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Extensions Of Intuitionistic Fuzzy Ideal Of Γ-Rings

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Abstract *In this paper, the concept of the extension of intuitionistic fuzzy ideals with respect to an element of a Γ-ring is introduced, and some of the basic properties are investigated. Finally, we explore the nature of extended ideals in the special case when the given intuitionistic fuzzy ideal is an intuitionistic fuzzy prime ideal or an intuitionistic fuzzy semi-prime ideal. Some related results have been examined.*

1 Introduction

Ideals play an important role in the algebraic structure of rings with wide-ranging applications in various fields. Chaudhari and Nemade [1] extended the concept of subtractive extension of an ideal to a commutative (m,n)-semiring with identity. This theory gets a more generalized form when Nobusawa [2] introduced the notion of a Γ-ring. By choosing a specific subset Γ, any traditional ring will become a particular case of a Γ-ring. Later Barnes [3] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. After these two papers were published, work has been done for extending various basic results proved for ring theory to Γ-rings theory. Paul [4] discussed different types of gamma ideals in Γ-ring and maintained them with corresponding operator rings of Γ-rings.

Jun [5] was the first to study the fuzzy ideal of a Γ-ring and obtain several characteristics for a fuzzy ideal to be a fuzzy prime ideal. Dutta and Chanda [6] characterize fuzzy prime ideals of Γ-rings via their operator rings. Ersoy [7] characterizes fuzzy, semi-prime ideals and various types of Γ-ideals and the relation between fuzzy semi-prime ideals and fuzzy prime ideals. Aggarwal et al. in [8] examined fuzzy primary ideals and a few theorems related to fuzzy prime ideals of the Γ-ring. Venkateshwarlu and others in [9] studied the extension of fuzzy ideals in Γ-semirings. Atanassaov first flipped the intuitionistic fuzzy (IF) sets concept [10, 11, 12], and this was a generalization of Zadeh [13] theory of fuzzy sets. Basnet [14] explores the basic algebraic structure of intuitionistic fuzzy set theory. Kim et al. in [15] carry forward this theory by intuitionistic fuzzification of the ideal of Γ-ring and theory related to prime ideals of Γ-ring and corresponding operator ring was given by Palaniappan et. al. in [16, 17, 18]. The authors in [19, 20, 21, 22] have studied the notion of IF characteristic ideals, IF structure space, IF prime radical of Γ-ring, IF f-primary ideals of Γ-rings, and finally IF primary decomposition of ideals for a commutative Noetherian Γ-ring was established in [23] and obtained many new results. Majumder discussed Atanassov's Intuitionistic Anti-Fuzzy Interior Ideals of semigroups in [24] which could help extend future research work. The objective of this paper is expressed in the abstract.

In the second part, we recall some groundwork related to intuitionistic fuzzy ideals (IFIs), intuitionistic fuzzy semiprime ideals (IFSPIs), and intuitionistic fuzzy prime ideals (IFPIs) which *are necessary for the framework of the concerned subject matter. In the third part, we try to extend an IFI with respect to an element of the* Γ*-ring.*

2 Preliminaries

Definition 2.1. ([2, 3]) "If $(H, +)$ and $(\Gamma, +)$ are additive Abelian groups. Then H is called a Γ-ring (in the sense of Barnes [3]) if there exist mapping $H \times \Gamma \times H \to H$ [image of (h_1, α, h_2) is denoted by $h_1\alpha h_2$, $h_1, h_2 \in H, \gamma \in \Gamma$] satisfying the following conditions:
(1) $h_1\alpha h_2 \in H$.
(2) $(h_1 + h_2)\alpha h_3 = h_1\alpha h_3 + h_2\alpha h_3$, $h_1(\alpha + \beta)h_2 = h_1\alpha h_2 + h_1\beta h_2$, $h_1\alpha(h_2 + h_3) = h_1\alpha h_2 + h_1\alpha h_3$.
(3) $(h_1\alpha h_2)\beta h_3 = h_1\alpha(h_2\beta h_3)$. for all $h_1, h_2, h_3 \in H$, and $\gamma \in \Gamma$."

Definition 2.2. ([2, 3]) "A Γ-ring H is said to be commutative if $h_1\gamma h_2 = h_2\gamma h_1$ for all $h_1, h_2 \in H, \gamma \in \Gamma$."

Definition 2.3. ([2, 3]) "Let H be a Γ-ring. Then a subset N of H is called left (resp. right) ideal if $H\Gamma N = \{h_1\alpha h_2 | h_1 \in H, \alpha \in \Gamma, h_2 \in N\} \subseteq N$ (resp.$N\Gamma H = \{h_1\alpha h_2 | h_1 \in N, \alpha \in \Gamma, h_2 \in H\} \subseteq N$). A left as well as right ideal is called an ideal."

Definition 2.4. ([4]) "Let H be a Γ-ring. A proper ideal L of H is called prime if for all pair of ideals S and T of H, $S\Gamma T \subseteq L$ implies that $S \subseteq L$ or $T \subseteq L$."

Theorem 2.5. *([5, 6]) "If K is an ideal of a Γ-ring H, the following conditions are equivalent:*
(i) K is a prime ideal of H;
(ii) If $a, b \in H$ and $a\Gamma H\Gamma b \subseteq K$ then $a \in K$ or $b \in K$."

Definition 2.6. ([7]) "Let H be a Γ-ring. A proper ideal L of H is called semiprime if for any ideal S of H, $S\Gamma S \subseteq L$ implies that $S \subseteq L$."

We now review some intuitionistic fuzzy logic concepts. We refer the reader to follow [12] and [14] for complete details.

Definition 2.7. ([11, 15]) "An intuitionistic fuzzy set G in H can be represented as an object of the form $G = \{\langle h, \mu_G(h), \nu_G(h)\rangle : h \in H\}$, where the functions $\mu_G : H \to [0, 1]$ and $\nu_G : H \to [0, 1]$ denote the degree of membership (namely $\mu_G(h)$) and the degree of non-membership (namely $\nu_G(h)$) of each element $h \in H$ to G respectively and $0 \leq \mu_G(h) + \nu_G(h) \leq 1$ for each $h \in H$."

Remark 2.8. ([11, 13, 15])
"(i) When $\mu_G(h) + \nu_G(h) = 1$, i.e., $\nu_G(h) = 1 - \mu_G(h), \forall h \in H$. Then G is called a fuzzy set.
(ii) An intuitionistic fuzzy set (IFS) $G = \{\langle h, \mu_G(h), \nu_G(h)\rangle : h \in H\}$ is shortly denoted by $G(h) = (\mu_G(h), \nu_G(h))$, for all $h \in H$."

