Palestine Journal of Mathematics

Vol 14(1)(2025) , 456–465

Some results for 3-prime near-rings with multiplicative generalized derivations

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16Y30; Secondary 16W25, 16N60.

Keywords and phrases: 3-prime near-ring, multiplicative generalized derivations, semigroup ideal, commuting map.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract. The purpose of this paper is to determine the structure of a 3-prime near-ring N admitting a multiplicative generalized derivation F and a multiplicative multiplier σ satisfying either of the following conditions: (i) $F([x,y]_\sigma) = \pm x^m(x \circ y)_\sigma x^n$, (ii) $F([x,y]_\sigma) = \pm x^m[x,y]_\sigma x^n$, (iii) $F(x \circ y)_\sigma = \pm x^m(x \circ y)_\sigma x^n$, (iv) $F(x \circ y)_\sigma = \pm x^m[x,y]_\sigma x^n$, (v) $F([x,y]_\sigma) = \pm[F(x),y]_\sigma$, (vi) $F(x \circ y)_\sigma = \pm(F(x) \circ y)_\sigma$ for all $x,y \in U$, where U is a nonzero semigroup ideal of N and m,n are non-negative integers. We also give examples to justify the hypothesis of our results.

1 Introduction

A right near-ring N is a triplet $(N,+,\cdot)$, where $+$ and $\cdot$ are two binary operations such that (i) $(N,+)$ is a group (not necessarily abelian), (ii) $(N,\cdot)$ is a semigroup and (iii) $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x,y,z \in N$. Consonantly, instead of (iii), if N satisfies left distributive law, then N is said to be a left near-ring. The most natural example of a right near-ring is the set of all identity preserving mappings acting from the left of an additive group G (not necessarily abelian) into itself, with pointwise addition and composition of mappings as multiplication. If these mappings act from the right on G, then we get a left near-ring (for more examples see Pilz [19]). A near-ring N is said to be zero-symmetric if $x0 = 0$ for all $x \in N$ (the right distributive law implies that $0x = 0$). Throughout the paper, N represents a zero-symmetric right-near-ring with $Z(N)$ as the multiplicative center of N. For any $x,y \in N$, the symbols $[x,y]$ and $(x \circ y)$ denote the Lie product $xy - yx$ and the Jordan product $xy + yx$, respectively. If $\sigma : N \to N$ is an arbitrary map, then we write $[x,y]_\sigma = \sigma(x)y - yx$ and $(x \circ y)_\sigma = \sigma(x)y + yx$ for all $x,y \in N$. A near-ring N is said to be 3-prime if $xNy = \{0\}$ for all $x,y \in N$ implies that $x = 0$ or $y = 0$. A nonempty subset U of a near-ring N is said to be a semigroup right (resp. semigroup left) ideal of N if $UN \subseteq U$ (resp. $NU \subseteq U$); and if U is both a semigroup right ideal and a semigroup left ideal, then it is said to be a semigroup ideal of N. If S is a nonempty subset of N, then a mapping $f : S \to N$ is said to be centralizing (resp. commuting) on S if $[f(x),x] \in Z(N)$ (resp. $[f(x),x] = 0$) for all $x \in S$. The notion of derivation in near-rings was initiated by Bell and Mason [6]. An additive mapping $d : N \to N$ is said to be a derivation on N if $d(xy) = d(x)y + xd(y)$ for all $x,y \in N$ or equivalently in [23], $d(xy) = xd(y) + d(x)y$ for all $x,y \in N$. Inspired by the definition of derivation in near-rings, Gölbaşi [17] defined generalized derivation in near-rings as follows: An additive mapping $F : N \to N$ is said to be a right (resp. left) generalized derivation associated with a derivation d on N if $F(xy) = F(x)y + xd(y)$ (resp. $F(xy) = d(x)y + xF(y)$) for all $x,y \in N$. Furthermore, F is said to be a generalized derivation associated with a derivation d on N if it is both a right generalized derivation and a left generalized derivation on N. Thus, the notion of generalized derivation covers the notion of multiplier for $d = 0$. There are many results claiming that 3-prime near-rings with certain restricted derivations and generalized derivations have ring-like behavior (for references see [1], [2], [3], [4], [6], [8], [9], [10], [11], [12], [17], [20], [21], [22], [23]).

