Palestine Journal of Mathematics

Vol 14(1)(2025) , 489–498

A Class of Unitary Modules with Stable Range 2

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Communicated by Ayman Badawi

MSC 2010 Classifications: 16D10, 19B10, 13A15, 13A99.

Keywords and phrases: B_J-stable module; B-stable module; B-stable ring; B-stability and the rank of a module.

The research of the third author was in part supported by grant no. 1403130016 from IPM

Abstract The notion and some properties of B-rings, in a natural way, are extended to B-stable and B_J-stable modules. Let $J(R)$ be the Jacobson radical of a commutative ring R with identity $1 \neq 0$. A sequence $(a_1, \ldots, a_{n+1})$, $n \geq 1$, of elements in R is said to be unimodular if $1 \in (a_1, \ldots, a_{n+1})$. A ring R is said to be a B-ring if for any unimodular sequence $(a_1, \ldots, a_{n+1})$, $n \geq 2$, with $(a_1, \ldots, a_{n-1}) \not\subseteq J(R)$, there exists an element b in R such that $(a_1, \ldots, a_n + ba_{n+1}) = R$. Our aim in this paper is to extend this notion in the context of modules. It is shown that a cyclic module over a B-ring (in particular, semilocal ring, Noetherian ring in which every prime ideal is maximal, Dedekind domain) is B_J-stable. Also, a torsion-free cyclic R-module is B-stable if and only if R is a B-stable ring. Any module of rank greater than or equal to 3 is never B_J-stable and consequently not B-stable. We show that a finitely generated R-module A is B_J-stable if $\mathrm{Z}(B)$ is finite for every submodule B of A with $B \not\subseteq J(A)$, where $\mathrm{Z}(B)$ denotes the set of all maximal submodules of A containing B.

1 Introduction

The main purpose of this paper is to study *B-stable and B_J-stable modules* (Definition 3.1), in general as a natural extension of *B-rings* [14], which is similar to the notion of rings having *stable range* 2 (Remark 1.2, Definition 1.1, and Lemma 3.2).

In this introductory section, we recall the definitions of *an n-stable module, n-stable ring, and a B-ring* from [19], [5], and [14], respectively, for the sake of reference and end the section with a brief description related to the organization of the following sections.

The concept of *stable range* was initiated by H. Bass in his investigation of *the stability properties of the general linear group in algebraic K-theory* [4]. In ring theory, stable range provides an arithmetic invariant for rings that is related to interesting issues such as cancellation, substitution, and exchange. The simplest case of stable range 1 has especially proved to be important in the study of many ring-theoretic topics.

In this paper, unless otherwise indicated, all rings considered are *commutative with identity* $1 \neq 0$ and all *modules are unitary*. Also by a *sequence* of elements of a ring R or of a module A, we mean a *finite sequence* and will use it implicitly without any confusion in the context. A sequence $(a_1, \ldots, a_{n+1})$, $n \geq 1$, of elements of a ring R is said to be *unimodular* if $1 \in (a_1, \ldots, a_{n+1})$. Such a sequence plays an important role in the study of projective modules (see [9]).

We now recall the definition of *an n-stable module* from [19], which is a natural extension of n-stable rings, and later in this paper will define and study some properties of B-stable and B_J-stable modules as a subclass of 2-stable modules (see also Lemma 3.2, Definition 3.1 of B-stable and B_J-stable modules; and B-stable rings as a special case of B-stable modules).

Definition 1.1. For any integer $n \geq 1$, a sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ of elements of an R-module A over a ring R is said to be a unimodular sequence whenever the submodule $\langle a_1, a_2, \ldots, a_n, a_{n+1} \rangle =$

A. A sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ of elements of A is said to be stable whenever

$$\langle a_1, a_2, \ldots, a_n, a_{n+1}\rangle = \langle a_1 + r_1 a_{n+1}, a_2 + r_2 a_{n+1}, \ldots, a_n + r_n a_{n+1}\rangle$$

for some $r_1, r_2, \ldots, r_n \in R$. For any fixed integer $n \geq 1$, A is said to be an n-stable module provided that any unimodular sequence of elements in A of size larger than n is stable. For convenience, an n-stable module is called stable] whenever $n = 1$. It follows that if A is an n-stable module, then also A is m-stable for any fixed integer $m \geq n$.