Proposition 2.9. *([11, 15]) "If G_1, G_2 be two intuitionistic fuzzy sets of H, then*
(i) $G_1 \subseteq G_2 \Leftrightarrow \mu_{G_1}(h) \leq \mu_{G_2}(h)$ and $\nu_{G_1}(h) \geq \nu_{G_2}(h), \forall h \in H$;
(ii) $G_1 = G_2 \Leftrightarrow G_1 \subseteq G_2$ and $G_2 \subseteq G_1$, i.e., $G_1(h) = G_2(h)$, for all $h \in H$."

Definition 2.10. ([19]) "For any subset I of H, the intuitionistic fuzzy characteristic function (IFCF) χ_I is an IFS of H, defined as $\chi_I(h) = (1, 0), \forall h \in I$ and $\chi_I\ (h) = (0, 1), \forall h \in H \setminus I$."

Definition 2.11. ([11, 12, 14]) "The crisp set $G_{(\alpha,\beta)} = \{h \in H : \mu_G(h) \geq \alpha$ and $\nu_G(h) \leq \beta\}$ is called the (α, β)-level cut set of G for $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$"

Remark 2.12. ([11, 12, 14]) "$G_* = \{h \in H : \mu_G(h) = \mu_G(0_H)$ and $\nu_G(h) = \nu_G(0_H)\}$."

Definition 2.13. ([15, 16]) "Let G be an IFS of a Γ-ring H. Then G is called an intuitionistic fuzzy ideal (IFI) of H if for all $m, n \in H, \alpha \in \Gamma$, the following are satisfied
(i) $\mu_G(m - n) \geq \mu_G(m) \wedge \mu_G(n)$;

(ii) $\mu_G(m\alpha n) \geq \mu_G(m) \vee \mu_G(n)$;
(iii) $\nu_G(m-n) \leq \nu_G(m) \vee \nu_G(n)$;
(iv)$\nu_G(m\alpha n) \leq \nu_G(m) \wedge \nu_G(n)$."

Definition 2.14. ([12, 14]) Let G_1, G_2 be two IFSs of a Γ-ring H. Then their cartesian product $G_1 \times G_2$ is an IFS on $H \times H$ and is defined as

$$\mu_{G_1\times G_2}(h,k) = \mu_{G_1}(h) \wedge \mu_{G_2}(k) \text{ and } \nu_{G_1\times G_2}(h,k) = \nu_{G_1}(h) \vee \nu_{G_2}(k),$$

for all $(h,k) \in H \times H$.

Theorem 2.15. *If G_1, G_2 are IFIs of a Γ-ring H, then $G_1 \times G_2$ is also an IFI of $H \times H$.*

Proof. Let $(h_1,k_1),(h_2,k_2) \in H \times H$, $\alpha \in \Gamma$. Then

$$\begin{aligned}
\mu_{G_1\times G_2}((h_1,k_1)-(h_2,k_2)) &= \mu_{G_1\times G_2}(h_1-h_2,k_1-k_2) = \mu_{G_1}(h_1-h_2) \wedge \mu_{G_2}(k_1-k_2)\\
&\geq [\mu_{G_1}(h_1) \wedge \mu_{G_1}(h_2)] \wedge [\mu_{G_2}(k_1) \wedge \mu_{G_2}(k_2)]\\
&= [\mu_{G_1}(h_1) \wedge \mu_{G_2}(k_1)] \wedge [\mu_{G_1}(h_2) \wedge \mu_{G_2}(k_2)]\\
&= \mu_{G_1\times G_2}((h_1,k_1)) \wedge \mu_{G_1\times G_2}((h_2,k_2)).
\end{aligned}$$

Similarly, we can show that $\nu_{G_1\times G_2}((h_1,k_1)-(h_2,k_2)) \leq \nu_{G_1\times G_2}((h_1,k_1)) \vee \mu_{G_1\times G_2}((h_2,k_2))$.
Also, $\mu_{G_1\times G_2}((h_1,k_1)\alpha(h_2,k_2)) = \mu_{G_1\times G_2}(h_1\alpha h_2, k_1\alpha k_1) = \mu_{G_1}(h_1\alpha h_2) \wedge \mu_{G_2}(k_1\alpha k_2)$.
Therefore, either $\mu_{G_1\times G_2}((h_1,k_1)\alpha(h_2,k_2)) = \mu_{G_1}(h_1\alpha h_2)$ or $\mu_{G_2}(k_1\alpha k_2)$, then $\mu_{G_1\times G_2}((h_1,k_1)\alpha(h_2,k_2)) \geq \mu_{G_1}(h_1) \vee \mu_{G_1}(h_2)$ or $\mu_{G_2}(k_1) \vee \mu_{G_2}(k_2)$. Thus we conclude that

$$\begin{aligned}
\mu_{G_1\times G_2}((h_1,k_1)\alpha(h_2,k_2)) &\geq [\mu_{G_1}(h_1) \vee \mu_{G_1}(h_2)] \wedge [\mu_{G_2}(k_1) \vee \mu_{G_2}(k_2)]\\
&\geq [\mu_{G_1}(h_1) \wedge \mu_{G_2}(k_1)] \vee [\mu_{G_1}(h_2) \wedge \mu_{G_2}(k_2)]\\
&= \mu_{G_1\times G_2}((h_1,k_1)) \vee \mu_{G_1\times G_2}((h_2,k_2)).
\end{aligned}$$

[Because $(a\vee b)\wedge(c\vee d) = [(a\vee b)\wedge c] \vee [(a\vee b)\wedge d] = [(a\wedge c)\vee(b\wedge c)] \vee [(a\wedge d)\vee(b\wedge d)] \geq (a\wedge c)\vee(b\wedge d)$ and $(a\wedge b)\vee(c\wedge d) = [(a\wedge b)\vee c]\wedge[(a\wedge b)\vee d] = [(a\vee c)\wedge(b\vee c)]\wedge[(a\vee d)\wedge(b\vee d)] \leq (a\vee c)\wedge(b\vee d), \forall a,b,c,d \in [0,1]$].
Similarly, we can prove that $\nu_{G_1\times G_2}((h_1,k_1)\alpha(h_2,k_2)) \leq \nu_{G_1\times G_2}((h_1,k_1)) \wedge \nu_{G_1\times G_2}((h_2,k_2))$.
Hence $G_1 \times G_2$ is an IFI of $H \times H$. □

Definition 2.16. ([16, 18]) "Let G_1, G_2 be two IFSs of a Γ-ring H. Then the product $G_1\Gamma G_2$ of G_1 and G_2 is an IFS on H and is defined by

$$G_1\Gamma G_2(h) = \begin{cases} (\sup_{h=h_1\gamma h_2}\{\mu_{G_1}(h_1)\wedge\mu_{G_1}(h_2)\}, \inf_{h=h_1\gamma h_2}\{\nu_{G_1}(h_1)\vee\nu_{G_2}(h_2)\}), & \text{if } h = h_1\gamma h_2\\ (0,1), & \text{otherwise"}\end{cases}$$

Definition 2.17. ([18]) "Let H be a Γ-ring. A non-constant IFI P of H is called an intuitionistic fuzzy prime ideal (IFPI) of H, if for all pair of IFIs G_1, G_2 of H, $G_1\Gamma G_2 \subseteq P$ implies that $G_1 \subseteq P$ or $G_2 \subseteq P$."

