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ON FOURTH-ORDER FULLY NONLINEAR IMPULSIVE BVPS ON A FINITE INTERVAL

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Abstract In this paper, we investigate the existence of solutions for a fourth-order fully nonlinear impulsive boundary value problem on a finite interval. We employ the Schauder fixed point theorem in our analysis. The impulsive conditions are defined by continuous functions, and the system's nonlinearity is fully continuous.

1 Introduction

Differential equations of various orders are widely regarded by mathematicians as indispensable tools for scientific inquiry, offering a versatile means to interpret and model a broad spectrum of phenomena across disciplines such as physics, chemistry, biology, and economics. References such as [13, 15, 20, 23] provide valuable insights into the application of these equations in diverse fields of study.

Additionally, the exploration of nonlinear Boundary Value Problems (BVPs) plays a pivotal role in enhancing our understanding of nonlinear physical phenomena. It also facilitates their practical application across BVPs of different orders. Notably, the investigation into the existence and uniqueness of solutions for these BVPs is crucial for comprehending the behavior of such phenomena. References like [3, 6, 8, 9, 10, 17, 21, 22] elucidate various applications aimed at achieving this goal.

In recent years, there has been a surge in interest in impulsive differential equations, owing to their ability to offer a robust framework for mathematically describing real-world scenarios. These equations find applications across diverse fields including control theory, physics, chemistry, population dynamics, biotechnology, and medicine. Noteworthy references such as [2, 4, 11, 12, 14, 16, 18, 26] delve into the significance and applications of impulsive differential equations.

Impulsive events, characterized by sudden state changes or impulses, are often modeled as short-term alterations occurring within the larger context of an ongoing process. Such impulsive phenomena are accurately captured by impulsive differential equations. Pertinent works, as documented in references [1, 5, 19, 24, 27], focus on fourth-order impulsive BVPs, offering valuable insights into their mathematical treatment and applications.

The present study is confined to the investigation of solutions for a fourth-order Boundary Value Problem (BVP) on a finite interval, incorporating impulsive conditions. We employ the Schauder fixed point theorem as a foundational tool in our analysis. Notably, this work represents the first comprehensive study of fourth-order Impulsive Boundary Value Problems featuring fully nonlinear and fully impulsive conditions.

The fourth-order differential equation under consideration is represented by:

$$v^{(4)}(t) + g(t, v(t), v'(t), v''(t), v'''(t)) = 0, \quad t \in [0,1]\backslash\{t_1, t_2, ..., t_n\}, \tag{1.1}$$

where $g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function. The boundary conditions are defined as:

$$v(0) = A, \quad v'(0) = B, \quad v''(0) = C, \quad v'''(1) = D, \tag{1.2}$$

where $A, B, C, D \in \mathbb{R}$. The impulsive conditions are delineated as follows:

$$\begin{cases} \triangle v(t_k) = I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)), \\ \triangle v'(t_k) = I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)), \\ \triangle v''(t_k) = I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)), \\ \triangle v'''(t_k) = I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)), \end{cases} \tag{1.3}$$

where $\triangle v^{(i-1)}(t_k) = v^{(i-1)}(t_k^+) - v^{(i-1)}(t_k^-)$, $I_{ik} \in C([0,1] \times \mathbb{R}^4, \mathbb{R})$, $i = 1,2,3,4$ with $0 < t_1 < t_2 < ... < t_n < 1$. Here, $J = [0,1]$, $J' = [0,1]\backslash\{t_1, \ldots, t_n\}$, $J_0 = [0, t_1]$, and $J_k = (t_k, t_{k+1}]$ for $k = 1, \ldots, n$, with $t_{n+1} = 1$.

2 Preliminary results

Consider the set

$$\begin{aligned} PC^m[0,1] = \{&v \in C^m([0,1], \mathbb{R}), \text{ for } t \neq t_k, \ v^{(i)}(t_k) = v^{(i)}(t_k^-), \ v^{(i)}(t_k^+) \text{ exist} \\ &\text{for } k = 1,2,3,\cdots,n, \text{ for } i = 0,1,\cdots,m\}. \end{aligned}$$

In this work we consider the space $X = PC^3[0,1]$ which is a Banach space with the norm $\|x\|_X := \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\}$, where

$$\|\omega^{(i)}\|_\infty = \sup_{0 \le t \le 1} |\omega^{(i)}(t)|, \ i = 0,1,2,3.$$

We give the solution of the linear impulsive problem associated with (1.1), (1.2) and (1.3).