Daif [13] introduced the notion of multiplicative derivation in rings. Further, Goldmann and Šemrl [18] gave the complete description of these mappings. Motivated by the definition of multiplicative derivation in rings, Daif and Tammam [15] extended the notion of multiplicative derivation to multiplicative generalized derivation in rings. Recently, Ashraf et al. [5] defined multiplicative derivation and multiplicative generalized derivation in near-rings as follows. A mapping (not necessarily additive) $d : N \to N$ is said to be a multiplicative derivation on a near-ring N if $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. A mapping (not necessarily additive) $F : N \to N$ is said to be a multiplicative right (resp. left) generalized derivation on a near-ring N if there exists a multiplicative derivation d on N such that $F(xy) = F(x)y + xd(y)$ (resp. $F(xy) = d(x)y + xF(y)$) for all $x, y \in N$.

In [14], Daif and Bell proved that if R is a 3-prime ring and I is a nonzero ideal of R. If R admits a derivation d such that $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, then R is commutative. Further, Dhara [16] proved that if R is a semi3-prime ring and F is a generalized derivation associated with a derivation d on R such that $F([x, y]) = \pm[x, y]$ or $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in I$, a nonzero ideal of R, then R contains a nonzero central ideal, provided $d(I) \neq \{0\}$. Moreover, he obtained that if R is a 3-prime ring, R must be commutative, provided $d \neq 0$. Further, Boua and Oukhtite [10] extended these results for 3-prime near-rings. More precisely, they proved that if N is a 3-prime near-ring with a nonzero derivation d such that $d([x, y]) = \pm[x, y]$ or $d(x \circ y) = \pm(x \circ y)$ for all $x, y \in N$, then N is a commutative ring. In [8], Boua obtained the commutativity of a 3-prime near-ring N in case of a semigroup ideal U of N satisfying one of the conditions: (i) $d([x, y]) = [d(x), y]$; (ii) $[d(x), y] = [x, y]$; (iii) $d(x \circ y) = d(x) \circ y$ and (iv) $d(x) \circ y = x \circ y$ for all $x, y \in U$. Recently, Shang [22] considered the more general situations for a generalized derivation F of a 3-prime near-ring N satisfying any one of the following: (i) $F([x, y]) = \pm x^k[x, y]x^l$; (ii) $F(x \circ y) = \pm x^k(x \circ y)x^l$ for all $x, y \in N$; where $k \geq 0$, $l \geq 0$ are non-negative integers and proved that N is a commutative ring. In this line of investigation, it is natural to look forward for some comparable results for multiplicative generalized derivation in 3-prime near-rings for more general constraints replacing $[x, y]$ and $(x \circ y)$ by $[x, y]_\sigma$ and $(x \circ y)_\sigma$ respectively. In this paper, we obtain the structure of a 3-prime near-ring N with multiplicative generalized derivation $F : N \to N$ associated with a nonzero multiplicative derivation d on N.

2 Some Preliminary Results

The following lemmas are necessary to develop and prove our main results:

Lemma 2.1. ([7], Lemma 1.2(i), (iii) and Lemma 1.3(iii)). Let N be a 3-prime near-ring.

(i) If $z \in Z(N) \setminus \{0\}$, then z is not a zero divisor.

(ii) If $z \in Z(N) \setminus \{0\}$ and $zx \in Z(N)$, then $x \in Z(N)$.

(iii) If z centralizes a nonzero semigroup left ideal, then $z \in Z(N)$.

Lemma 2.2. ([7], Lemma 1.3(i) and Lemma 1.4(i)) Let N be a 3-prime near-ring and U be a nonzero semigroup ideal of N.

(i) If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

(ii) If $x \in N$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.

Lemma 2.3. ([19], Proposition 1.5) If N is a near-ring, then $-xy = (-x)y$ for all $x, y \in N$.

Lemma 2.4. ([7], Lemma 1.5) If N is a 3-prime near-ring and $Z(N)$ contains a nonzero semigroup left ideal or a semigroup right ideal, then N is a commutative ring.

3 Results on the commutativity of 3-prime near-rings admitting multiplicative generalized derivations

In this section, we will extend numerous existing results in the literature (see, [3], [8], [11], [21], [22]) in different directions by working on multiplicative generalized derivations with more specific constraints by including other special types of mappings in near-rings.