We now recall the definition of *an n-stable ring* from [5].

Remark 1.2. A sequence $a_1, a_2, \ldots, a_n, a_{n+1}$, $n \geq 1$, of elements of a ring R is said to be stable whenever the ideal

$$(a_1, a_2, \ldots, a_n, a_{n+1}) = (a_1 + r_1 a_{n+1}, a_2 + r_2 a_{n+1}, \ldots, a_n + r_n a_{n+1})$$

for some $r_1, r_2, \ldots, r_n \in R$. For a fixed integer $n \geq 1$, a ring R is said to be n-stable provided that any unimodular sequence of elements in R of size strictly larger than n is stable. Note that the term "R is n-stable" is used for convenience in [15] and [17] is exactly the same as the term *"n is in the stable range of R"* as used in [5].

The following example provides a relation between the *dimension* and the stable range of a commutative ring.

Example 1.3. In Theorem 3.4 of [6], it is shown that any n-dimensional commutative integral domain [resp., ring] is $n+1$- [resp., $n+2$-] stable.

We now recall the definition of a *B-ring* from [14] (see also Definition 3.1 of B-stable and B_J-stable modules; and B-stable rings).

• Let $J(R)$ be the *Jacobson radical* of a commutative ring R. A ring R is said to be a *B-ring* if for any unimodular sequence $(a_1, \ldots, a_{n+1})$ of elements of R, $n \geq 2$, with $(a_1, \ldots, a_{n-1}) \not\subseteq J(R)$, there exists an element b in R such that $(a_1, \ldots, a_n + ba_{n+1}) = R$.

Remark 1.4. As in [5], we use $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, to denote both a *sequence* and the *ideal* generated by the elements of the sequence; but the context will always make our meaning clear. Also, we follow [5] for the term *"unimodular sequence"* instead of *"primitive vector"* as used in [14]. For a detailed study of *B-rings, B-semirings, and stable range in commutative rings*; see [14], [15], [16], [17], [18], [20], [12], [13], [1], [8], and [5].

• The organization of this paper is as follows: In Section 2, we recall some definitions and prove some results related to unitary modules for the sake of completeness and will use them in the sequel. In Section 3, we study some algebraic properties of B-stable and B_J-stable modules. It is shown that in our study of B-stable [resp. B_J-stable] cyclic R-modules, we need only consider sequences of size three instead of arbitrary $(n+1)$-tuples (Theorem 3.6). The homomorphic image of a cyclic B-stable [resp. B_J-stable] module is again a B-stable [resp. B_J-stable] module (Theorem 3.8). It is shown that a cyclic module over a B-ring [in particular, semilocal ring, Noetherian ring in which every prime ideal is maximal, Dedekind domain (Theorem 3.16)] is B_J-stable (Corollary 3.9. It is shown that a torsion-free cyclic R-module is B-stable if and only if R is a B-stable ring (Theorem 3.12); and also a multiplication module over a semilocal ring is B_J-stable (Corollary 3.17). Further, in this section, we will show that for any submodule $B \subseteq J(A)$ of a finitely generated R-module A, A is B-stable if and only if A/B is B-stable. In Section 4, we show that a finitely generated R-module A is B_J-stable whenever $\mathrm{Z}(B)$ is finite with $B \not\subseteq J(A)$, where $\mathrm{Z}(B)$ denotes the set of all maximal submodules of A containing the submodule B of A (Theorem 4.1); and discuss some B-stability condition of a cyclic module with some restriction on $\mathrm{Z}(a)$ when $a \notin J(A)$ (Theorem 4.3). Finally, we conclude the paper with a relation between the B-stability and rank of A (Theorem 4.5), i.e., it is shown that any module of rank greater than or equal to 3 is never B_J-stable (consequently, not a B-stable module).