Theorem 2.18. *([18]) "If P is an IFPI of a Γ-ring H, then the following conditions hold:*
(i) $\mu_P(0_H) = 1, \nu_P(0_H) = 0$,
(ii) P_* is a prime ideal of H,
(iii) $Img(P) = \{(1,0),(\theta,\eta)\}$, where $0 \leq \theta,\eta < 1$ such that $\theta+\eta \leq 1$."

Definition 2.19. ([17, 19]) "A non-constant IFI P of a Γ-ring H is said to be an intuitionistic fuzzy semi-prime ideal (IFSPI) if for any IFI G_1 of H, $G_1\Gamma G_1 \subseteq P$, implies that $G_1 \subseteq P$."

Proposition 2.20. *Let P be a non-constant IFI of a Γ-ring H, then the following conditions are equivalent:*
(i) P is an IFSPI of H
(ii) For any $a \in H$,

$$\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 m\gamma_2 a)\} = \mu_P(a) \textit{ and } \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 m\gamma_2 a)\} = \nu_P(a).$$

Proof. $(i) \Rightarrow (ii)$ Let P be an IFSPI of H. Since P is an IFI of H, it follows that

$$\mu_P(a\gamma_1 m\gamma_2 a) \geq \mu_P(a) \text{ and } \nu_P(a\gamma_1 m\gamma_2 a) \leq \nu_P(a), \forall m \in H, \gamma_1, \gamma_2 \in \Gamma.$$

If possible, suppose that $\mu_P(a\gamma_1 m\gamma_2 a) > \mu_P(a)$ and $\nu_P(a\gamma_1 m\gamma_2 a) < \nu_P(a)$, for some $a \in H$. Let $\langle a\rangle$ be the ideal generated a. Define the IFS C on H by

$$\mu_C(x) = \begin{cases} t, & \text{if } x \in \langle a\rangle \\ 0, & \text{otherwise} \end{cases}; \quad \nu_C(x) = \begin{cases} s, & \text{if } x \in \langle a\rangle \\ 1, & \text{otherwise.} \end{cases}$$

where $t, s \in (0, 1)$ such that $t + s \leq 1$. Then C is an IFI of H.
Consider $x \in H$ such that $x \neq u\gamma v$, for some $u, v \in \langle a\rangle$, then $C\Gamma C(x) = (0, 1)$ and $C\Gamma C(x) = (\sup_{x=u\gamma v,u,v\in\langle a\rangle}\{\mu_C(u) \wedge \mu_C(v)\}, \inf_{x=u\gamma v,u,v\in\langle a\rangle}\{\mu_C(u) \vee \mu_C(v)\})$.

Now any $u \in \langle a\rangle$ is of the form $u = \sum_{i=1}^{p} m_i^{'}\gamma_i^{'}a\gamma_i^{''}m_i^{''}, m_i^{'}, m_i^{''} \in H, \gamma_i^{'}, \gamma_i^{''} \in \Gamma$ and $p \in Z^+$. Similarly, $v = \sum_{j=1}^{q} m_j^{'}\gamma_j^{'}a\gamma_j^{''}m_j^{''}, m_j^{'}, m_j^{''} \in H, \gamma_j^{'}, \gamma_j^{''} \in \Gamma$ and $q \in Z^+$.
Now $u\gamma v = (\sum_{i=1}^{p} m_i^{'}\gamma_i^{'}a\gamma_i^{''}m_i^{''})(\sum_{j=1}^{q} m_j^{'}\gamma_j^{'}a\gamma_j^{''}m_j^{''}, m_j^{'})$. Since P is an IFI of H, it follows that
$\mu_P(x) = \mu_P(u\gamma v) \geq \mu_P(a\gamma_1 m^{'}\gamma_2 a) \geq Inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 m\gamma_2 a)\} > t = \mu_{C\Gamma C}(x)$, for some $m^{'} \in H$. Similarly, we can show $\nu_P(x) < \nu_{C\Gamma C}(x)$. So we get $C\Gamma C \subseteq P$. As P is an IFSPI of H, it follows that $C \subseteq P$.
Hence $t = \mu_C(a) \leq \mu_P(a)$ and $s = \nu_C(a) \geq \nu_P(a)$, a contradiction. Consequently we have $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 m\gamma_2 a)\} = \mu_P(a)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 m\gamma_2 a)\} = \nu_P(a)$.

$(ii) \Rightarrow (i)$ Let us assume that P be an IFI of H satisfying for any $a \in H$,

$$\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 m\gamma_2 a)\} = \mu_P(a) \text{ and } \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 m\gamma_2 a)\} = \nu_P(a).$$

Let C be an IFI of H such that $C\Gamma C \subseteq P$ and $C \nsubseteq P$. Then $\exists^s\, b \in H$ such that $\mu_C(b) > \mu_P(b)$ and $\nu_C(b) < \nu_P(b)$. Now $\mu_P(b\gamma_1 m\gamma_2 b) \geq \mu_{C\Gamma C}(b\gamma_1 m\gamma_2 b) \geq \mu_C(b)$ and $\nu_P(b\gamma_1 m\gamma_2 b) \leq \nu_{C\Gamma C}(b\gamma_1 m\gamma_2 b) \leq \nu_C(b)$, $\forall m \in H, \gamma_1, \gamma_2 \in \Gamma$. So $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(b\gamma_1 m\gamma_2 b)\} \geq \mu_C(b)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(b\gamma_1 m\gamma_2 b)\} \leq \nu_C(b)$. Thus $\mu_P(b) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_P(b\gamma_1 m\gamma_2 b)\} \geq \mu_C(b) > \mu_P(b)$ and $\nu_P(b) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_P(b\gamma_1 m\gamma_2 b)\} \leq \nu_C(b) < \nu_P(b)$, a contradiction. So P is an IFSPI of H. □

3 Extensions of Intuitionistic Fuzzy Ideal of Γ-Ring

In this section, we try to extend an IFI with respect to an element of Γ*-ring, and thus introduce the notion of extensions of IFI of* Γ*-ring. A suitable characterization of IFPIs and IFSPIs will be developed. Throughout the paper, H will always be regarded as a commutative* Γ*-ring.*

Definition 3.1. Let G be an IFS of a Γ-ring H, $h \in H$, then the IFS $< h, G >$ on H defined by

$$\mu_{<h,G>}(h^{'}) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'})\}\ ;$$
$$\nu_{<h,G>}(h^{'}) = \sup_{m\in H\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 m\gamma_2 h^{'})\}$$

is called the extension of G with respect to h.