Theorem 3.1. *Let N be a 3-prime near-ring, U be a nonzero semigroup ideal of N and $m \geq 0$, $n \geq 0$ non-negative integers. If N admits a right multiplicative generalized derivation F associated with a nonzero multiplicative derivation d which commutes with a nonzero left multiplicative multiplier σ such that $F([x,y]_\sigma) = \pm x^m [x,y]_\sigma x^n$ for all $x, y \in U$, then N is a commutative ring.*

Proof. By hypothesis,

$$F([x,y]_\sigma) = \pm x^m [x,y]_\sigma x^n \quad \text{for all} \quad x, y \in U. \tag{3.1}$$

Replacing y by yx in (3.1), we get

$$F([x,y]_\sigma x) = \pm x^m [x,y]_\sigma x^{n+1},$$

$$F([x,y]_\sigma)x + [x,y]_\sigma d(x) = \pm x^m [x,y]_\sigma x^{n+1} \quad \text{for all} \quad x, y \in U.$$

Using hypothesis, we arrive at $[x,y]_\sigma d(x) = 0$, which implies that

$$\sigma(x)yd(x) = yxd(x) \quad \text{for all} \quad x, y \in U. \tag{3.2}$$

Substituting zy for y in (3.2), where $z \in N$ and using (3.2), we obtain

$$\sigma(x)zyd(x) = zyxd(x) = z\sigma(x)yd(x),$$

which gives

$$[\sigma(x), z]yd(x) = 0 \quad \text{for all} \quad x, y \in U, z \in N,$$

i.e.,

$$[\sigma(x), z]Ud(x) = \{0\} \quad \text{for all} \quad x \in U, z \in N.$$

Applying Lemma 2.2(i), we get

$$\sigma(x) \in Z(N) \quad \text{or} \quad d(x) = 0 \quad \text{for all} \quad x \in U. \tag{3.3}$$

If there is an element $x_0 \in U$, such that $\sigma(x_0) \in Z(N)$, then from assumption we get

$$F([x, \sigma(x_0)y]_\sigma) = \pm x^m [x, \sigma(x_0)y]_\sigma x^n \quad \text{for all} \quad x, y \in U.$$

It follows

$$F([x,y]_\sigma \sigma(x_0)) = \pm x^m [x,y]_\sigma x^n \sigma(x_0) \quad \text{for all} \quad x, y \in U.$$

By definition of F,

$$F([x,y]_\sigma)\sigma(x_0) + [x,y]_\sigma d(\sigma(x_0)) = \pm x^m [x,y]_\sigma x^n \sigma(x_0) \quad \text{for all} \quad x, y \in U.$$

Using (3.1) to get

$$[x,y]_\sigma d(\sigma(x_0)) = 0 \quad \text{for all} \quad x, y \in U.$$

Therefore,

$$\sigma(x)yd(\sigma(x_0)) = yxd(\sigma(x_0)) \quad \text{for all} \quad x, y \in U. \tag{3.4}$$

Replace y by ty, in (3.4), where $t \in N$, and use it to get $[\sigma(x), t]Ud(\sigma(x_0)) = \{0\}$ for all $x \in U$, $t \in N$, by Lemma 2.2(i), we get

$$\sigma(x) \in Z(N) \quad \text{or} \quad d(\sigma(x_0)) = 0 \quad \text{for all} \quad x \in U. \tag{3.5}$$

If $\sigma(x) \in Z(N)$ for all $x \in U$, then $\sigma(xs) = \sigma(x)s \in Z(N)$ for all $x \in U$, $s \in N$, by Lemma 2.1(ii), we conclude that either $\sigma(x) = 0$ for all $x \in U$ or $s \in Z(N)$ for all $s \in N$, the first case leads to $\sigma(vx) = \sigma(v)x = 0$ for all $x \in U$, $v \in N$ and Lemma 2.2(ii) assures that $\sigma = 0$; a contradiction while the second case implies that N is a commutative ring by Lemma 2.4. According to the last result, equation (3.5) becomes N is a commutative ring or $d(\sigma(x_0)) = 0$. In view of (3.5), equation (3.3) becomes

$$N \text{ is a commutative ring} \quad \text{or} \quad d(\sigma(x)) = 0 \quad \text{for all} \quad x \in U. \tag{3.6}$$