2 Background on Modules

In this section, we will recall some definitions and prove some results concerning unitary modules over a commutative ring with identity $1 \neq 0$ for the sake of completeness, which will be used in the sequel. If B is a submodule of an R-module A, we denote the *annihilator ideal* of A/B by

$$(B : A) = \{r \in R \mid rA \subseteq B\}$$

and note that if the commutative ring R has an identity, then for any submodule (ideal) B of the R-module $R = A$, $(B : A)$ is precisely B. A proper submodule M of A is said to be *maximal* provided that for any submodule N of A with $M \subseteq N \subseteq A$, then either $M = N$ or $N = A$. It is not difficult to show that any submodule M of A is maximal if and only if for each $X \in A \setminus M$, $\langle x, M\rangle = A$. We define the *Jacobson radical* of an R-module A, denoted by $J(A)$, to be the intersection of all maximal submodules of A. If A has no maximal submodules, then we set $J(A) = A$. It is not difficult to show that $J(A) = A$ if and only if $A = \langle 0\rangle$ (apply Proposition 2.1(i)). An element u of an R-module A is said to be a *unit* provided that u is not contained in any maximal submodule of A. A *minimal generating set* of an R-module A is a subset X of A such that $\langle X\rangle = A$ and no proper subset of X spans A. Clearly any minimal generating set of a *finitely generated* module A is finite. For a finitely generated R-module A, we say that A is of *rank* m (m a positive integer) if A has a minimal generating set of m elements and does not have a minimal generating set of fewer than m elements. We extend the definition of *"rank"* as given in [7] for *free modules* to an arbitrary finitely generated module A, the rank of A being the *cardinality of a minimal generating set* of A of least size. The two definitions coincide in the case of free modules, since any basis is clearly a minimal generating set, and no minimal generating set of size less than that of a basis exists.

We end this section with the following proposition.

Proposition 2.1. *Let A be a (nonzero) finitely generated R-module over a ring R. Then the following assertions hold.*

(i) *Every proper submodule of A is contained in a maximal submodule of A. In particular, A has a maximal submodule.*

(ii) *If B is a maximal submodule of A, then $(B : A)$ is a maximal ideal of R.*

(iii) *Assume that $A = \langle\{a_i\}_{i\in I}\rangle$, where I is an arbitrary index set. If $x \in J(A)$, $k \in I$, and $r \in R$; then $A = \langle a_k - rx, \{a_i\}_{i\neq k}\rangle$.*

(iv) *Let $B \subseteq J(A)$ be a submodule of a finitely generated R-module A. If $A/B = \langle\{a_i + B\}_{i\in I}\rangle$ for an arbitrary index set I, then $A = \langle\{a_i\}_{i\in I}\rangle$.*

Proof. For the proof of (i) and (ii), See Theorem 2.8 in [2] and Proposition 4 in [10], respectively. Actually, result (i) is easily obtained, simply by using a proof quite similar to the standard proof given for maximal ideals in a ring with unity.

For the proof of (iii), let $x \in J(A)$ and $A = \langle\{a_i\}_{i\in I}\rangle$. Choose any r in R and any k in I. Consider the submodule B of A, $B = \langle a_k - rx, \{a_i\}_{i\neq k}\rangle$. If $B \neq A$, then by part (i), there exists a maximal submodule M of A such that $B \subseteq M$. But $rx \in J(A) \subseteq M$, so that $a_k = (a_k - rx) + rx \in M$. Hence $A = \langle\{a_i\}_{i\in I}\rangle \subseteq M$. This contradicts maximality of M as a proper subset of A. Thus $B = A$.