Lemma 2.1. *Let $\eta \in L^1[0,1]$. Then the linear impulsive BVP*

$$v^{(4)}(t) + \eta(t) = 0, \quad t \in J', \tag{2.1}$$

with conditions (1.2) and (1.3), has a unique solution in X expressed as

$$\begin{aligned} v(t) =& A + Bt + \frac{C}{2}t^2 + \frac{D}{6}t^3 + \sum_{0<t_k<t} I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\ &+ \sum_{0<t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(t - t_k) \\ &+ \sum_{0<t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \left(\frac{t^2}{2} + \frac{t_k^2}{2} - t_k t\right) \\ &+ \int_0^t \left(\frac{t^2+s^2}{2} - ts\right) \left(\int_s^1 \eta(\mu) d\mu - \sum_{1>t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right) ds. \end{aligned} \tag{2.2}$$

Proof. For $t \in (0, t_1]$, by (2.1), we find that

$$v'''(t) = v'''(0) - \int_0^t \eta(s) ds$$

and

$$v'''(t_1^-) = v'''(0) - \int_0^{t_1} \eta(s) ds \tag{2.3}$$

and, for $t \in [t_1, t_2]$,

$$v'''(t) = v'''(t_1^+) - \int_{t_1}^t \eta(s)ds. \tag{2.4}$$

By (2.3) and (2.4), we have

$$v'''(t_1^-) + v'''(t) = v'''(0) + v'''(t_1^+) - \int_0^t \eta(s)ds.$$

Hence

$$\begin{aligned} v'''(t) &= v'''\left(t_1^+\right) - v'''\left(t_1^-\right) + v'''(0) - \int_0^t \eta(s)ds \\ &= I_{41}\left(t_1, v\left(t_1\right), v'\left(t_1\right), v''\left(t_1\right), v'''\left(t_1\right)\right) + v'''(0) - \int_0^t \eta(s)ds. \end{aligned}$$

Repeating this process for every positive t, we find

$$v'''(t) = v'''(0) - \int_0^t \eta(s)ds + \sum_{0<t_k<t} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right).$$

Thus, we obtain

$$D = v'''(0) + \sum_{k=1}^n I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) - \int_0^1 \eta(s)ds.$$

It follows that

$$v'''(0) = D + \int_0^1 \eta(s)ds - \sum_{k=1}^n I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right).$$

Hence

$$\begin{aligned} v'''(t) =& D + \int_t^1 \eta(s)ds + \sum_{0<t_k<t} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \\ &- \sum_{k=1}^n I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right). \end{aligned}$$

Equivalently, one has

$$v'''(t) = D + \int_t^1 \eta(s)ds - \sum_{1>t_k>t} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right). \tag{2.5}$$

Integrating (2.5) gives

$$v''\left(t_1^-\right) = C + \int_0^{t_1} \left(D + \int_s^1 \eta(\tau)d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right)\right) ds$$

and

$$v''(t) = v''\left(t_1^+\right) + \int_{t_1}^t \left(D + \int_s^1 \eta(\tau)d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right)\right) ds.$$

Hence

$$\begin{aligned} v''(t) =& v''\left(t_1^-\right) + I_{31}\left(t_1, v\left(t_1\right), v'\left(t_1\right), v''\left(t_1\right) v'''\left(t_1\right)\right) \\ &+ \int_{t_1}^t \left(D + \int_s^1 \eta(\tau)d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right) v'''\left(t_k\right)\right)\right) ds. \end{aligned}$$

For all positive t, one has

$$\begin{aligned} v''(t) =& C + I_{31}\left(t_1, v\left(t_1\right), v'\left(t_1\right), v''\left(t_1\right), v'''\left(t_1\right)\right) \\ &+ \int_0^t \left(D + \int_s^1 \eta(\tau) d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) ds \\ =& C + \sum_{0<t_k<t} I_{3k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \\ &+ \int_0^t \left(D + \int_s^1 \eta(\tau) d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) ds. \end{aligned}$$