If $d(\sigma(x)) = 0$ for all $x \in U$, it follows $d(\sigma(xz)) = d(\sigma(x)z) = \sigma(x)d(z) = 0$ for all $x \in U$, $z \in N$. Thus, $\sigma(x)d(z) = 0$ for all $x \in U$, $z \in N$. Also, $\sigma(xw)d(z) = \sigma(x)wd(z) = 0$ for all $x \in U$, $z, w \in N$. i.e. $\sigma(x)Nd(z) = \{0\}$ for all $x \in U$, $z \in N$, 3-primeness of N implies $\sigma(x) = 0$ for all $x \in U$, which is easily implies $\sigma = 0$; a contradiction. Hence, we obtain that N is a commutative ring. □

Theorem 3.1 directly leads to the following corollaries

Corollary 3.2. ([21], Theorem 2.2) *Let N be a 3-prime near-ring. If there exist non-negative integers $m \geq 0$, $n \geq 0$ and N admits a nonzero derivation d such that $d([x,y]) = \pm x^m[x,y]x^n$ for all $x, y \in N$, then N is a commutative ring.*

Corollary 3.3. ([22], Theorem 1) *Let N be a 3-prime near-ring. If there exist non-negative integers $m \geq 0$, $n \geq 0$ and N admits a right generalized derivation F associated with a nonzero derivation d such that $F([x,y]) = \pm x^m[x,y]x^n$ for all $x, y \in N$, then N is a commutative ring.*

Theorem 3.4. *Let N be a 3-prime near-ring, U be a nonzero semigroup ideal of N and $m \geq 0$, $n \geq 0$ non-negative integers. If N admits a right multiplicative generalized derivation F associated with a nonzero derivation d which commutes with a nonzero left multiplier σ such that $F(x \circ y)_\sigma = \pm x^m(x \circ y)_\sigma x^n$ for all $x, y \in U$, then N is a commutative ring.*

Proof. Suppose that

$$F(x \circ y)_\sigma = \pm x^m(x \circ y)_\sigma x^n \text{ for all } x, y \in U. \tag{3.7}$$

Substituting yx in place of y in (3.7) and using $(x \circ yx)_\sigma = (x \circ y)_\sigma x$, we get

$$F((x \circ y)_\sigma x) = \pm x^m(x \circ y)_\sigma x^{n+1} \text{ for all } x, y \in U,$$

which gives,

$$F(x \circ y)_\sigma x + (x \circ y)_\sigma d(x) = \pm x^m(x \circ y)_\sigma x^{n+1} \text{ for all } x, y \in U.$$

Now using (3.7), we find that $(x \circ y)_\sigma d(x) = 0$, which implies that

$$\sigma(x)yd(x) = -yxd(x) \text{ for all } x, y \in U. \tag{3.8}$$

Replacing y by ry for $r \in N$ in (3.8), using (3.8) and Lemma 2.3, we get

$$\begin{aligned} ryxd(x) &= r(-(\sigma(x)yd(x))) \\ &= r(-\sigma(x))yd(x) \\ &= (-\sigma(x))ryd(x) \text{ for all } x, y \in U, r \in N. \end{aligned}$$

This implies that,

$$[r, -\sigma(x)]Ud(x) = \{0\} \text{ for all } x \in U, r \in N.$$

Applying Lemma 2.2(i), we obtain

$$-\sigma(x) \in Z(N) \text{ or } d(x) = 0 \text{ for all } x \in U. \tag{3.9}$$

If there is an element $x_0 \in U$, such that $-\sigma(x_0) \in Z(N)$, then from assumption we get

$$F((u \circ (-\sigma(x_0))v)_\sigma) = \pm u^m(u \circ (-\sigma(x_0))v)_\sigma u^n \text{ for all } u, v \in U.$$

It follows

$$F((u \circ v)_\sigma(-\sigma(x_0))) = \pm u^m(u \circ v)_\sigma u^n(-\sigma(x_0)) \text{ for all } u, v \in U.$$

By definition of F,

$$F(u \circ v)_\sigma(-\sigma(x_0)) + (u \circ v)_\sigma d(-\sigma(x_0)) = \pm u^m(u \circ v)_\sigma u^n(-\sigma(x_0)) \text{ for all } u, v \in U.$$

Using (3.7) to get

$$(u \circ v)_\sigma d(-\sigma(x_0)) = 0 \text{ for all } u, v \in U. \tag{3.10}$$

Therefore,

$$(-\sigma(u))vd(-\sigma(x_0)) = vud(-\sigma(x_0)) \text{ for all } u, v \in U. \tag{3.11}$$

