrices.*

(ii). KVA *and* AVK *are s-k-unitary matrices.*

(iii). $|A|$ *is unit modulus.*

(iv). If A is a real s-k-symmetric matrix such that $A^2 = I$ *then A is a s-k-unitary matrix.*

(v). If A is a s-k-unitary matrix, then A will be a s-k-normal matrix.

(vi). If A is skew s-k-Hermitian, then e^A *will be s-k-orthogonal.*

Proof. (i). $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = (KVA^*VK)^T = KV(A^*)^TVK = KV\left(A^T\right)^*VK$

$$\left(\overline{A}\right)^{-1} = \overline{\left(A^{-1}\right)} = \overline{(KVA^*VK)} = \left(KV\overline{A^*}VK\right) = KV\left(\overline{A}\right)^*VK$$

$$\left(A^{-1}\right)^{-1} = (KVA^*VK)^{-1} = (KV(A^*)^{-1}VK) = KV\left(A^{-1}\right)^*VK$$

(ii). A is s-k-unitary matrix $\Rightarrow (KVA^*VK) = A^{-1}$

Now, $(KVA)^{-1} = A^{-1}VK = (KVA^*VK)VK = KV(KVA)^*VK$

$\Rightarrow KVA$ is s-k-unitary

Now, $(AVK)^{-1} = KVA^{-1} = KV(KVA^*VK) = KV(AVK)^*VK$

$\Rightarrow AVK$ is s-k-unitary.

(iii). A is s-k-unitary matrix $\Rightarrow KVA^*VK = A^{-1}$

$A(KVA^*VK) = I$

$|A(KVA^*VK)| = |I|$

since, $|K| = \pm 1$, $|V| = \pm 1$

$|A||\overline{A}| = 1$

$||A||^2 = 1$

$|A| = \pm 1$ or $\pm \iota$

(iv). A is real s-k-symmetric matrix $\Rightarrow (KVA^TVK) = A$

$A(KVA^TVK) = A^2$

$A\left(KVA^TVK\right) = I$

A is real so $A^T = A^*$

$A(KVA^*VK) = I$

$\Rightarrow A$ is s-k-unitary matrix.

(v). A is s-k-unitary matrix $\Rightarrow KVA^*VK = A^{-1}$
$A(KVA^*VK) = I$
also ,$(KVA^*VK)A = I$
$\Rightarrow A(KVA^*VK) = (KVA^*VK)A = I$
$\Rightarrow A$ is s-k-normal matrix.

(vi). A is skew s-k-Hermitian $\Rightarrow (KVA^*VK) = -A$
$\exp(KVA^*VK)$=$\exp(-A)$
Now, $\exp(A)\exp(KVA^*VK)$= $\exp(A)\exp(-A)$=I
$\Rightarrow \exp(A)$ is s-k-unitary matrix.

□

Theorem 2.2. *For A, $B \in \mathbb{C}_{n\times n}$, A and B are s-k-unitary matrices such that $A(KVB^*VK) = (KVB^*VK)A$, and $B(KVA^*VK) = (KVA^*VK)B$. If*

(i). $A(KVB^*VK) + B(KVA^*VK) = -I$, then $A+B$ is s-k-unitary.
(ii). $A(KVB^*VK) + B(KVA^*VK) = I$, then $A-B$ is s-k-unitary.

Proof. (i). A and B are s-k-unitary matrices;therefore, $KVA^*VK = A^{-1}$,$KVB^*VK = B^{-1}$.
We have to prove that $(A+B)(KV(A+B)^*VK) = I$.

$$(A+B)(KV(A+B)^*VK) = (A+B)(KV(A^*+B^*)VK)$$
$$= A(KVA^*VK) + A(KVB^*VK) + B(KVA^*VK) + B(KVB^*VK)$$
$$= I + A(KVB^*VK) + B(KVA^*VK) + I$$
$$= I + (-I) + I = I$$

Similarly, $(KV(A+B)^*VK)(A+B) = I$
$(A+B)(KV(A+B)^*VK) = (KV(A+B)^*VK)(A+B) = I$
Hence, $A+B$ is s-k-unitary.