Proposition 3.2. *If G is an IFI of a commutative* Γ*-ring H and $h \in H$, then the extension $< h, G >$ of G with respect to "h" is also an IFI of H.*

Proof. Let $m_1, m_2 \in H$, $\gamma \in \Gamma$, we have

$$\begin{aligned}
\mu_{<h,G>}(m_1 - m_2) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2(m_1 - m_2))\} \\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 m_1 - h\gamma_1 m\gamma_2 m_2))\} \\
&\geq \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 m_1) \wedge \mu_G(h\gamma_1 m\gamma_2 m_2)\} \\
&= \{\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}(\mu_G(h\gamma_1 m\gamma_2 m_1))\} \wedge \{\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}(\mu_G(h\gamma_1 m\gamma_2 m_1))\} \\
&= \mu_{<h,G>}(m_1) \wedge \mu_{<h,G>}(m_2).
\end{aligned}$$

Thus $\mu_{<h,G>}(m_1 - m_2) \geq \mu_{<h,G>}(m_1) \wedge \mu_{<h,G>}(m_2)$. Similarly we can show that $\nu_{<h,G>}(m_1 - m_2) \leq \nu_{<h,G>}(m_1) \vee \nu_{<h,G>}(m_2)$. Also,

$$\begin{aligned}
\mu_{<h,G>}(m_1\gamma m_2) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2(m_1\gamma m_2))\} \\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma_1 m\gamma_2 m_1)\gamma m_2)\} \\
&\geq \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 m_1)\} \\
&= \mu_{<h,G>}(m_1)
\end{aligned}$$

Since H is a commutative Γ-ring $m_1\gamma m_2 = m_2\gamma m_1$, for all $m_1, m_2 \in H, \gamma \in \Gamma$.

$$\begin{aligned}
\mu_{<h,G>}(m_1\gamma m_2) &= \mu_{<h,G>}(m_2\gamma m_1) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2(m_2\gamma m_1))\} \\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma_1 m\gamma_2 m_2)\gamma m_1)\} \\
&\geq \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 m_2)\} \\
&= \mu_{<h,G>}(m_2)
\end{aligned}$$

Thus $\mu_{<h,G>}(m_1\gamma m_2) \geq \mu_{<h,G>}(m_1) \vee \mu_{<h,G>}(m_2)$. Similarly, we can show $\nu_{<h,G>}(m_1\gamma m_2) \leq \mu_{<h,G>}(m_1) \wedge \nu_{<h,G>}(m_2)$. Hence $< h, G >$ is an IFI of H. □

Remark 3.3. The converse of the Proposition (3.2) need not be true, see the following example:

Example 3.4. Consider $H = \Gamma = Z_9 = \{0, 1, 2, 3, ..., 8\}$ under the addition modulo 9 and multiplication modulo 9 operations. Then H is a Γ-ring. Define an IFS G of H as

$$\mu_G(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0.4, & \text{if } h \in \{3, 6\} \\ 0.7, & \text{otherwise} \end{cases}; \quad \nu_G(h) = \begin{cases} 0, & \text{if } h = 0 \\ 0.5, & \text{if } h \in \{3, 6\} \\ 0.2, & \text{otherwise.} \end{cases}$$

It is easy to see that G is not an IFI of H, for $\mu_G(4-1) = \mu_G(3) = 0.4 \ngeq 0.7 = \mu_G(4) \wedge \mu_G(1)$. However the extension of G with respect to 3 , i.e., the IFS $< 3 + G >$ is defined as

$$\mu_{<3+G>}(h) = \begin{cases} 1, & \text{if } h \in \{0, 3, 6\} \\ 0.4, & \text{otherwise} \end{cases}; \quad \nu_{<3+G>}(h) = \begin{cases} 0, & \text{if } h \in \{0, 3, 6\} \\ 0.5, & \text{otherwise.} \end{cases}$$

is an IFI of H.

Proposition 3.5. *If $\{G_i : i \in J\}$ be an arbitrary family of IFIs of a Γ-ring H and "h" be any element of H, then $< h, \bigcap_{i\in J} G_i >= \bigcap_{i\in J} < h, G_i >$.*

Proof. Let $h^{'} \in H$ be any element, then we have

$$\begin{aligned} \mu_{<h,\bigcap_{i\in J} G_i>}(h^{'}) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\bigcap_{i\in J} G_i}(h\gamma_1 m\gamma_2 h^{'})\} \\ &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\inf_{i\in J}\{\mu_{G_i}(h\gamma_1 m\gamma_2 h^{'})\}\} \\ &= \inf_{i\in J}\{\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{G_i}(h\gamma_1 m\gamma_2 h^{'})\}\} \\ &= \inf_{i\in J}\{\mu_{<h,G_i>}(h^{'})\} \\ &= \mu_{\inf_{i\in J}<h,G_i>}(h^{'}). \end{aligned}$$

Similarly, we can show that $\nu_{<h,\bigcap_{i\in J} G_i>}(h^{'}) = \nu_{\sup_{i\in J}<h,G_i>}(h^{'})$.
Hence $< h, \bigcap_{i\in J} G_i >= \bigcap_{i\in J} < h, G_i >$. □

Proposition 3.6. *If G_1, G_2 are two IFIs of a Γ-ring H, then for any $(h,k) \in H \times H$, $< (h,k), G_1 \times G_2 >$ is also an IFI of $H \times H$.*

Proof. Since G_1, G_2 be two IFIs of a Γ-ring H. Then by Theorem (2.15), $G_1 \times G_2$ is an IFI of $H \times H$ and then result follows by Proposition (3.2). □