Replace v by tv, in (3.11), where $t \in N$, and using it to get $[-\sigma(u), t]Ud(-\sigma(x_0)) = \{0\}$ for all $u \in U$, $t \in N$, by Lemma 2.2(i), we get

$$-\sigma(u) \in Z(N) \quad \text{or } d(-\sigma(x_0)) = 0 \quad \text{for all } \ u \in U. \tag{3.12}$$

If $-\sigma(u) \in Z(N)$ for all $u \in U$, then $-\sigma(us) = (-\sigma(u))s \in Z(N)$ for all $u \in U$, $s \in N$, by Lemma 2.1(ii), we conclude that either $(-\sigma(u)) = 0$ for all $u \in U$ or $s \in Z(N)$ for all $s \in N$, the first case leading to $-\sigma(ru) = (-\sigma(r))u = 0$ for all $u \in U$, $r \in N$, and Lemma 2.2(ii) assures that $\sigma = 0$; a contradiction, while the second case implies that N is a commutative ring by Lemma 2.4. According to the last result, equation (3.12) becomes N is a commutative ring or $d(-\sigma(x_0)) = 0$. In view of (3.9), we can easily write

$$N \text{ is a commutative ring} \quad \text{or} \quad d(\sigma(x)) = 0 \quad \text{for all } \ x \in U. \tag{3.13}$$

If $d(\sigma(x)) = 0$ for all $x \in U$, it follows $d(\sigma(xz)) = \sigma(x)d(z) = 0$ for all $x \in U$, $z \in N$. Thus, $\sigma(x)d(z) = 0$ for all $x \in U$, $z \in N$. Also, $\sigma(xw)d(z) = \sigma(x)wd(z) = 0$ for all $x \in U$, $z, w \in N$. So, $\sigma(x)Nd(z) = \{0\}$ for all $x \in U$, $z \in N$, 3-primeness of N implies $\sigma(x) = 0$ for all $x \in U$, which easily gives $\sigma = 0$; a contradiction. Hence, we obtain that N is a commutative ring. □

The following corollaries are direct consequences of Theorem 3.4.

Corollary 3.5. ([21], Theorem 2.4*) Let N be a 3-prime near-ring. If there exist non-negative integers $m \geq 0$, $n \geq 0$ and N admits a nonzero derivation d such that $d(x \circ y) = \pm x^m (x \circ y)x^n$ for all $x, y \in N$, then N is a commutative ring.*

Corollary 3.6. ([22], Theorem 2*) Let N be a 3-prime near-ring. If there exist non-negative integers $m \geq 0$, $n \geq 0$ and N admits a right generalized derivation F associated with a nonzero derivation d such that $F(x \circ y) = \pm x^m (x \circ y)x^n$ for all $x, y \in N$, then N is a commutative ring.*

Theorem 3.7. *Let N be a 3-prime near-ring, U be a nonzero semigroup ideal of N and $m \geq 0$, $n \geq 0$ non-negative integers. If N admits a right multiplicative generalized derivation F associated with a nonzero multiplicative derivation d which commutes with a nonzero left multiplicative multiplier σ such that $F([x, y]_\sigma) = \pm x^m (x \circ y)_\sigma x^n$ for all $x, y \in U$, then N is a commutative ring of characteristic two.*

Proof. Suppose that

$$F([x, y]_\sigma) = \pm x^m (x \circ y)_\sigma x^n \quad \text{for all } \ x, y \in U. \tag{3.14}$$

Replacing y by yx in (3.14), using $[x, yx]_\sigma = [x, y]_\sigma x$ and $(x \circ yx)_\sigma = (x \circ y)_\sigma x$, we obtain

$$F([x, y]_\sigma x) = \pm x^m (x \circ y)_\sigma x^{n+1} \quad \text{for all } \ x, y \in U,$$

it follows that

$$F([x, y]_\sigma)x + [x, y]_\sigma d(x) = \pm x^m (x \circ y)_\sigma x^{n+1} \quad \text{for all } \ x, y \in U.$$

By hypothesis, we have $[x, y]_\sigma d(x) = 0$ for all $x, y \in U$.

$$\sigma(x)yd(x) = yxd(x) \quad \text{for all } \ x, y \in U. \tag{3.15}$$

Since (3.15) is the same as (3.2), arguing in a similar way as in Theorem 3.1, we conclude that N is a commutative ring. In this case we have

$$\begin{aligned}
\pm x^m (x \circ y)_\sigma t x^n &= \pm x^m (x \circ yt)_\sigma x^n \\
&= F([x, yt]_\sigma) \\
&= F([x, y]_\sigma t) \\
&= F([x, y]_\sigma)t + [x, y]_\sigma d(t) \\
&= \pm x^m (x \circ y)_\sigma t x^n + [x, y]_\sigma d(t) \quad \text{for all } \ x, y \in U, t \in N.
\end{aligned}$$