For the proof of (iv), let $A = \langle g_1, g_2, \dots, g_n\rangle$ with $B \subseteq J(A)$ and $A/B = \langle\{a_i + B\}_{i\in I}\rangle$. Then for some finite subset I_0 of I, we have $g_k + B = \sum r_i^{(k)} a_i + B$, where $i \in I0$ for each $k = 1, 2, \dots, n$. Hence, $g_k = \sum r_i^{(k)} a_i + b_k$ for some b_k in $B \subseteq J(A)$, $i \in I_0$. Now by part (iii), $A = \langle g_1 - b_1, \dots, g_n - b_n\rangle$ and for each k, $g_k - b_k$ in $\langle\{a_i\}_{i\in I_0}\rangle \subseteq \langle\{a_i\}_{i\in I}\rangle$. □

3 Some Properties of B-stable and B_J-stable Modules

In this section, we merely focus on some algebraic properties of B-stable and B_J-stable modules, where B-stable modules can be regarded as a generalization of a special case of 2-stable rings and B_J-stable modules are exactly a natural extension of B-rings to unitary modules (see the introduction and Definition 3.1). It is shown that in our study of B-stable [resp. B_J-stable] cyclic R-modules, we need only consider sequences of size three instead of arbitrary $(n+1)$-tuples (Theorem 3.6). The homomorphic image of a cyclic B-stable [resp. B_J-stable] module is again a B-stable [resp. B_J-stable] module (Theorem 3.8). It is shown that a cyclic module over a B-ring is B_J-stable (Corollary 3.9 and also a torsion-free cyclic R-module is B-stable if and only if R is a B-stable ring (Theorem 3.12). Further, we will show that for any submodule $B \subseteq J(A)$ of a finitely generated R-module A, A is B-stable if and only if A/B is B-stable. Finally, we end the section by applying Corollary 3.9 to show that a cyclic module over a semilocal ring, Noetherian ring in which every prime ideal is maximal, and Dedekind domain is B_J-stable (see Remark 3.15 and Theorem 3.16); and also a multiplication module over a semilocal ring is B_J-stable (Corollary 3.17).

We now state the definitions of the B-*stable sequence*, B-stable module, B_J-stable module, and B-stable ring.

Definition 3.1. Let A be an R-module over a ring R. A sequence $(a_1,\dots,a_{n+1})$, $n \geq 2$, of elements in A is said to be B-stable if there exists an element $R \in R$ such that

$$\langle a_1,\dots,a_{n+1}\rangle = \langle a_1,\dots,a_n + ra_{n+1}\rangle.$$

A sequence $(a_1,\dots,a_{n+1})$, $n \geq 1$, of elements in A is said to be *unimodular* if $A = \langle a_1,\dots,a_{n+1}\rangle$. An R-module A is said to be a B-stable [resp. B_J-stable] module whenever any unimodular sequence $(a_1,\dots,a_{n+1})$, $n \geq 2$, of elements in A [resp. with $\langle a_1,a_2,\dots,a_{n-1}\rangle \nsubseteq J(A)$] is B-stable. A ring R is said to be a B-stable ring whenever for any unimodular sequence $(a_1,\dots,a_{n+1})$, $n \geq 2$, of elements in R, there exists an element r in R such that the ideal $(a_1,\dots,a_n + ra_{n+1}) = R$.

Lemma 3.2. *Let A be a B_J-stable module over a ring R. Then A is a 2-stable module (Definition 1.1).*

Proof. Without loss of generality, let $A = \langle a_1,a_2,a_3\rangle$. If $a_1 \notin J(A)$, then $A = \langle a_1,a_2+ra_3\rangle$ by hypothesis for some $r \in R$. Now suppose $a_1 \in J(A)$. Then we claim that $A = \langle (a_1+a_3,a_2+0a_3\rangle$. Otherwise, by Proposition 2.1(i), $\langle a_1+a_3,a_2\rangle$ is contained in a maximal submodule M of A, which implies $A \subseteq M$ since $a_1 \in J(A)$, yielding a contradiction. □

Remark 3.3. Clearly, any B-stable module is a 2-stable module (Definition 1.1), for instance, if $A = \langle a_1,a_2,a_3\rangle$ is a B-stable module, then $A = \langle a_1+0a_3,a_2+ra_3\rangle$ for some $r \in R$. Actually, the class of B_J-stable modules is contain in the class of 2-stable modules by Lemma 3.2 and, obviously, each B-stable module is a B_J-stable module.