Integrating the last equation, we have that

$$v'\left(t_1^-\right) = B + \left(C + \sum_{t_k<t_1} I_{3k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) t_1$$
$$+ \int_0^{t_1} \int_0^r \left(D + \int_s^1 \eta(\tau) d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) dsdr$$

and

$$v'(t) = v'\left(t_1^+\right) + \left(C + \sum_{0<t_k<t} I_{3k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) (t - t_1)$$
$$+ \int_{t_1}^{t} \int_0^r \left(D + \int_s^1 \eta(\tau) d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) dsdr.$$

Since $v'\left(t_1^+\right) - v'\left(t_1^-\right) = I_{21}\left(t_1, v\left(t_1\right), v'\left(t_1\right), v''(t_1), v'''(t_1)\right)$, one has

$$\begin{aligned} v'(t) =& B + \left(C + \sum_{0<t_k<t} I_{3k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) (t - t_k) \\ &+ \int_0^t \int_0^r \left(D + \int_s^1 \eta(\tau) d\tau - \sum_{1>t_k>s} I_{4k}\left(t_k, v\left(t_k\right), v'\left(t_k\right), v''\left(t_k\right), v'''\left(t_k\right)\right) \right) dsdr \\ &+ I_{21}\left(t_1, v\left(t_1\right), v'\left(t_1\right), v''(t_1), v'''(t_1)\right). \end{aligned}$$

By repeating the same process as described above, we arrive at

$$\begin{aligned} v'(t) =& B + Ct + \sum_{0<t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\ &+ \sum_{0<t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(t - t_k) \\ &+ \int_0^t (t-s) \left(D + \int_s^1 \eta(\tau) d\tau - \sum_{0>t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) ds \\ =& B + Ct + \frac{t^2}{2} D + \sum_{0<t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\ &+ \sum_{0<t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(t - t_k) \\ &+ \int_0^t (t-s) \left(\int_s^1 \eta(\tau) d\tau - \sum_{1>t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) ds. \end{aligned}$$

Similarly, we find that

$$\begin{aligned} v(t) =& A + Bt + \frac{C}{2}t^2 + \frac{D}{6}t^3 + \sum_{0<t_k<t} I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\ &+ \sum_{0<t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(t-t_k) \\ &+ \sum_{0<t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \left(\frac{t^2}{2} + \frac{t_k^2}{2} - t_k t\right) \\ &+ \int_0^t \left(\frac{t^2+s^2}{2} - ts\right) \left(\int_s^1 \eta(\mu)d\mu - \sum_{1>t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right) ds. \end{aligned}$$

□

To apply the Schauder fixed point theorem, we rely on the following lemma.

Lemma 2.2 ([7]). *A set $M \subset X$ is relatively compact if the following two conditions hold:*

(i) all functions from M are uniformly bounded;

(ii) all functions from M are equicontinuous.

Theorem 2.3 ([25] Schauder's Fixed Point Theorem). *Let E be a Banach space, and $\Omega \subset E$ a nonempty, bounded, closed, and convex subset of E. Let $T : \Omega \to E$ be a completely continuous operator with $T(\Omega) \subset \Omega$. Then T has a fixed point.*

3 Main result

The existence of at least one solution for problem (1.1), (1.2), (1.3).

Theorem 3.1. *Let $g : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function. If there exist $\rho_1 > 0$ and a positive M_{ρ_1} such that*

$$\rho_1 \geq |A| + |B| + |C| + |D| + \sum_{i=1}^{4} \sup_{\|v\|_X < \rho_1} \sum_{k=1}^{n} \Big| I_{ik}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| + M_{\rho_1} \quad (3.1)$$

and

$$||g(., v, v', v'', v''')||_{L^1[0,1]} \leq M_{\rho_1},$$

for all $v \in X$ satisfying $\|v\|_X < \rho_1$, thus problem (1.1), (1.2), (1.3) has at least a solution $v \in X$.