This implies that $(\sigma(x)-x)yd(t)=0$ for all $x,y\in U$, $t\in N$. So, $(\sigma(x)-x)Ud(t)=\{0\}$ for all $x\in U$, $t\in N$. Using Lemma 2.2(i) and $d\neq 0$, we conclude that $\sigma(x)=x$ for all $x\in U$. Replacing x by tx, where $t\in N$, we arrive at $(\sigma(t)-t)x=0$ for all $x\in U$, $t\in N$, so that $(\sigma(t)-t)U=\{0\}$ for all $t\in N$, using Lemma 2.2(ii), we get $\sigma=id_N$. Using this result and the commutativity of N, equation (3.14) becomes $2x^{m+n+1}y=0$ for all $x,y\in U$ and applying 3-primeness of N, we conclude $2U=\{0\}$, which easily gives $2N=\{0\}$. □

Theorem 3.7 directly results in the following corollary.

Corollary 3.8. ([3], Theorem 3.1*) Let N be a 3-prime near-ring, U be a nonzero semigroup ideal of N and $m\geq 0$, $n\geq 0$ non-negative integers. If N admits a right multiplicative generalized derivation F associated with a nonzero multiplicative derivation d which commutes with a nonzero left multiplicative multiplier σ such that $F([x,y])=\pm x^m(x\circ y)x^n$ for all $x,y\in U$, then N is a commutative ring.*

Theorem 3.9. *Let N be a 3-prime near-ring and U be a nonzero semigroup ideal of N and $m\geq 0$, $n\geq 0$ non-negative integers. If N admits a right multiplicative generalized derivation F associated with a nonzero derivation d which commutes with a nonzero left multiplier σ such that $F(x\circ y)_\sigma=\pm x^m[x,y]_\sigma x^n$ for all $x,y\in U$, then N is a commutative ring of characteristic two.*

Proof. Let

$$F(x\circ y)_\sigma=\pm x^m[x,y]_\sigma x^n \text{ for all } x,y\in U. \tag{3.16}$$

Substituting yx for y in (3.16), we find that

$$\begin{aligned} F(x\circ yx)_\sigma &= F(x\circ yx)_\sigma \\ &= F((x\circ y)_\sigma x) \\ &= \pm x^m[x,yx]_\sigma x^n \\ &= \pm x^m[x,y]_\sigma x^{n+1}, \end{aligned}$$

which implies that,

$$F(x\circ y)_\sigma x+(x\circ y)_\sigma d(x)=\pm x^m[x,y]_\sigma x^{n+1} \text{ for all } x,y\in U.$$

Using the hypothesis, we get

$$(x\circ y)_\sigma d(x)=0 \text{ for all } x,y\in U,$$

$$\sigma(x)yd(x)=-yxd(x) \text{ for all } x,y\in U. \tag{3.17}$$

Since (3.8) and (3.17) are similar, therefore arguing in the similar manner as in case of Theorem 3.4, we obtain N is a commutative ring. In this case, we get

$$\begin{aligned} \pm x^m[x,y]_\sigma tx^n &= \pm x^m[x,yt]_\sigma x^n \\ &= F(x\circ yt)_\sigma \\ &= F((x\circ y)_\sigma t) \\ &= F((x\circ y)_\sigma)t+(x\circ y)_\sigma d(t) \\ &= \pm x^m[x,y]_\sigma tx^n+(x\circ y)_\sigma d(t) \text{ for all } x,y\in U,t\in N, \end{aligned}$$

which implies that $(\sigma(x)+x)yd(t)=0$ for all $x,y\in U$, $t\in N$. So, $(\sigma(x)+x)Ud(t)=\{0\}$ for all $x\in U$, $t\in N$. Using Lemma 2.2(i) and $d\neq 0$, we conclude that $\sigma(x)=-x$ for all $x\in U$. Replacing x by nx, where $n\in N$, we arrive at $(\sigma(n)+n)x=0$ for all $x\in U$, $n\in N$, so that $(\sigma(n)+n)U=\{0\}$ for all $n\in N$, using Lemma 2.2(ii), we get $\sigma=-id_N$. Using this result and the commutativity of N, equation (3.14) becomes $-2x^{m+n+1}y=0$ for all $x,y\in U$ and applying 3-primeness of N, we conclude $2U=\{0\}$, which easily gives $2N=\{0\}$. □

The following corollary is the direct consequence of Theorem 3.9.