Theorem 3.7. *If G_1, G_2 are two IFIs of a Γ-ring H. Then for every $(h,k) \in H \times H$, we have*

$$< (h,k), G_1 \times G_2 >=< h, G_1 > \times < k, G_2 >.$$

Proof. For any $(h^{'}, k^{'}) \in H \times H$, we have

$$\begin{aligned} \mu_{<(h,k),G_1\times G_2>}(h^{'},k^{'}) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{G_1\times G_2}((h,k)\gamma_1 m\gamma_2 (h^{'},k^{'}))\} \\ &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\mu_{G_1\times G_2}((h\gamma_1 m\gamma_2 h^{'}, k\gamma_1 m\gamma_2 k^{'})) \\ &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}[\min\{\mu_{G_1}(h\gamma_1 m\gamma_2 h^{'}), \mu_{G_2}(k\gamma_1 m\gamma_2 k^{'})\}] \\ &= \min[\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\mu_{G_1}(h\gamma_1 m\gamma_2 h^{'}), \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\mu_{G_2}(k\gamma_1 m\gamma_2 k^{'})] \\ &= \min\{\mu_{<h,G_1>}(h^{'}), \mu_{<k,G_2>}(k^{'})\} \\ &= \mu_{<h,G_1>\times<k,G_2>}(h^{'},k^{'}). \end{aligned}$$

Similarly, we can show that $\nu_{<(h,k),G_1\times G_2>}(h^{'},k^{'}) = \nu_{<h,G_1>\times<k,G_2>}(h^{'},k^{'})$.
Hence $< (h,k), G_1 \times G_2 >=< h, G_1 > \times < k, G_2 >$. □

Proposition 3.8. *Assume that H is a commutative Γ-ring. If G is an IFI of H and $h \in H$. Then the following conditions hold*

(i) $G \subseteq< h, G >$

(ii) $< (h\gamma)^{n-1}h, G >\subseteq< (h\gamma)^n h, G >$, *where* $\gamma \in \Gamma$

(iii) If $h \in Supp(G)$, then $Supp(< h, G >) = H$, where $Supp(G)$ is defined by $Supp(G) = \{h \in H : \mu_G(h) > 0, \nu_G(h) < 1\}$.

Proof. **(1)** Let $h^{'} \in H$. Since G is an IFI of H, we have $\mu_{<h,G>}(h^{'}) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'})\} \geq \mu_G(h^{'})$ and $\nu_{<h,G>}(h^{'}) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 m\gamma_2 h^{'})\} \leq \nu_G(h^{'}), \forall h^{'} \in H$
Thus $G \subseteq< h, G >$.

(2) Let $n \in \mathbf{N}$, $h^{'} \in H$. Since G is an IFI of H, we have

$$\begin{aligned} \mu_{<(h\gamma)^n h,G>}(h^{'}) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma)^n h\gamma_1 m\gamma_2 h^{'})\} \\ &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma(h\gamma)^{n-1} h\gamma_1 m\gamma_2 h^{'})\} \\ &\geq \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G((h\gamma)^{n-1} h\gamma_1 m\gamma_2 h^{'})\} \\ &= \mu_{<(h\gamma)^{n-1}h,G>}(h^{'}). \end{aligned}$$

Thus $\mu_{<(h\gamma)^n h,G>}(h^{'}) \geq \mu_{<(h\gamma)^{n-1}h,G>}(h^{'})$. In the same manner it can be seen that $\nu_{<(h\gamma)^n h,G>}(h^{'}) \leq \nu_{<(h\gamma)^{n-1}h,G>}(h^{'})$, for all $h^{'} \in H$. Thus $<(h\gamma)^{n-1}h, G> \subseteq <(h\gamma)^n h, G>$.

(3) Since $< h, G >$ is an IFI of H, we have for $h^{'} \in H$
$\mu_{<h,G>}(h^{'}) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'})\} \geq \mu_G(h) > 0$ and
$\nu_{<h,G>}(h^{'}) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h\gamma_1 m\gamma_2 h^{'})\} \leq \nu_G(h) < 1$. It imply $h^{'} \in Supp(< h, G >)$. So $H \subseteq Supp(< h, G >)$. But $Supp(< h, G >) \subseteq H$ always, so $Supp(< h, G >) = H$. □

Theorem 3.9. *Assume that H is a Γ-ring and G is an IFPI of H. Then for all $h_1, h_2 \in H$ $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \mu_G(h_1) \vee \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \nu_G(h_1) \wedge \nu_G(h_2)$. Conversely, let G is an IFI of a Γ-ring H such that $Img(G) = \{(1, 0), (\theta, \eta)\}$, where $0 \leq \theta, \eta < 1$ with $\theta + \eta \leq 1$ and $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \mu_G(h_1) \vee \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \nu_G(h_1) \wedge \nu_G(h_2)$ holds for all $h_1, h_2 \in H$, then G is an IFPI of H.*

Proof. Let G be an IFPI of H. Then (i) $\mu_G(0_H) = 1, \nu_G(0_H) = 0$ (ii) G_* is a prime ideal of H (iii) $Img(G) = \{(1, 0), (\theta, \eta)\}$, where $0 \leq \theta, \eta < 1$ s.t. $\theta + \eta \leq 1$.
Clearly $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 1$ or θ; $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 0$ or η.

Case(i) Let $\mu_G(h_1) \vee \mu_G(h_2) = 1$. Suppose $\mu_G(h_1) = 1$, then $\nu_G(h_1) = 0$. This implies that $h_1 \in G_*$. Since G_* is an ideal of H so $h_1\gamma_1 m\gamma_2 h_2 \in G_*$, for all $\gamma_1, \gamma_2 \in \Gamma$ and for all $m, h_2 \in H$. Therefore $\mu_G(h_1\gamma_1 m\gamma_2 h_2) = 1$ and $\nu_G(h_1\gamma_1 m\gamma_2 h_2) = 0$, for all $\gamma_1, \gamma_2 \in \Gamma$, $m, h_2 \in H$. Hence $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 1 = \mu_G(h_1) \vee \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 0 = \nu_G(h_1) \wedge \nu_G(h_2)$.

Case(ii) Let $\mu_G(h_1) \vee \mu_G(h_2) = \theta$. Then atleast one of $\mu_G(h_1)$ or $\mu_G(h_2)$ is θ. Suppose $\mu_G(h_1) = \theta$ and so $\nu_G(h_1) = \eta$. This implies $h_1 \notin G_*$. Hence $h_1\Gamma H\Gamma h_2 \nsubseteq G_*$. Thus $\exists^{'}s$, $\gamma_1, \gamma_2 \in \Gamma$ and $m \in H$ such that $h_1\gamma_1 m\gamma_2 h_2 \notin G_*$. Hence $\mu_G(h_1\gamma_1 m\gamma_2 h_2) \neq 1$ and $\nu_G(h_1\gamma_1 m\gamma_2 h_2) \neq 0$. As $Img(G) = \{(1, 0), (\theta, \eta)\}$, so we have $\mu_G(h_1\gamma_1 m\gamma_2 h_2) = \theta$ and $\nu_G(h_1\gamma_1 m\gamma_2 h_2) = \eta$. Thus $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \theta = \mu_G(h_1) \vee \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \eta = \nu_G(h_1) \wedge \nu_G(h_2)$.