Proof. Problem (1.1), (1.2), (1.3) has at least one solution:
Let us define the operator $T : X \to X$ $k = 1, 2 \cdots, n$ by

$$\begin{aligned} Tv(t) =& A + Bt + \frac{C}{2}t^2 + \frac{D}{6}t^3 + \sum_{t_k<t} I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\ &+ \sum_{t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(t-t_k) \\ &+ \sum_{t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \left(\frac{t^2}{2} + \frac{t_k^2}{2} - t_k t\right) \\ &+ \int_0^t \left(\frac{t^2+s^2}{2} - ts\right) \left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right) ds, \end{aligned}$$

where

$$G_v(s) := g(s, v(s), v'(s), v''(s), v'''(s)),$$

then, for all $v \in X$ such that $\|v\|_X < \rho$, there are $a_{1,\rho}, a_{2,\rho}, a_{3,\rho}, a_{4,\rho}, M_\rho \in \mathbb{R}_+$ such that

$$\sup_{\|v\|_X<\rho} \sum_{k=1}^{n} \Big| I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \le a_{1,\rho} < +\infty,$$

$$\sup_{\|v\|_X<\rho} \sum_{k=1}^{n} \Big| I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \le a_{2,\rho} < +\infty,$$

$$\sup_{\|v\|_X<\rho} \sum_{k=1}^{n} \Big| I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \le a_{3,\rho} < +\infty,$$

$$\sup_{\|v\|_X<\rho} \sum_{k=1}^{n} \Big| I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \le a_{4,\rho} < +\infty,$$

and

$$\sup_{\|v\|_X<\rho} \sup_{t\in[0,1]} \Big| G_v(t) \Big| \le M_\rho.$$

By Lemma 2.1, the fixed points of T are solutions of problem (1.1), (1.2), (1.3). So it is sufficient to prove that T has a fixed point.

(i) T is continuous:
Let $v_m \to v$ in X, there exists $\rho > 0$ such that $\sup_m \|v_m\|_X < \rho$.
Then,

$$\begin{aligned}
&\sup_{0\le t\le 1} |(Tv_m)'''(t) - (Tv)'''(t)| \\
&= \sup_{0\le t\le 1} \Big| \int_t^1 G_{v_m}(\mu)d\mu - \sum_{t_k>t} I_{4k}(t_k, v_m(t_k), v_m'(t_k), v_m''(t_k), v_m'''(t_k)) \\
&- \int_t^1 G_v(\mu)d\mu + \sum_{t_k>t} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \\
&\le \sup_{0\le t\le 1} \int_t^1 \Big| G_{v_m}(\mu) - G_v(\mu) \Big| d\mu \\
&+ \sup_{0\le t\le 1} \sum_{t_k>t} \Big| I_{4k}(t_k, v_m(t_k), v_m'(t_k), v_m''(t_k), v_m'''(t_k)) - I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \\
&\le \int_0^1 \Big| G_{v_m}(\mu) - G_v(\mu) \Big| d\mu + \sum_{k=1}^{n} \Big| I_{4k}(t_k, v_m(t_k), v_m'(t_k), v_m''(t_k), v_m'''(t_k)) \\
&\quad - I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big|.
\end{aligned}$$

By Lebesgue Dominated Convergence Theorem, we have

$$\int_0^1 \Big| G_{v_m}(\mu) - G_v(\mu) \Big| d\mu \to 0,$$

as $m \to +\infty$. Moreover,

$$\sum_{k=1}^{n} \Big| I_{4k}(t_k, v_m(t_k), v_m'(t_k), v_m''(t_k), v_m'''(t_k)) - I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \to 0,$$

as $m \to +\infty$, so,

$$\|(Tv_m)''' - (Tv)'''\|_\infty \to 0,$$

as $m \to +\infty$. Also

$$\sup_{0\le t\le 1}\Big|(Tv_m)''(t)-(Tv)''(t)\Big|$$
$$=\sup_{0\le t\le 1}\Big|\sum_{t_k<t} I_{3k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))$$
$$-\sum_{t_k<t} I_{3k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))$$
$$+\int_0^t\left(\left(\int_s^1 G_{v_m}(\mu)d\mu-\sum_{t_k>s} I_{4k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))\right)\right.$$
$$\left.-\left(\int_s^1 G_v(\mu)d\mu-\sum_{t_k>s} I_{4k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\right)ds\right|$$
$$\le \sup_{0\le t\le 1}\Big|\sum_{t_k<t} I_{3k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))$$
$$-\sum_{t_k<t} I_{3k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\Big|$$
$$+\sup_{0\le t\le 1}\int_0^t \|(Tv_m)'''-(Tv)'''\|_\infty ds.$$