Example 3.4. A simple R-module A is a B-stable module and consequently, a B_J-stable module (a nonzero unitary R-module A is simple if its only sub- modules are 0 and A). That is, $A = \langle a_1,a_2,\dots,a_n,a_{n+1}\rangle = \langle a_1,a_2,\dots,a_n+0a_{n+1}\rangle$, where $n \geq 2$ and $a_i \neq 0$ for each $1 \leq i \leq n+1$.

Proposition 3.5. *Let $n \geq 2$ be a fixed integer. Then any unimodular sequence $a_1,\dots,a_{n+1}$ of an R-module A is B-stable whenever a_n or a_{n+1} is in $J(A)$.*

Proof. Without loss of generality, assume $a_n \in J(A)$. Hence if $\langle a_1,\dots,a_n+a_{n+1}\rangle \neq A$, then by Proposition 2.1(i), there exists a maximal submodule M of A such that $\langle a_1,\dots,a_n+a_{n+1}\rangle \subseteq M$ which implies $A \subseteq M$, and this contradicts the maximality of M. □

Theorem 3.6. *Let A be a cyclic module over a commutative ring R. Then A is B-stable [resp. B_J-stable] if and only if any unimodular sequence (a_1,a_2,a_3) [resp. with $a_1 \notin J(A)$] is B-stable. In other words, there exists an element r in R such that $\langle a_1,a_2+ra_3\rangle = \langle a_1,a_2,a_3\rangle$.*

Proof. The necessary part is quite clear. We just give a proof for the B_J-stable case and leave the other part to the reader. To prove the sufficient part, let $A = \langle a\rangle$ be a cyclic R-module. Let $(a_1,\dots,a_{n+1}), n \geq 2$, be a unimodular sequence in A with the condition $\langle a_1,\dots,a_{n-1}\rangle \nsubseteq J(A)$. Without loss of generality, assume $a_1 \notin J(A)$. Now,

$$a \in \langle a_1,a_2,\dots,a_{n+1}\rangle$$

implies $a = \sum r_i a_i$ for some $r_1, \ldots, r_{n+1} \in R$. Thus, $a \in \langle a_1, a_n, l \rangle$, where

$$l = r_2 a_2 + r_3 a_3 + \cdots + r_{n-1} a_{n-1} + r_{n+1} a_{n+1}.$$

Now by the hypothesis, there exists s in R such that

$$a \in \langle a_1, a_n + sl \rangle \subseteq \langle a_1, a_2, \ldots, a_{n-1}, a_n + s r_{n+1} a_{n+1} \rangle.$$

□

In view of the above theorem, we need only consider triples instead of arbitrary $(n+1)$-tuples in our study of B-stable [resp. B_J-stable] cyclic R-modules. Note that Rahimi in [18, Theorem 2] has proved that a ring R is a B-ring if and only if any unimodular sequence (a_1, a_2, a_3) of elements of R with $a_1 \notin J(R)$ is B-stable. In other words, there exists an element r in R such that the ideal $(a_1, a_2 + ra_3) = (a_1, a_2, a_3)$.

The results in the following remark are obtained from [11] and we will use them in the proof of Theorem 3.8 and omit their proofs since they could be verified directly from the definition.

Remark 3.7. Let A be a finitely generated R-module and B an R-module. Then the following assertions hold.

(i) The element $u \in A$ is unit if and only if $\langle u \rangle = A$ ([11, Theorem 1.4]).

(ii) Let $\phi : A \to B$ be an R-module epimorphism. Then $u \in A$ a unit implies that $\phi(u)$ is a unit in B ([11, Theorem 1.5]).