(ii). We have to prove that $(A-B)(KV(A-B)^*VK) = I$

$$(A-B)(KV(A-B)^*VK) = (A-B)(KV(A^*-B^*)VK)$$
$$=A(KVA^*VK) - A(KVB^*VK) - B(KVA^*VK) + B(KVB^*VK)$$
$$= I - (A(KVB^*VK) + B(KVA^*VK)) + I$$
$$= I - I + I = I$$

Similarly, $(KV(A-B)^*VK)(A-B) = I$
$(A-B)(KV(A-B)^*VK) = (KV(A-B)^*VK)(A-B) = I$
Hence, $A-B$ is s-k-unitary.

□

Theorem 2.3. *Let $A, B \in \mathbb{C}_{n\times n}$. If A and B are s-k-unitary matrices, then AB is also s-k-unitary matrices.*

Proof. A is s-k-unitary $\Rightarrow KVA^*VK = A^{-1}$
B is s-k-unitary $\Rightarrow KVB^*VK = B^{-1}$
Now,

$$KV(AB)^*VK = KV(B^*A^*)VK$$
$$=(KVB^*VK)(KVA^*VK)$$
$$=B^{-1}A^{-1}$$
$$=(AB)^{-1}$$

Hence, AB is a s-k unitary matrix. □

Theorem 2.4. *Let $A \in \mathbb{C}_{n\times n}$. If A is s-k-unitary, then A^n is a s-k-unitary matrix.*

Proof. Case I: If n = 0,

$A^0 = I$ is a s-k-unitary matrix.

Case II: If n is a positive integer,

$$(A^n)^{-1} = (A \ldots\ldots\ldots\ldots A)^{-1}$$
$$= A^{-1} \ldots\ldots\ldots\ldots A^{-1} \text{ (n times)}$$
$$= (KVA^*VK)(KVA^*VK) \ldots\ldots\ldots\ldots (KVA^*VK)$$
$$= (KVA^*(VKKV)A^*(VKKV) \ldots\ldots\ldots (VKKV)A^*VK$$
$$= KV(A^*A^* \ldots\ldots\ldots A^*)VK \quad \text{[since } V^2 = I, K^2 = I]$$
$$= KV(A^n)^*VK$$

Hence, A^n is a s-k-unitary matrix.
Case III: If n is a negative integer,
Firstly, let us prove that n = -1.

$$KVA^*VK = A^{-1}$$
$$(KVA^*VK)^{-1} = \left(A^{-1}\right)^{-1}$$
$$(KV(A^*)^{-1}VK) = \left(A^{-1}\right)^{-1}$$
$$KV\left(A^{-1}\right)^*VK) = \left(A^{-1}\right)^{-1}$$

A^{-1} is s-k-unitary.
Let n be any negative integer.
Let m = -n.
Since A^{-1} is s-k-unitary matrix, $\left(A^{-1}\right)^m$ is s-k-unitary $(m>0)$ by using case 2, which we have already proved.
i.e., $A^n = A^{-m}$ = $\left(A^{-1}\right)^m$
Hence, A^n is s-k-unitary matrix. □

Equivalent conditions of s-k unitary matrices:

Theorem 2.5. *Let $A \in \mathbb{C}_{n\times n}$, then any two of the following statements implies the other:*

(i). A is unitary.

(ii). A is s-k unitary.

(iii). $A^*KV = KVA^*$

Proof. i) and ii) $\implies$ iii)
A is s-k-unitary matrix $\implies KVA^*VK = A^{-1}$
$KVA^*VK = A^*$ [since A is unitary].
Post-multiplying both sides by K, we get $KVA^*VKK = A^*K$.
$KVA^*VI = A^*K$ [since k^2=I]
Post-multiplying both sides by V, we get $KVA^*VV = A^*KV$.
$KVA^* = A^*KV$ [since V^2=I]
ii) and iii) $\implies$ i)
A is s-k-unitary matrix $\implies KVA^*VK = A^{-1}$
$A^*KVVK = A^{-1}$ [by (iii)]
$A^*KIK = A^{-1}$
$A^*KK = A^{-1}$
$A^* = A^{-1}$ [since V^2=I, K^2=I]
$\implies A$ is a unitary matrix.
iii) and i) $\implies$ ii)
Since $A^*KV = KVA^*$
Post- multiplying both sides by V, we get $A^*K = KVA^*V$ [since $V^2 = I$].
Again, after multiplying both sides by K, we get $A^* = KVA^*VK$ [since $K^2 = I$].
Since A is a unitary matrix, therefore, $A^{-1} = KVA^*VK$.
$\implies A$ is a s-k-unitary matrix. □

Theorem 2.6. *Let $A \in \mathbb{C}_{n\times n}$. If A is s-k-unitary matrix, then AA^* and A^*A are s-k-unitary matrices.*

Proof. We have $(AA^*)^{-1} = (A^*)^{-1}A^{-1} = \left(\left(A^{-1}\right)^*\right)A^{-1}$

$$(KVA^*VK)^*(KVA^*VK) = (KVAVK)(KVA^*VK) = (KVAV(KK)VA^*VK)$$
$$=(KVA(VV)A^*VK) = (KVAA^*VK) = (KV(AA^*)^*VK)$$

Therefore, AA^* is s-k-unitary matrix.