We have

$$\sum_{k=1}^n\Big|I_{3k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))-I_{3k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\Big|\to 0,$$

as $m \to +\infty$, and

$$\sup_{0\le t\le 1}\int_0^t \|(Tv_m)'''-(Tv)'''\|_\infty ds \le \|(Tv_m)'''-(Tv)'''\|_\infty \to 0,$$

as $m \to +\infty$. Moreover, we can prove that $\|(Tv_m)'-(Tv)'\|_\infty \to 0$, by the same way:

$$\sup_{0\le t\le 1}\Big|(Tv_m)'(t)-(Tv)'(t)\Big|$$
$$\le \sup_{0\le t\le 1}\sum_{t_k<t}\Big|I_{2k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))-I_{2k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\Big|$$
$$+\sup_{0\le t\le 1}\sum_{t_k<t}(t-t_k)\Big|I_{3k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))-I_{3k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\Big|$$
$$+\sup_{0\le t\le 1}\int_0^t (t-s)\Big|\left(\int_s^1 G_{v_m}(\mu)d\mu-\sum_{t_k>s} I_{4k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))\right)$$
$$-\left(\int_s^1 G_v(\mu)d\mu-\sum_{t_k>s} I_{4k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\right)\Big|ds$$
$$\le \sum_{k=1}^n\Big|I_{2k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))-I_{2k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\Big|$$
$$+\sum_{k=1}^n\Big|I_{3k}(t_k,v_m(t_k),v_m'(t_k),v_m''(t_k),v_m'''(t_k))-I_{3k}(t_k,v(t_k),v'(t_k),v''(t_k),v'''(t_k))\Big|$$
$$+\|(Tv_m)'''-(Tv)'''\|_\infty\to 0,$$

as $m \to +\infty$. We have by the same technique

$$\sup_{0\le t\le 1}\Big|Tv_m(t)-Tv(t)\Big|\to 0,$$

as $m \to +\infty$.

(ii) T is compact:

Let $U \subset X$ be any bounded subset, therefore there is $\rho > 0$ such that $\|v\|_X < \rho$ for all $v \in U$ and

$$L_\rho = (M\rho + a_{4,\rho}).$$

For each $v \in U$, one has

$$\begin{aligned}\|(Tv)'''\|_\infty &= \sup_{0\le t\le 1}\Big| D + \left(\int_t^1 G_v(\mu)d\mu - \sum_{t_k>t} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right)\Big| \\ &\le |D| + L_\rho < +\infty.\end{aligned}$$

$$\begin{aligned}\|(Tv)''\|_\infty = \sup_{0\le t\le 1}|(Tv)''(t)| &\le |C| + |D| + \sup_{0\le t\le 1}\Big|\sum_{t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\ &+ \sup_{0\le t\le 1}\int_0^t \Big|\left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right)\Big| ds \\ &\le |C| + |D| + \Big|\sum_{t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| + \sup_{0\le t\le 1}\int_0^t L_\rho ds \\ &\le |C| + |D| + a_{3,\rho} + L_\rho < +\infty.\end{aligned}$$

$$\begin{aligned}\|(Tv)'\|_\infty = \sup_{0\le t\le 1}|(Tv)'(t)| &\le |B| + |C| + \frac{|D|}{2} + \sup_{0\le t\le 1}\Big|\sum_{t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\ &+ \sup_{0\le t\le 1}\Big|\sum_{t_k<t}(t-t_k) I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\ &+ \sup_{0\le t\le 1}\int_0^t (t-s)\Big|\left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right)\Big| ds \\ &\le |B| + |C| + \frac{|D|}{2} + \Big|\sum_{t_k<t} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\ &+ \Big|\sum_{t_k<t} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\ &+ \sup_{0\le t\le 1}\int_0^t (t-s)L_\rho ds \le |B| + |C| + \frac{|D|}{2} + a_{2,\rho} + a_{3,\rho} + \frac{L_\rho}{2} < +\infty.\end{aligned}$$

$$\|(Tv)\|_\infty = \sup_{0\le t\le 1}|(Tv)(t)| \le |A| + |B| + \frac{|C|}{2} + \frac{|D|}{6} + a_{1,\rho} + a_{2,\rho} + a_{3,\rho} + L_\rho < +\infty.$$