(iii) Let A be an R-module (not necessarily finitely generated) such that A has a unit. Then $x \in J(A)$ if and only if $u - rx$ is a unit in A for any element $r \in R$ and any unit u in A ([11, Theorem 1.6]).

Theorem 3.8. *The homomorphic image of a B-stable [resp. B_J-stable] cyclic R-module is B-stable [resp. B_J-stable].*

Proof. We just give a proof for the B_J-stable case and leave the other part to the reader. Let B be a submodule of an B_J-stable cyclic R-module $A = \langle a \rangle$. Suppose $\langle a_1 + B, \ldots, a_{n+1} + B \rangle$ is a unimodular sequence in A/B with $\langle a_1 + B, \ldots, a_{n-1} \rangle \nsubseteq J(A/B)$. Without loss of generality, suppose that $a_1 + B \notin J(A/B)$. Hence, for appropriate $r_1, \ldots, r_{n+1} \in R$,

$$a + B = \sum_{i=1}^{n+1} r_i a_i + B$$

implies

$$a = \sum_{i=1}^{n} r_i a_i + r_{n+1} a_{n+1} + b$$

for some $b \in B$. Now, by applying the results in Remark 3.7, it follows that $a_1 \notin J(A)$. In this case, the rest of the proof follows directly from the definition. □

Corollary 3.9. *If A is a cyclic module over a B-stable ring [resp. B-ring], then A is a B-stable [resp. B_J-stable] module.*

Proof. The proof follows directly from the above result and the fact that a cyclic module $A = Ra$ over a ring R is the homomorphic image of the R-module R. That is, the map $R \to Ra$ given by $r \mapsto ra$ is an R-module epimorphism. Note that a B-stable ring [resp. B-ring] R can be regarded as a B-stable [resp. B_J-stable] module over itself. □

Note that Rahimi in [18, Theorem 3] has proved that the homomorphic image of a B-ring is always a B-ring.

Lemma 3.10. *Let A be an R-module. Then $J(R)A \subseteq J(A)$.*

Proof. Let M be any maximal submodule of the R-module A. Then $J(R) \subseteq (M : A)$ by Proposition 2.1(ii) and thus $J(R)A \subseteq (M : A)A \subseteq M$. Since M is any arbitrary maximal submodule of A, then $J(R)A \subseteq J(A)$. □

We now prove the result of the above corollary by using the definitions of a B-stable and B_J-stable modules for torsion-free cyclic modules without using Theorem 3.8.

Theorem 3.11. *Let $A = Ra$ be a torsion-free cyclic R-module over a B-stable ring [resp. B-ring] R. Then A is a B-stable [resp. B_J-stable] module.*

Proof. Here, we just make an argument for the B_J-stable case and leave the other part to the reader. Let $A = \langle a \rangle$ be a torsion-free cyclic R-module. Without loss of generality, let $a \in \langle a_1, a_2, a_3 \rangle$ with $a_1 \notin J(A)$ and $a_i = r_i a$ for some $r_i \in R$ $(i = 1, 2, 3)$. Thus, by torsion-freeness of A, $1 \in (r_1, r_2, r_3)$. Now, by Lemma 3.10, $r_1 \notin J(R)$ since $a_1 \notin J(A)$; and hence $1 \in (r_1, r_2 + sr_3)$ for some $s \in R$ by hypothesis. Therefore, $a \in \langle a_1, a_2 + sa_3 \rangle$ and the proof is complete. □

Note that the converse of the above result is also true for the B-stable case and we state it in the following form.