Also,
$$(A^*A)^{-1} = A^{-1}\,(A^*)^{-1}$$
$$= A^{-1}\left(\left(A^{-1}\right)^*\right)$$
$$=(KVA^*VK)(KVA^*VK)^* = (KVA^*VK)\,(KVAVK)$$
$$=(KVA^*V(KK)VAVK) = (KVA^*\,(VV)\,AVK)$$
$$=(KVA^*AVK)$$
$$= (KV(A^*A)^*VK)$$

Therefore, A^*A is s-k-unitary matrix. □

Theorem 2.7. *Let $A \in \mathbb{C}_{n\times n}$, then any two of the following statements implies the other:*

(i). A is involutory.

(ii). A is s-k Hermitian.

(iii). A is s-k unitary.

Proof. i) and ii) $\implies$ iii)
A is involutery; $\Rightarrow A^2 = I$.
$AA = I$
$A\,(KVA^*VK) = I$ [since A is s-k-Hermitian].
$KVA^*VK = A^{-1}$
$\Rightarrow A$ is s-k-unitary.
ii) and iii) $\implies$ i)
A is the s-k-unitary matrix $KVA^*VK = A^{-1}$.
$A = A^{-1}$ [since A is s-k-Hermitian].
$A^2 = I$
$\implies A$ is involutery.
iii) and i) $\implies$ ii)
A is involutery; $\Rightarrow A^2 = I$
$AA = I$
$A = A^{-1}$
$= KVA^*VK$ [since A is s-k-unitary]
$\Rightarrow A$ is s-k-Hermitian. □

Theorem 2.8. *Let $A \in \mathbb{C}_{n\times n}$. If A is s-k-unitary matrix and $KVA = AKV$, then KVA is unitary.*

Proof. A is s-k-unitary matrix $\Rightarrow KVA^*VK = A^{-1}$.
$KVA^*V^*K^* = A^{-1}$ [since $V^* = V, K^* = K$]
$KV(KVA)^* = A^{-1}$
Pre-multiplying both sides by A, we get $AKV(KVA)^* = AA^{-1}$.
$KVA(KVA)^* = I$

Also, A is the s-k-unitary matrix $KVA^*VK = A^{-1}$.
$(KVA^*VK)A = I$
Post - multiplying both sides by K and V, respectively,
$(KVA^*VK)AKV = KV$
Pre- multiplying both sides by K^{-1} and V^{-1}, respectively,
$(A^*VK)AKV = I$
$(A^*V^*K^*)AKV = I$
$(KVA)^*AKV = I$
$(KVA)^*KVA = I$
Thus, $KVA(KVA)^* = (KVA)^*KVA = I$
$\Longrightarrow$ KVA is unitary. □

Theorem 2.9. *For any matrix A, it commutes with KV if and only if A^* commutes with VK.*

Proof. Suppose A commute with KV. Therefore, $AKV = KVA \Leftrightarrow (AKV)^* = (KVA)^*$
$VKA^* = A^*VK$ [since $V^* = V, K^* = K$]. □

Example: Let $A = \begin{bmatrix} 0 & \imath & 0 \\ 0 & 0 & \imath \\ \imath & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$AKV = \begin{bmatrix} 0 & \imath & 0 \\ 0 & 0 & \imath \\ \imath & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \imath & 0 & 0 \\ 0 & \imath & 0 \\ 0 & 0 & \imath \end{bmatrix}$$

$$KVA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \imath & 0 \\ 0 & 0 & \imath \\ \imath & 0 & 0 \end{bmatrix} = \begin{bmatrix} \imath & 0 & 0 \\ 0 & \imath & 0 \\ 0 & 0 & \imath \end{bmatrix}$$

$$VKA^* = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -\imath \\ -\imath & 0 & 0 \\ 0 & -\imath & 0 \end{bmatrix} = \begin{bmatrix} -\imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & -\imath \end{bmatrix}$$

$$A^*VK = \begin{bmatrix} 0 & 0 & -\imath \\ -\imath & 0 & 0 \\ 0 & -\imath & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -\imath & 0 & 0 \\ 0 & -\imath & 0 \\ 0 & 0 & -\imath \end{bmatrix}$$

Hence, $AKV = KVA \Leftrightarrow VKA^* = A^*VK$.