Therefore,

$$\|Tv\|_X = \max\{\|Tv\|_\infty, \|(Tv)'\|_\infty, \|(Tv)''\|_\infty, \|(Tv)'''\|_\infty\} < +\infty,$$

that is, TU is uniformly bounded. As T is continuous and from (3.1), if we take the closed bounded ball $U \subset X$ of center 0 and radius $r = \rho_1$, then we have $TU \subset U$.

TU is equicontinuous, because, for $[0, L] \subset [0, 1]$ and $r_1, r_2 \in [0, L] \cap J_k$ such that $k = 0, 1, \ldots, n$ and $r_1 < r_2$, we have

$$
\begin{aligned}
&\Big|(Tv)'''(r_2) - (Tv)'''(r_1)\Big| \\
&= \Big| \int_{r_2}^{1} G_v(\mu)d\mu - \sum_{t_k > r_2} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\
&\quad - \int_{r_1}^{1} G_v(\mu)d\mu + \sum_{t_k > r_1} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \\
&\le \Big| - \int_{r_1}^{r_2} G_v(\mu)d\mu + \sum_{r_2 > t_k > r_1} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| \to 0,
\end{aligned}
$$

as $r_1 \to r_2$.

$$
\begin{aligned}
&|(Tv)''(r_2) - (Tv)''(r_1)| \\
&= \Big| \sum_{t_k < r_2} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) - \sum_{t_k < r_1} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\
&\quad + Dr_2 - Dr_1 + \int_0^{r_2} \left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k > s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) ds \\
&\quad - \int_0^{r_1} \left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k > s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) ds \Big| \\
&\le \sum_{r_1 < t_k < r_2} \Big| I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \Big| + \Big| Dr_2 - Dr_1 \Big| \\
&\quad + \int_{r_1}^{r_2} \Big| \left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k > s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) \Big| ds \\
&\le \sum_{r_1 < t_k < r_2} \Big| I_{3k}(t_k, v(t_k), v'(t_k), , v''(t_k), v'''(t_k)) \Big| + \int_{r_1}^{r_2} L_{\rho_1} ds \to 0,
\end{aligned}
$$

as $r_1 \to r_2$. Also,

$$
\begin{aligned}
&|(Tv)'(r_2) - (Tv)'(r_1)| \\
&= \Big| Cr_2 + \sum_{t_k < r_2} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\
&\quad + \sum_{t_k < r_2} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(r_2 - t_k) \\
&- Cr_1 - \sum_{t_k < r_1} I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \\
&\quad - \sum_{t_k < r_1} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(r_1 - t_k) + \frac{Dt_2^2}{2} - \frac{Dt_1^2}{2} \\
&\quad + \int_0^{r_2} (r_2 - s) \left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k > s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) ds \\
&\quad - \int_0^{r_1} (r_1 - s) \left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k > s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) \right) ds \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{r_1<t_k<r_2} \Big|I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| + \Big|r_2 - r_1\Big|\Big|C\Big| \\
&\quad + \sum_{r_1<t_k<r_2} \Big|I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))r_2\Big| \\
&\quad + \sum_{t_k<r_1} \Big|I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))(r_2 - r_1)\Big| \\
&\quad + \sum_{r_1<t_k<r_2} \Big|I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))t_k\Big| + \Big|\frac{Dt_2^2}{2} - \frac{Dt_1^2}{2}\Big| \\
&\quad + \int_0^{r_1} \Big|r_2 - r_1\Big|\Big|\left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right)\Big|ds \\
&\quad + \int_{r_1}^{r_2} \Big|r_2 - s\Big|\Big|\left(\int_s^1 G_v(\mu)d\mu - \sum_{t_k>s} I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right)\Big|ds \\
&\leq \sum_{r_1<t_k<r_2} \Big|I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\
&\quad + r_2 \sum_{r_1<t_k<r_2} \Big|I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\
&\quad + \quad \Big|r_2 - r_1\Big|\Big|C| + \Big|r_2 - r_1\Big| \sum_{t_k<r_1} \Big|I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\Big| \\
&\quad + \Big|\sum_{r_1<t_k<r_2} I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))t_k\Big| + \Big|\frac{Dt_2^2}{2} - \frac{Dt_1^2}{2}\Big| \\
&\quad + \int_0^{r_1} \Big|r_2 - r_1\Big|L_{\rho_1} ds + \int_{r_1}^{r_2} \Big|r_2 - s\Big|L_{\rho_1} ds \to 0,
\end{aligned}
$$