Corollary 3.10. ([3], Theorem 3.2) *Let N be a 3-prime near-ring and U be a nonzero semigroup ideal of N. If there exist non-negative integers $m \geq 0$, $n \geq 0$ and N admits a right generalized derivation F associated with a nonzero derivation d such that $F(x \circ y) = \pm x^m [x, y] x^n$ for all $x, y \in U$, then N is a commutative ring.*

The following example shows that the 3-primeness hypothesis in Theorem 3.1 and Theorem 3.7 is essential.

Example 3.11. Let S be a zero-symmetric right near-ring. Consider

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\} \text{ and } U = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in S \right\}.$$

It can be easily seen that N is a non 3-prime zero-symmetric right near-ring with respect to matrix addition and matrix multiplication and U is a nonzero semigroup ideal of N.

Now define the mappings $F, d, \sigma : N \to N$ by

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z^2 \\ 0 & 0 & 0 \end{pmatrix} , \; d \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\sigma \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that F is a right multiplicative generalized derivation associated with a multiplicative derivation d which commutes with a nonzero left multiplicative multiplier σ on N satisfying (i) $F([x, y]_\sigma) = \pm x^m (x \circ y)_\sigma x^n$, (ii) $F([x, y]_\sigma) = \pm x^m [x, y]_\sigma x^n$ for all $x, y \in U$. However, N is not commutative.

Theorem 3.12. *Let N be a 3-prime near-ring and U be a nonzero semigroup ideal of N. If N admits a nonzero left multiplicative multiplier σ and a right multiplicative generalized derivation F associated with a nonzero multiplicative derivation d such that F is commuting on U and $F([x, y]_\sigma) = \pm [F(x), y]_\sigma$ for all $x, y \in U$, then N is a commutative ring.*

Proof. Assume that

$$F([x, y]_\sigma) = [F(x), y]_\sigma \text{ for all } x, y \in U. \tag{3.18}$$

Replacing y by yx in (3.18), we get

$$\begin{aligned} F([x, yx]_\sigma) &= F([x, y]_\sigma x) \\ &= [F(x), yx]_\sigma \text{ for all } x, y \in U, \end{aligned}$$

i.e.,

$$F([x, y]_\sigma)x + [x, y]_\sigma d(x) = \sigma(F(x))yx - yxF(x) \text{ for all } x, y \in U.$$

Since F is commuting on U, therefore the last expression gives that

$$\begin{aligned} F([x, y]_\sigma)x + [x, y]_\sigma d(x) &= \sigma(F(x))yx - yF(x)x \\ &= [F(x), y]_\sigma x \text{ for all } x, y \in U, \end{aligned}$$

which reduces to,

$$[x, y]_\sigma d(x) = 0 \text{ for all } x, y \in U. \tag{3.19}$$

This implies that $\sigma(x)yd(x) = yxd(x)$ for all $x, y \in U$. Replacing y by ry in the last expression and using it again, we obtain $[\sigma(x), r]yd(x) = 0$ for all $x, y \in U$, $r \in N$. Using Lemma 2.2(i), we get $\sigma(x) \in Z(N)$ or $d(x) = 0$ for all $x \in U$. Arguing in the similar manner as in case of Theorem 3.1, we can get the result.

Using the same techniques, we can prove the result for the case $F([x, y]_\sigma) = -[F(x), y]_\sigma$ for all $x, y \in U$. □

The following corollaries are the direct consequences of Theorem 3.12.

Corollary 3.13. ([8], Theorem 2.7*) Let N be a 3-prime near-ring and U be a nonzero semigroup ideal of N. If N admits a nonzero derivation d such that $d([x, y]) = [d(x), y]$ for all $x, y \in U$, then N is a commutative ring.*

Corollary 3.14. ([11], Theorem 2.6*) Let N be a 3-prime near-ring. If N admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [F(x), y]$ for all $x, y \in N$, then N is a commutative ring.*