Theorem 2.10. *Let $A \in \mathbb{C}_{n\times n}$ be a s-k-unitary matrix and $AKV = KVA$, then A is unitary matrix.*

Proof. A is a s-k-unitary matrix $\Rightarrow KVA^*VK = A^{-1}$
$KVVKA^* = A^{-1}$. [using theorem 2.9]
$KIKA^* = A^{-1}$ [since $V^2 = I$]
$KKA^* = A^{-1}$ [since $K^2 = I$]
$IA^* = A^{-1}$
$A^* = A^{-1}$
Hence, A is unitary matrix. □

3 Partial ordering of s-k-unitary matrices

In this section, we have shown that the star partial ordering and Loewner partial ordering remain invariant under 'KV'. In other words, s-k type generalization these partial orderings. First, we explain the concept of the s-k-invariant partial ordering on matrices. Following that, certain theorems related to partial ordering on s-k-unitary matrices are developed.

Theorem 3.1. *For $A, B \in \mathbb{C}_{n\times n}$, K is the associated permutation matrix of k , and V is the permutation matrix with units on the secondary diagonal, then*

(i). $A \overset{\leq}{L} B \Leftrightarrow VKA \overset{\leq}{L} VKB \Leftrightarrow AVK \overset{\leq}{L} BVK$.

(ii). $A \overset{\leq}{L} B \Leftrightarrow KVA \overset{\leq}{L} KVB \Leftrightarrow AKV \overset{\leq}{L} BKV$.

Proof. Let $A, B \in \mathbb{C}_{n\times n}$, K be the associated permutation matrix, and V is the permutation matrix with units at secondary diagonal

(i). $A \overset{\leq}{L} B \Leftrightarrow \rho(B^\dagger A) \leq 1$ and $R(A) \subseteq R(B)$ [using eq. (1.1)]
$\Leftrightarrow \rho(B^\dagger KVVKA) \leq 1$ and $A = BB^\dagger A$ [using eq.(1.2)]
$\Leftrightarrow \rho(B^\dagger KVVKA) \leq 1$ and $VKA = (VKB)(B^\dagger KV)VKA$
$\Leftrightarrow \rho((VKB)^\dagger(VKA)) \leq 1$ and $R(VKA) \subseteq R(VKB)$
Therefore, $A \overset{\leq}{L} B \Leftrightarrow VKA \overset{\leq}{L} VKB$.
Similarly, $A \overset{\leq}{L} B \Leftrightarrow \rho(B^\dagger A) \leq 1$ and $R(A) \subseteq R(B)$
$\Leftrightarrow \rho(KVB^\dagger AVK) \leq 1$ and $A = BB^\dagger A$
$\Leftrightarrow \rho((BVK)^\dagger(AVK)) \leq 1$ and $AVK = (BVK)(BVK)^\dagger AVK$
$\Leftrightarrow \rho((BVK)^\dagger(AVK)) \leq 1$ and $R(AVK) \subseteq R(BVK)$
Therefore, $A \overset{\leq}{L} B \Leftrightarrow AVK \overset{\leq}{L} BVK$

(ii). $A \underset{L}{\leq} B \Leftrightarrow \rho(B^{\dagger}A) \leq 1$ and $R(A) \subseteq R(B)$ [using eq. (1.1)]
$\Leftrightarrow \rho(B^{\dagger}VKKVA) \leq 1$ and $A = BB^{A}$ [using eq. (1.2)]
$\Leftrightarrow \rho(B^{\dagger}VKKVA) \leq 1$ and $KVA = (KVB)(B^{\dagger}VK)KVA$
$\Leftrightarrow \rho((KVB)^{\dagger}(KVA)) \leq 1$ and $R(KVA) \subseteq R(KVB)$
Therefore, $A \underset{L}{\leq} B \Leftrightarrow KVA \underset{L}{\leq} KVB$.
Similarly, $A \underset{L}{\leq} B \Leftrightarrow \rho(B^{\dagger}A) \leq 1$ and $R(A) \subseteq R(B)$
$\Leftrightarrow \rho(VKB^{\dagger}AKV) \leq 1$ and $A = BB^{\dagger}A$
$\Leftrightarrow \rho((BKV)^{\dagger}(AKV)) \leq 1$ and $AKV = (BKV)(BKV)^{\dagger}AKV$
$\Leftrightarrow \rho((BKV)^{\dagger}(AKV)) \leq 1$ and $R(AKV) \subseteq R(BKV)$
Therefore, $A \underset{L}{\leq} B \Leftrightarrow AKV \underset{L}{\leq} BKV$.