Conversely, to prove the converse it suffices to show that G_* is a prime ideal of H. Let $h_1, h_2 \in H$ such that $h_1\Gamma H\Gamma h_2 \subseteq G_*$. Then for all $\gamma_1, \gamma_2 \in \Gamma$, $m \in H$, $h_1\gamma_1 m\gamma_2 h_2 \in G_*$. So $\mu_G(h_1\gamma_1 m\gamma_2 h_2) = 1$ and $\nu_G(h_1\gamma_1 m\gamma_2 h_2) = 0$, for all $\gamma_1, \gamma_2 \in \Gamma$ and $m \in H$.
Hence $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 1$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 0$. So $\mu_G(h_1) \vee \mu_G(h_2) = 1$ and $\nu_G(h_1) \wedge \nu_G(h_2) = 0$. This implies that $\mu_G(h_1) = 1, \nu_G(h_1) = 0$ or $\mu_G(h_2) = 1, \nu_G(h_2) = 0$, i.e., $h_1 \in G_*$ or $h_2 \in G_*$. Thus G_* is a prime ideal of H. Hence G is an IFPI of H (By Theorem (2.18)). □

Proposition 3.10. *Let G is an IFPI of Γ-ring H and $h \in H$, then $\mu_{<h,G>}(k) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{<h\gamma_1 m\gamma_2 h,G>}(k)\}$ and $\nu_{<h,G>}(k) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{<h\gamma_1 m\gamma_2 h,G>}(k)\}$, for all $k \in H$.*

Proof. Now

$$\begin{aligned}
\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}(\mu_{<h\gamma_1 m\gamma_2 h,G>}(k)) &= \inf_{m\in H,\gamma_1^{'},\gamma_2^{'}\in\Gamma}\{\inf_{m\in H}(h^{'}\gamma_1^{'} m\gamma_2^{'} k)\}, \text{ where } h^{'} = h\gamma_1 m\gamma_2 h \\
&= \inf_{m\in H,\gamma_1^{'},\gamma_2^{'}\in\Gamma}\{\mu_G(h^{'}) \vee \mu_G(k)\} \text{ as } G \text{ is an IFPI} \\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h) \vee \mu_G(k)\} \\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h)\} \vee \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(k)\} \\
&= \mu_G(h) \vee \mu_G(h) \vee \mu_G(k) = \mu_G(h) \vee \mu_G(k) \\
&= \inf_{m\in H,\gamma_3,\gamma_4\in\Gamma}\{\mu_G(h\gamma_3 m\gamma_4 k)\} \text{ as } G \text{ is an IFPI} \\
&= \mu_{<h,G>}(k).
\end{aligned}$$

Similarly, other results can also be proved. □

Definition 3.11. If H is a Γ-ring. $N \subseteq H$ and $h \in H$, we define

$$< h, N >= \{h^{'} \in H | h\Gamma H \Gamma h^{'} \subseteq N\}$$

Proposition 3.12. *Let H be a Γ-ring and $\emptyset \neq N \subseteq H$. Then for every $h \in H$, we have*

$$< h, \chi_N >= \chi_{<h,N>}.$$

Proof. Let $h^{'} \in H$. Now $\mu_{<h,\chi_N>}(h^{'}) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\chi_N}(h\gamma_1 m\gamma_2 h^{'})\} = 1$ or 0 and $\nu_{<h,\chi_N>}(h^{'}) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{\chi_N}(h\gamma_1 m\gamma_2 h^{'})\} = 0$ or 1.

Case(i) If $\mu_{<h,\chi_N>}(h^{'}) = 1$ and so $\nu_{<h,\chi_N>}(h^{'}) = 0$ and therefore,
$\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\chi_N}(h\gamma_1 m\gamma_2 h^{'})\} = 1$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{\chi_N}(h\gamma_1 m\gamma_2 h^{'})\} = 0$.
This implies $h\gamma_1 m\gamma_2 h^{'} \in N$, for all $\gamma_1, \gamma_2 \in \Gamma, m \in H$ and so $h^{'} \in< h, N >$.
Hence $\mu_{\chi_{<h,N>}}(h^{'}) = 1, \nu_{\chi_{<h,N>}}(h^{'}) = 0$.
Thus in this case we have $< h, \chi_N >= \chi_{<h,N>}$.

Case(ii) If $\mu_{<h,\chi_N>}(h^{'}) = 0$ and so $\nu_{<h,\chi_N>}(h^{'}) = 1$ and therefore,
$\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_{\chi_N}(h\gamma_1 m\gamma_2 h^{'})\} = 0$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_{\chi_N}(h\gamma_1 m\gamma_2 h^{'})\} = 1$.
Hence $h\gamma_1 m\gamma_2 h^{'} \notin N$, for some $\gamma_1, \gamma_2 \in \Gamma, m \in H$.
This implies $h^{'} \notin< h, N >$. Hence $\mu_{\chi_{<h,N>}}(h^{'}) = 0, \nu_{\chi_{<h,N>}}(h^{'}) = 1$.
Thus in this case we have $< h, \chi_N >= \chi_{<h,N>}$. Hence the result proved. □

Theorem 3.13. *Let G be an IFPI of a Γ-ring H, $h \in H$ be such that $h \notin G_*$, then $< h, G >= G$. Conversely, let G be an IFI of H such that $Img(G) = \{(1, 0), (\theta, \eta)\}$, where $0 \leq \theta, \eta < 1$ with $\theta + \eta \leq 1$. If $< h, G >= G$, for some $h \in H$ for which $G(h) = (\theta, \eta)$, then G is an IFPI of H.*

Proof. Let G be an IFPI of H. Then (i) $\mu_G(0_H) = 1, \nu_G(0_H) = 0$ (ii) G_* is a prime ideal of H (iii) $Img(G) = \{(1, 0), (\theta, \eta)\}$, where $0 \leq \theta, \eta < 1$ with $\theta + \eta \leq 1$. Let $h_2 \in H$.

Case(i) If $h_2 \in G_*$, then $h_1\gamma_1 m\gamma_2 h_2 \in G_*$ for all $\gamma_1, \gamma_2 \in \Gamma, m, h_1 \in H$. So, $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 1 = \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 0 = \nu_G(h_2)$. That is $\mu_{<h_1,G>}(h_2) = \mu_G(h_2)$ and $\nu_{<h_1,G>}(h_2) = \nu_G(h_2)$, i.e., $< h_1, G > (h_2) = G(h_2)$.