Theorem 3.12. *A torsion-free cyclic R-module is B-stable if and only if R is a B-stable ring.*

Proof. The proof follows directly from the definition (see also the argument in the proof of the above theorem). □

Theorem 3.13. *Let $B \subseteq J(A)$ be a submodule of a finitely generated R-module A over a ring R. Then A is a B-stable module if and only if A/B is a B-stable module.*

Proof. The proof follows directly from the definition and Proposition 2.1(iv) above. For the necessary part, let A be a B-stable module and

$$(a_1 + B, \dots, a_{n+1} + B)$$

be a unimodular sequence in A/B with $n \geq 2$. By Proposition 2.1(iv),

$$A = \langle a_1, \dots, a_{n+1} \rangle.$$

Clearly,

$$a_{n+1} \in \langle a_1, \dots, a_n, a_{n+1} \rangle$$

implies

$$a_{n+1} \in \langle a_1, \dots, a_n + ra_{n+1} \rangle$$

for some $r \in R$ by hypothesis. Thus,

$$a_{n+1} + B \in \langle a_1 + B, \dots, a_n + ra_{n+1} + B \rangle$$

and hence,

$$a_n + B \in \langle a_1 + B, a_2 + B, \dots, a_n + ra_{n+1} + B \rangle,$$

which implies

$$\langle a_1 + B, a_2 + B, \dots, a_n + ra_{n+1} + B \rangle = A/B.$$

Conversely, suppose that A/B is a B-stable module and $\langle a_1, \dots, a_n, a_{n+1} \rangle = A$ with $n \geq 2$. Thus,

$$\langle a_1 + B, \dots, a_n + B, a_{n+1} + B \rangle = A/B$$

implies

$$\langle a_1 + B, a_2 + B, \dots, a_n + ra_{n+1} + B \rangle = A/B$$

for some r in R by hypothesis. Now by Proposition 2.1(iv),

$$A = \langle a_1, a_2, \dots, a_n + ra_{n+1} \rangle.$$

□

We end this section with a few results related to some special cases of rings such as semilocal rings and Dedekind domains that are B-rings and applying Corollary 3.9 to show that the cyclic modules over them are B_J-stable (Theorem 3.16).

Theorem 3.14. *A cyclic module over a local ring (a ring with a unique maximal ideal) is B_J-stable.*

Proof. The result follows directly from Corollary 3.9 and the fact that a local ring is a B-ring. But, here we just prove this directly by using the definition of a B_J-stable module. Let $A = Ra$ be a cyclic module over a local ring R. Suppose $\langle a_1, a_2, a_3 \rangle = \langle a \rangle$ with $a_1 \notin J(A)$. Therefore, $a_i = r_i a$ for some $r_i \in R$ $(1 \le i \le 3)$. Since by Lemma 3.10, $J(R)A \subseteq J(A)$, then $r_1 \notin J(R)$. Hence, $1 \in (r_1, r_2, r_3) = (r_1, r_2 + r_3)$ since r_1 is a unit in R. Consequently, $a \in \langle a_1, a_2 + a_3 \rangle$ and the proof is complete. □

Remark 3.15. Actually, we recall from Corollary 3.9, a more general result than the above theorem, that a cyclic module A over a B-ring is a B_J-stable module. Also, in the remark following Theorem 2.2 of [14], it concludes from that theorem that quasi-semi-local rings (rings with a finite number of maximal ideals) and Noetherian rings in which every proper prime ideal is maximal (in particular, Dedekind domains) are B-rings. Therefore, we can write the following theorem as an application of Corollary 3.9.

Theorem 3.16. *A cyclic module over a semilocal ring (a ring with a finite number of maximal ideals) [resp. Noetherian ring in which every proper prime ideal is maximal (in particular, Dedekind domain)] is a B_J-stable module.*

From this result and the fact that every multiplication module (i.e., an R-module M in which every submodule of M is of the form IM, for some ideal I of R) over a semilocal ring is cyclic [3, Proposition 4], we can have the following corollary.

Corollary 3.17. *Any multiplication module over a semilocal ring is a B_J-stable module.*

4 Some Special Cases

For convenience, we introduce the notation $\mathrm{Z}(B)$ to mean the set of maximal submodules of an R-module A containing the submodule B of A; and $\mathrm{Z}(a)$ will denote the set of maximal submodules of A containing the element a. In this section, we show that A is a B_J-stable module whenever $\mathrm{Z}(B)$ is finite with $B \nsubseteq J(A)$ (Theorem 4.1) and discuss some B-stability condition of a cyclic module with some restriction on $\mathrm{Z}(a)$ when $a \notin J(A)$ (Theorem 4.3). Finally, we conclude the paper by showing that any module of rank greater than or equal to 3 is never B_J-stable and consequently, not a B-stable module (Theorem 4.5).