□

Theorem 3.2. *For $A, B \in \mathbb{C}_{n\times n}$, K is the associated permutation matrix of k, and V is the permutation matrix with units at secondary diagonal, then*

$$A \underset{rs}{\leq} B \Leftrightarrow VKA \underset{rs}{\leq} VKB \Leftrightarrow AVK \underset{rs}{\leq} BVK$$

Proof. $A \underset{rs}{\leq} B \Leftrightarrow$ rank $(B-A)$ = rank (B)- rank (A)
$\Leftrightarrow$ rank $VK(B-A)$ = rank (VKB) - rank (VKA)
$\Leftrightarrow$ rank $(VKB - VKA)$ = rank (VKB) - rank (VKA)
$\Leftrightarrow VKA \underset{rs}{\leq} VKB$
Similarly, $A \underset{rs}{\leq} B \Leftrightarrow$ rank $(B-A)$ = rank (B) - rank(B)-rank(A)
$\Leftrightarrow$ rank$(B-A)VK$ = rank (BVK)-rank (AVK)
$\Leftrightarrow$ rank $(BVK - AVK)$ = rank(BVK) - rank (AVK)
$\Leftrightarrow AVK \underset{rs}{\leq} BVK$
Therefore, $A \underset{rs}{\leq} B \Leftrightarrow VKA \underset{rs}{\leq} VKB \Leftrightarrow AVK \underset{rs}{\leq} BVK$. □

Remark 3.3. Loewner partial ordering is preserved under unitary similarity.
$A \underset{L}{\leq} B \Leftrightarrow P^{*}AP \underset{L}{\leq} P^{*}BP$, where P is unitary matrix .

Theorem 3.4. *Loewner partial ordering is preserved on s-k-unitary similarity.*

Proof. $A \underset{L}{\leq} B \Leftrightarrow VKA \underset{L}{\leq} VKB$ [using theorem (3.1)]
$P^{*}VKAP \underset{L}{\leq} P^{*}VKBP$ [the above remark]
$KVP^{*}VKAP \underset{L}{\leq} KVP^{*}VKBP$ [using theorem (3.1)]
$(KVP^{-1}VK)AP \underset{L}{\leq} (KVP^{-1}VK)BP$ [since P is unitary matrix]
If, $C = (KVP^{-1}VK)AP$, then C is s-k-unitary, similar to A.
If,$D = (KVP^{-1}VK)BP$, then D is s-k-unitary, similar to B.
Therefore, $C \underset{L}{\leq} D$ and hence Loewner partial ordering is preserved on s-k-unitary similarity. □

Theorem 3.5. *Let $A, B \in \mathbb{C}_{n\times n}$, if $A \underset{L}{\leq} B$ then $B - A$ is s-k-Hermitian.*

Proof. $A \underset{L}{\leq} B \Leftrightarrow KVA \underset{L}{\leq} KVB$ [using theorem (3.1)]
$\Rightarrow KVB - KVA \geq 0$
$\Rightarrow KV(B-A) \geq 0$
Hence, $B - A$ is s-k-Hermitian positive definite.
$\Rightarrow KV(B-A)^{*}VK = (B-A)$
$\Rightarrow B - A$ is s-k-Hermitian. □

Theorem 3.6. *If A and B are s-k-unitary and Hermitian matrices then $A \underset{L}{\leq} B$ iff $A^{-1} \underset{L}{\leq} B^{-1}$.*