Theorem 3.15. *Let N be a 3-prime near-ring, U be a nonzero semigroup ideal of N. If N admits a right multiplicative generalized derivation F associated with a nonzero derivation d which commutes with a nonzero left multiplier σ such that $F(x \circ y)_\sigma = \pm(F(x) \circ y)_\sigma$ for all $x, y \in U$ and F is commuting on U, then N is a commutative ring.*

Proof. By hypothesis

$$F(x \circ y)_\sigma = (F(x) \circ y)_\sigma \text{ for all } x, y \in U. \tag{3.20}$$

Replacing y by yx in (3.20), we get

$$\begin{aligned} F(x \circ yx)_\sigma &= F((x \circ y)_\sigma x) \\ &= (F(x) \circ yx)_\sigma \text{ for all } x, y \in U, \end{aligned}$$

which implies that

$$F(x \circ y)_\sigma x + (x \circ y)_\sigma d(x) = \sigma(F(x))yx + yxF(x) \text{ for all } x, y \in U.$$

Since F is commuting on U, we get

$$\begin{aligned} F(x \circ y)_\sigma x + (x \circ y)_\sigma d(x) &= \sigma(F(x))yx + yF(x)x \\ &= (F(x) \circ y)_\sigma x \text{ for all } x, y \in U, \end{aligned}$$

Using (3.20), the last expression reduces to $(x \circ y)_\sigma d(x) = 0$ for all $x, y \in U$, which implies

$$\sigma(x)yd(x) = -yxd(x) \text{ for all } x, y \in U. \tag{3.21}$$

Since (3.21) is same as (3.8), arguing in the similar manner as in case of Theorem 3.4, we can obtain the result.
Using the same techniques, we can prove the result for the case $F((x \circ y)_\sigma) = -(F(x) \circ y)_\sigma$ for all $x, y \in U$. □

The following corollary is the direct consequence of Theorem 3.15.

Corollary 3.16. ([8], Theorem 2.10*) Let N be a 2-torsion free 3-prime near-ring and U be a nonzero semigroup ideal of N. If N admits a derivation d such that $d(x \circ y) = (d(x) \circ y)$ for all $x, y \in U$, then $d = 0$.*

The following example demonstrates that the 3-primeness hypothesis in Theorem 3.12 is not superfluous.

Example 3.17. Suppose that S is a zero-symmetric right near-ring and let

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\} \text{ and } U = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in S \right\}.$$

Then N is a zero-symmetric right near-ring with respect to matrix addition and matrix multiplication and U is a nonzero semigroup ideal of N but N is not 3-prime.

Define the mappings $F, d, \sigma : N \to N$ by

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & xz & 0 \\ 0 & 0 & 0 \\ 0 & z^2 & 0 \end{pmatrix}, \quad d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & yx & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\sigma\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z^2 & 0 \end{pmatrix}.$$

It can be easily verified that F is a right multiplicative generalized derivation associated with a multiplicative derivation d which commutes with a nonzero left multiplicative multiplier σ on N satisfying $F([x,y]_\sigma) = \pm[F(x), y]_\sigma$ for all $x, y \in U$. However, N is not commutative.

The following example shows the necessity of 3-primeness hypothesis in Theorems 3.4, 3.9 and 3.15.

Example 3.18. Suppose that S is a zero-symmetric right near-ring. Consider

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid 0, x, y, z \in S \right\} \text{ and } U = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, y \in S \right\}.$$

Then it is easy to check that N is a zero-symmetric right near-ring with respect to matrix addition and matrix multiplication and U is a nonzero semigroup ideal of N but N is not 3-prime.

Define the mappings $F, d, \sigma : N \to N$ by

$$F\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & xy & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, \quad d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\sigma\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that F is a right multiplicative generalized derivation associated with a derivation d which commutes with a nonzero left multiplier σ on N satisfying (i) $F(x \circ y)_\sigma = \pm x^m (x \circ y)_\sigma x^n$, (ii) $F(x \circ y)_\sigma = \pm x^m [x, y]_\sigma x^n$, (iii) $F(x \circ y)_\sigma = \pm (F(x) \circ y)_\sigma$ for all $x, y \in U$. However, N is not commutative.

4 Conclusion remarks

Finally, we conclude our discussion by posing two crucial questions:
(i) Can we prove these results, if we are replacing the multiplicative derivation d by any arbitrary map f on N?
(ii) Our hypothesis are dealt with the 3-prime near-rings, so, can we investigate the commutativity of semi 3-prime near-rings?. So, this is an interesting avenue for future research.

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Received: 2024-02-15
Accepted: 2024-06-27