Case(ii) Let $h_2 \notin G_*$. As G_* is a prime ideal of H, $h_1\gamma_1 m\gamma_2 h_2 \notin G_*$, for some $\gamma_1, \gamma_2 \in \Gamma, m, h_1 \in H$. So $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \theta = \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \eta = \nu_G(h_2)$, i.e., $\mu_{<h_1,G>}(h_2) = \mu_G(h_2)$ and $\nu_{<h_1,G>}(h_2) = \nu_G(h_2)$, i.e., $< h_1, G > (h_2) = G(h_2)$. So in both cases, we get $< h_1, G >= G$.

Conversely, let $h_1, h_2 \in H$.
Case(i) Let $\mu_G(h_1) = \theta, \nu_G(h_1) = \eta$. Now $\mu_G(h_2) = \mu_{<h_1,G>}(h_2) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\}$ and $\nu_G(h_2) = \nu_{<h_1,G>}(h_2) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\}$. Since $Img(G) = \{(1, 0), (\theta, \eta)\}$, where $0 \leq \theta, \eta < 1$ such that $\theta + \eta \leq 1$. Now $\mu_G(h_2) \geq \theta = \mu_G(h_1)$ and $\nu_G(h_2) \leq \theta = \nu_G(h_1)$. So $\mu_G(h_1) \vee \mu_G(h_2) = \mu_G(h_2)$ and $\nu_G(h_1) \wedge \nu_G(h_2) = \nu_G(h_2)$. Therefore we have $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \mu_G(h_1) \vee \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = \nu_G(h_1) \wedge \nu_G(h_2)$.
Case(ii) Let $\mu_G(h_1) = 1, \nu_G(h_1) = 0$, then $h_1 \in G_*$. As G is an IFI of H, G_* is an ideal of H. Hence $h_1\gamma_1 m\gamma_2 h_2 \in G_*$, for all $\gamma_1, \gamma_2 \in \Gamma, m, h_2 \in H$. So $\inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(x\gamma_1 m\gamma_2 y)\} = 1 = \mu_G(h_1) \vee \mu_G(h_2)$ and $\sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\nu_G(h_1\gamma_1 m\gamma_2 h_2)\} = 0 = \nu_G(h_1) \wedge \nu_G(h_2)$, for all $h_1, h_2 \in H$. Hence by the converse of Theorem (3.9) G is an IFPI of H. □

Theorem 3.14. *If G is an IFPI of a Γ-ring H, $h \in H$ be such that $h \in G_*$, then $< h, G >= \chi_H$.*

Proof. Let G be an IFPI of H. Then (i) $\mu_G(0_H) = 1, \nu_G(0_H) = 0$ (ii) G_* is a prime ideal of H (iii) $Img(G) = \{(1, 0), (\theta, \eta)\}$, where $0 \leq \theta, \eta < 1$ such that $\theta + \eta \leq 1$.
Let $h^{'} \in H$ and $h \in G_*$, then $h\gamma_1 m\gamma_2 h^{'} \in G_*$, for all $\gamma_1, \gamma_2 \in \Gamma, m, h^{'} \in H$. So

$$\mu_{<h,G>}(h^{'}) = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'})\} = 1 = \mu_{\chi_H}(h^{'}) \text{ and}$$
$$\nu_{<h,G>}(h^{'}) = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'})\} = 0 = \nu_{\chi_H}(h^{'}), \text{ for all } h^{'} \in H.$$

Hence $< h, G >= \chi_H$. □

Corollary 3.15. *If I is a prime ideal of Γ-ring H, then for $h \notin I$, $< h, \chi_I >= \chi_I$.*

Proof. Let I be a prime ideal of H. Then χ_I is an IFPI of H. Now $h \notin I$ implies $h \notin (\chi_I)_*$, we have by Theorem (3.13) $< h, \chi_I >= \chi_I$. □

Theorem 3.16. *If G is an IFS of a Γ-ring H such that $< h, G >= G, \forall h \in H$, then G is constant.*

Proof. For $h_1, h_2 \in H$ we have

$$\begin{aligned}
\mu_G(h_1) &= \mu_{<h_2,G>}(h_1), \text{ as } < h_2, G >= G \text{ for every } h_2 \in H \\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_2\gamma_1 m\gamma_2 h_1)\} = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h_1\gamma_1 m\gamma_2 h_2)\} \\
&= \mu_{<h_1,G>}(h_2) \\
&= \mu_G(h_2).
\end{aligned}$$

Thus $\mu_G(h_1) = \mu_G(h_2)$. Similarly we can show $\nu_G(h_1) = \nu_G(h_2)$, for all $h_1, h_2 \in H$. Hence G is constant. □

Proposition 3.17. *Let G be an IFPI of a Γ-ring H and $h \in H$. Then either $< h, G >$ is an IFPI of H or $< h, G >$ is constant.*

Proof. Let G be an IFPI of H and $h \in H$
Case(i) If $h \notin G_*$. By Theorem (3.13) $< h, G >= G$. This proves that $< h, G >$ is an IFPI of H.
Case(ii) If $h \in G_*$. Then $h\gamma_1 m\gamma_2 h' \in G_*$, for all $\gamma_1, \gamma_2 \in \Gamma, m, h' \in H$. Hence $\mu_{<h,G>}(h') = \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h')\} = 1; \nu_{<h,G>}(h') = \sup_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h')\} = 0$, for all $h' \in H$. This proves $< h, G >$ is a constant. □

Proposition 3.18. *Let G be an IFS of a Γ-ring H. Then G is an IFSPI of H if and only if $G(h\gamma h) = G(h)$, for all $h \in H$ and for all $\gamma \in \Gamma$.*

Proof. Let $G(h\gamma x) = G(h)$, for all $h \in H$ and for all $\gamma \in \Gamma$. Let G' be an IFI of H such that $G'\Gamma G' \subseteq G$. Let $G' \nsubseteq G$. Then $\exists^s\ h' \in H$ such that $\mu_{G'}(h') > \mu_G(h')$ and $\nu_{G'}(h') < \nu_G(h')$.

Now $\mu_{G'\Gamma G'}(h'\gamma h') \geq \mu_{G'}(h') > \mu_G(h')$ and $\nu_{G'\Gamma G'}(h'\gamma h') \leq \nu_{G'}(h') < \mu_G(h')$.
Again $\mu_G(h') = \mu_G(h'\gamma h') \geq \mu_{G'\Gamma G'}(h'\gamma h')$ and $\nu_G(h') = \nu_G(h'\gamma h') \leq \nu_{G'\Gamma G'}(h'\gamma h')$. This implies that $G'\Gamma G' = G$, which is a contradiction. Hence $G' \subseteq G$. Thus G is an IFSPI of H.