Proof. $A \underset{L}{\leq} B \Leftrightarrow KVA \underset{L}{\leq} KVB$ [using theorem(3.1)]
$\Leftrightarrow KVA^{*} \underset{L}{\leq} KVB^{*}$ [since A and B Hermitian]
$\Leftrightarrow KVA^{*}VK \underset{L}{\leq} KVB^{*}VK$ [using theorem(3.1)]
$\Leftrightarrow A^{-1} \underset{L}{\leq} B^{-1}$ [since A and B are s-k-unitary]
Conversely, if $A^{-1} \underset{L}{\leq} B^{-1}$
$KVA^{*}VK \underset{L}{\leq} KVB^{*}VK$.
Post- multiplying both sides by K, we get $KVA^{*}V \underset{L}{\leq} KVB^{*}V$ [since $K^2 = I$]
Post- multiplying both sides by V, we get $KVA^{*} \underset{L}{\leq} KVB^{*}$ [since $V^2 = I$]
Since A and B are Hermitian matrices,so, $KVA \underset{L}{\leq} KVB$.
$\Rightarrow A \underset{L}{\leq} B$. [using theorem(3.1)] □

Theorem 3.7. *If A and B are s-k-unitary matrices then $A \underset{*}{\leq} B$ iff $A^{-1} \underset{*}{\leq} B^{-1}$.*

Proof. Assuming that $A \underset{*}{\leq} B$, then, we have
i) $A^{*}A = A^{*}B$ ii) $AA^{*} = BA^{*}$.
From i) $KVA^{*}AVK = KVA^{*}BVK$
$KVA^{*}VKKVAVK = KVA^{*}VKKVBVK$ [since $V^2 = I$ and $K^2 = I$]
$(KVA^{*}VK)(KVAVK) = (KVA^{*}VK)(KVBVK)$
$A^{-1}(KVAVK) = A^{-1}(KVBVK)$ [since A is s-k-unitary]

Taking transpose conjugate on both sides, we get
$(KVAVK)^*(A^{-1})^* = (KVBVK)^*(A^{-1})^*$
$(KVA^*VK)(A^{-1})^* = (KVB^*VK)(A^{-1})^*$

$$A^{-1}\left(A^{-1}\right)^* = B^{-1}\left(A^{-1}\right)^* \quad (3.1)$$

From ii) $AA^* = BA^*$
$KVAA^*VK = KVBA^*VK$
$KVAVKKVA^*VK = KVBVKKVA^*VK$
$(KVAVK)(KVA^*VK) = (KVBVK)(KVA^*VK)$
$(KVAVK)A^{-1} = (KVBVK)A^{-1}$ [since A is s-k-unitary]
Taking transpose conjugate on both sides, we get
$(A^{-1})^*(KVAVK)^* = (A^{-1})^*(KVBVK)^*$
$(A^{-1})^*(KVA^*VK) = (A^{-1})^*(KVB^*VK)$

$$\left(A^{-1}\right)^* A^{-1} = \left(A^{-1}\right)^* B^{-1} \quad (3.2)$$

[since A and B are s-k-unitary]
By equations (3.1) and (3.2) we have
$A^{-1}(A^{-1})^* = B^{-1}(A^{-1})^*$ and $(A^{-1})^*A^{-1} = (A^{-1})^*B^{-1}$
Therefore, $A \underset{*}{\leq} B \Rightarrow A^{-1} \underset{*}{\leq} B^{-1}$
Similarly, we can prove $A^{-1} \underset{*}{\leq} B^{-1} \Rightarrow A \underset{*}{\leq} B$. □

Theorem 3.8. *If A and B are s-k unitary matrices such that $A \underset{L}{\leq} B$ and $A \underset{*}{\leq} B$, then $A = B$.*

Proof. Suppose, $A \underset{*}{\leq} B$, we have
i) $A^*A = A^*B$ ii) $AA^* = BA^*$
Therefore, $A^*(B-A) = 0$
Taking transpose conjugate on both sides, we get

$$(B-A)^* A = 0 \quad (3.3)$$

Since, $A \underset{L}{\leq} B$, we have $KV(B-A)^*VK = (B-A)$ [using theorem (3.5)
Taking transpose conjugate on both sides, we get
$KV(B-A)VK = (B-A)^*$
Using above results in equation (3.3, we get $(KV(B-A)VK)A = 0$
$(KVBVK)A - (KVAVK)A = 0$
$(KVBVK)A = (KVAVK)A$
Taking transpose conjugate on both sides, we get $A^*(KVB^*VK) = A^*(KVA^*VK)$
$A^*B^{-1} = A^*A^{-1}$
$\Rightarrow A^{-1} = B^{-1}$
$\Rightarrow A = B$.
Hence proved. □