as $r_1 \to r_2$. and we have by the same technique $|(Tv)(r_2) - (Tv)(r_1)| \to 0$ as as $r_1 \to r_2$.

So, by Lemma 2.2, the set TU is relatively compact.

As T is completely continuous, by Schauder Fixed Point Theorem, T has at least one fixed point $v \in X$. □

4 Example

Consider the impulsive boundary value problem:

$$
v^{(4)}(t) + g(t, v(t), v'(t), v''(t), v'''(t)) = 0, \quad \text{a.e. } t \in [0, 1]\setminus \{0.1, 0.8\}, \tag{4.1}
$$

where $g(t, x, y, z, w) = \frac{1}{10}(\frac{y^2x}{y^2+7t+2} + \cos(tx)y + \frac{x^2z}{x^2+4\sqrt{t+1}} + \frac{z^4w}{z^4+3})$, $t \in [0, 1]$ and

$$
v(0) = 0,\ v'(0) = 1 \quad,\ v''(0) = 0.5 \quad v'''(1) = 0, \tag{4.2}
$$

with the impulsive conditions for $t_k \in \{0.1, 0.8\}$,

$$
\begin{cases}
120I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) = \frac{v'^2(t_k)v(t_k)}{v'^2(t_k)+2} + \frac{v''^4(t_k)v'(t_k)}{v''^4(t_k)+1} \\
\qquad + \frac{v'^2(t_k)v''(t_k)}{v'^2(t_k)+5} + \frac{\sqrt{t_k}v'^4(t_k)v'''(t_k)}{v'^4(t_k)+4}, \\
120I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) = \sin(v''(t_k))v(t_k) + \frac{v'^3(t_k)}{v'^2(t_k)+1} \\
\qquad + v''(t_k) + t_k v'''(t_k), \\
120I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) = v(t_k) + v'(t_k) + \frac{\sin(t_k v(t_k))v''(t_k)}{v''^2(t_k)+3} \\
\qquad + \frac{v'''^3(t_k)}{v'''^2(t_k)+1}, \\
120I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k)) = v(t_k) + t_k^2 \cos(v(t_k))v'(t_k) \\
\qquad + v''(t_k) + v'''(t_k).
\end{cases} \tag{4.3}
$$

The impulsive conditions (4.3) satisfy (3.1), for all $v \in X$ such that $\|v\|_X < \rho$. Indeed,

$$\begin{cases} \sum_{k=1}^{2} \left|I_{1k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right| \leq \frac{\rho}{10} < +\infty, \\ \sum_{k=1}^{2} \left|I_{2k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right| \leq \frac{\rho}{10} < +\infty, \\ \sum_{k=1}^{2} \left|I_{3k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right| \leq \frac{\rho}{10} < +\infty, \\ \sum_{k=1}^{2} \left|I_{4k}(t_k, v(t_k), v'(t_k), v''(t_k), v'''(t_k))\right| \leq \frac{\rho}{10} < +\infty, \end{cases}$$

and $|g(t, v(t), v'(t), v''(t), v'''(t))| \leq \frac{4}{10}\rho$,

$$\int_0^1 |g(t, v(t), v'(t), v''(t), v'''(t))|dt \leq \frac{4}{10}\rho.$$

Take $\rho = \rho_1 = 20$,

$$\frac{3}{2} + \frac{8\rho_1}{10} = 17.5 \leq 20. \tag{4.4}$$

Then, problem (4.1), (4.2), (4.3) has at least one solution $v \in X$.

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