Conversely, let G be an IFSPI of H. Now for any $h \in H$, we have

$$\begin{aligned}
\mu_G(h) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h)\}(\text{ from Proposition (2.18)}) \\
&\geq \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h)\} \\
&\geq \mu_G(h\gamma_i h).
\end{aligned}$$

Again $\mu_G(h\gamma_i h) \geq \mu_G(h)$. Thus $\mu_G(h\gamma_i h) = \mu_G(h)$. In the similar manner it can be seen that $\nu_G(h\gamma_i h) = \nu_G(h)$. That is $G(h\gamma h) = G(h)$ for all $h \in H, \gamma \in \Gamma$. □

Proposition 3.19. *Assume for a commutative Γ-ring, G be an IFSPI of H. Then $< h, G >$ is an IFSPI of H for every $h \in H$.*

Proof. Let G be an IFSPI of H and $h \in H$. Then by Proposition (3.2) $< h, G >$ is an IFI of H.

For each $h^{'} \in H, \gamma \in \Gamma$, we have

$$\begin{aligned}
\mu_{<h,G>}(h^{'}) &= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'})\}\\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G\{(h\gamma_1 m\gamma_2 h^{'})\gamma(h\gamma_1 m\gamma_2 h^{'})\}\}(\text{ as } G \text{ is an IFSPI })\\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G\{(h\gamma_1 m\gamma_2 h^{'})\gamma(h^{'}\gamma_1 h\gamma_2 m)\}\}\\
&\geq \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2 h^{'}\gamma h^{'})\}\\
&= \inf_{m\in H,\gamma_1,\gamma_2\in\Gamma}\{\mu_G(h\gamma_1 m\gamma_2(h^{'}\gamma h^{'}))\}\\
&= \mu_{<h,G>}(h^{'}\gamma h^{'}).
\end{aligned}$$

Again $\mu_{<h,G>}(h^{'}\gamma h^{'}) \geq \mu_G(h^{'})$, as $< h, G >$ is IFI of H. Thus $\mu_{<h,G>}(h^{'}\gamma h^{'}) = \mu_G(h^{'})$. In the similar manner it can be seen that $\nu_{<h,G>}(h^{'}\gamma h^{'}) = \nu_G(h^{'})$ for all $h^{'} \in H, \gamma \in \Gamma$, by Proposition (3.18), $< h, G >$ is an IFSPI of H. □

Corollary 3.20. *Assume that for a comm. Γ-ring H and $\{G_i : i \in J\} \neq \emptyset$ family of IFSPIs of H. If $\mu_G(h) = \inf_{i\in J}\{\mu_{G_i}(h)\}$ and $\nu_G(h) = \sup_{i\in J}\{\nu_{G_i}(h)\}$. Then for any $h \in H$, $< h, G >$ is an IFSPI of H.*

Proof. G is an IFS of H. Let $m_1, m_2 \in H, \gamma \in \Gamma$, then

$$\begin{aligned}
\mu_G(m_1 - m_2) &= \inf_{i\in J}\{\mu_{G_i}(m_1 - m_2)\}\\
&\geq \inf_{i\in J}\{\mu_{G_i}(m_1) \wedge \mu_{G_i}(m_1)\}\\
&= \{\inf_{i\in J}\{\mu_{G_i}(m_1)\}\} \wedge \{\inf_{i\in J}\{\mu_{G_i}(m_2)\}\}\\
&= \mu_G(m_1) \wedge \mu_G(m_2).
\end{aligned}$$

Similarly, we can show that $\nu_G(m_1 - m_2) \leq \nu_G(m_1) \vee \nu_G(m_2)$. Also

$$\begin{aligned}
\mu_G(m_1\gamma m_2) &= \inf_{i\in J}\{\mu_{G_i}(m_1\gamma m_2)\}\\
&\geq \inf_{i\in J}\{\mu_{G_i}(m_1) \vee \mu_{G_i}(m_1)\}\\
&= \{\inf_{i\in J}\{\mu_{G_i}(m_1)\}\} \vee \{\inf_{i\in J}\{\mu_{G_i}(m_2)\}\}\\
&= \mu_G(m_1) \vee \mu_G(m_2).
\end{aligned}$$

Similarly, we can show that $\nu_G(m_1\gamma m_2) \geq \nu_G(m_1) \wedge \nu_G(m_2)$. Thus G is an IFI of H. Let $h \in H$, $\gamma \in \Gamma$, we have $\mu_G(h) = \inf_{i\in J}\{\mu_{G_i}(h)\} = \inf_{i\in J}\{\mu_{G_i}(h\gamma h)\} = \mu_G(h\gamma h)$, as each G_i is IFSPIs of H. Similarly, we can show that $\nu_G(h) = \nu_G(h\gamma h)$, for all $\gamma \in \Gamma$. Then by Proposition (3.18), G is an IFSPI of H and the result follows by using Proposition (3.19). □

Corollary 3.21. *Assume that for a Γ-ring H which is commutative and $\{P_i : i \in J\} \neq \emptyset$ family of semi-prime ideals of H and $P = \bigcap_{i\in J} P_i \neq \emptyset$. Then $< h, \chi_P >$ is an IFSPI of H for every $h \in H$.*

Proof. Since $P = \bigcap_{i\in J} P_i$, is a semi-prime ideal of H (By [5], Theorem 4). Then χ_P is an IFSPI of H. Thus by Proposition (3.19) $< h, \chi_P >$ is an IFSPI of H. □

Conclusion

This article defines, the concept of extension of an IFS G of a Γ-ring H with respect to an element $h \in H$, denoted by $< h, G >$, has been defined and their properties studied. It is shown that if G is an IFI of H, then $< h, G >$ is also an IFI; however, the converse need not be true. The study of the extension of an IFS G, in the special case when G is an IFPI or an

IFSPI, has been explored. It is shown that when G is an IFPI, then the extension ideal $< h, G >$ *is either an IFPI or is a constant depending upon* $h \notin G_*$ *or* $h \in G_*$*, respectively, where* $G_* = \{h \in H : \mu_G(h) = \mu_G(0)$ *and* $\nu_G(h) = \nu_G(0)\}$*. Also, when the IFI G is an IFSPI, then the extension ideal* $< h, G >$ *is always an IFSPI. Some more related results have been examined.*

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