Theorem 3.9. *For s-k unitary matrices A and B, if $R(A) = R(A^*)$ and $R(B) = R(B)^*$, then, $A* \leq B, \Rightarrow A^{-1} \leq *B^{-1}$.*

Proof. $A* \leq B \Leftrightarrow A^*A = A^*B$ and $R(A) \subseteq R(B)$
$\Rightarrow KVA^*AVK = KVA^*BVK$
$\Rightarrow KVA^*VKKVAVK = KVA^*VKKVBVK$ [since $V^2 = I$ and $K^2 = I$]
$\Rightarrow (KVA^*VK)(KVAVK) = (KVA^*VK)(KVBVK)$
$\Rightarrow A^{-1}(KVAVK) = A^{-1}(KVBVK)$ [since A is s-k-unitary]
Using transpose conjugate on both sides of the above equation, we get
$(KVAVK)^*(A^{-1})^* = (KVBVK)^*(A^{-1})^*$
$(KVA^*VK)(A^{-1})^* = (KVB^*VK)(A^{-1})^*$

$$(A^{-1})(A^{-1})^* = (B^{-1})(A^{-1})^* \quad (3.4)$$

Given, $R(A) = R(A^*)$, $R(B) = R(B^*)$,
Therefore,

$$R(A)^* \subseteq R(B)^* \quad (3.5)$$

From equations (3.4) and (3.5)
$A* \leq B \Rightarrow A^{-1} \leq *B^{-1}$. □

Theorem 3.10. *If A and B are s-k-unitary matrices such that $R(A) = R(A^*)$ and $R(B) = R(B)^*$, then $A \leq *B \Rightarrow A^{-1}* \leq B^{-1}$.*

Proof. $A \leq *B \Leftrightarrow AA^* = BA^*$ and $R(A^*) \subseteq R(B^*)$
$\Rightarrow KVAA^*VK = KVBA^*VK$
$\Rightarrow KVAVKKVA^*VK = KVBVKKVA^*VK$ [since $V^2 = I$ and$K^2 = I$]
$\Rightarrow (KVAVK)(KVA^*VK) = (KVBVK)(KVA^*VK)$
$\Rightarrow (KVAVK)A^{-1} = (KVBVK)A^{-1}$
Taking transpose conjugate on both sides of above equation, we get
$(A^{-1})^*(KVAVK)^* = (A^{-1})^*(KVBVK)^*$
$(A^{-1})^*(KVA^*VK) = (A^{-1})^*(KVB^*VK)$

$$(A^{-1})^*(A^{-1}) = (A^{-1})^*(B^{-1}) \quad (3.6)$$

$R(A) = R(A^*)$ and $R(B) = R(B^*)$
Therefore,

$$R(A) \subseteq R(B) \tag{3.7}$$

From equations(3.6) and (3.7)
$A \leq * B \Rightarrow A^{-1} * \leq B^{-1}$.
Hence proved. □

Theorem 3.11. *Let A be any matrix and B be any Hermitian matrix, then $A* \leq B \Rightarrow A^* \leq * B^*$.*

Proof. $A* \leq B \Leftrightarrow A^*A = A^*B$and $R(A) \subseteq R(B)$
$A^*A = A^*B$
$\Rightarrow A^*A = A^*B^*$ [since B is Hermitian]
Taking transpose conjugate on both sides, we get
$\Rightarrow A^*(A^*)^* = (B^*)^*(A^*)^*$
$\Rightarrow A^*(A^*)^* = B^*(A^*)^*$ [since B is Hermitian]
and $R(A) \subseteq R(B)$
$\Rightarrow R(A^*) \subseteq R(B^*)$
Therefore, $A* \leq B \Rightarrow A^* \leq *B^*$. □

4 Conclusion

In this paper, we have proved some basic characterizations and equivalent conditions of s-k-unitary matrices. Theorems related to their sum, difference, and product are derived. We proved the invariance concept for Loewner and minus partial ordering; similarly, we can do the same for star partial ordering as well. All of these partial orders are discussed for s-k-unitary matrices . We found that all properties and partial orders are preserved by s-k-type generalizations, or that the results of s-k-unitary matrices are relatable to unitary matrices. These results can also be applied to other generalized matrices such as s-k-hermitian,s-k-orthogonal,k-idempotent,k-involutory etc.

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