Theorem 4.1. *If A is a finitely generated module over a ring R such that for every submodule $B \nsubseteq J(A)$, $\mathrm{Z}(B)$ is finite, then A is a B_J-stable module.*

Proof. The proof is essentially the same as the proof of Theorem 2.2 of [14]. Let $A = \langle a_1, \ldots, a_n \rangle$, $n \ge 3$, and $\langle a_1, \ldots, a_{n-2} \rangle \nsubseteq J(A)$. By the hypothesis on A, $\mathrm{Z}(B)$ is finite, where $B = \langle a_1, \ldots, a_{n-2} \rangle$. Let $\mathrm{Z}(B) = \{M_1, \ldots, M_k\}$, and note that if r in R and $a_{n-1} + ra_n \notin M_i$ for each $i = 1, 2, \ldots, k$, then $(a_1, \ldots, a_{n-2}, a_{n-1} + ra_n)$ is a unimodular sequence by Proposition 2.1(i). For any M_i in $\mathrm{Z}(B)$ such that $a_n \in M_i$, we have $a_{n-1} + ra_n \notin M_i$, for all r in R; otherwise, $a_{n-1} \in M_i$, and $\langle a_1, \ldots, a_n \rangle \subseteq M_i$ which contradicts the hypothesis that $(a_1, \ldots, a_n)$ is a unimodular sequence. For those M_i's in $Z(B)$ for which $a_n \notin M_i$, we have $\langle a_n, M_i \rangle = A$. Hence, there exists an $x_i \in R$ such that $a_n x_i \equiv a_{n-1} (\mathrm{mod}\ M_i)$. For these M_i's, we can find by virtue of Proposition 2.1(ii) and the Chinese Remainder Theorem, an element r in R such that $r \equiv 1 - x_i (\mathrm{mod}\ (M_i : A))$ provided that $(M_i : A)$'s are distinct. Therefore, $ra_n \equiv a_n - a_n x_i \equiv a_n - a_{n-1} (\mathrm{mod}\ M_i)$. Consequently, It follows that $(a_{n-1} + ra_n) \notin M_i$ since $a_n \notin M_i$, $i = 1, \ldots, k$. Hence $(a_1, \ldots, a_{n-2}, a_{n-1} + ra_n)$ is a unimodular sequence. Note that for the case when $a_n \notin M_i$ just for only one i, $1 \le i \le k$, we can write $1 - x_i \equiv 1 - x_i (\mathrm{mod}\ (M_i : A))$ to get $(1 - x_i)a_n \equiv a_n - x_i a_n (\mathrm{mod}\ M_i) \equiv a_n - a_{n-1} (\mathrm{mod}\ M_i)$. Hence, $a_{n-1} + (1 - x_i)a_n \equiv a_n (\mathrm{mod}\ M_i)$, and by the same argument, $(a_{n-1} + (1 - x_i)a_n) \notin M_i$

since $a_n \notin M_i$. Finally, we complete the proof by concerning another special case when the ideals $(M_i : A)$ are equal for some of those M_i's not containing a_n. Without loss of generality, suppose $a_n \notin M_i$ and

$$(M_1 : A) = \cdots = (M_l : A)$$

for $1 \leq i \leq l \leq k$. Hence, $\langle a_n, M_i\rangle = A$ and $a_{n-1} = x_i a_n + m_i$ for some $x_i \in R$ and $m_i \in M_i$ with $i = 1, 2, \ldots, l$. Clearly,

$$1-x_1 \equiv 1-x_1(\text{mod}\,(M_1 : A)), 1-x_2 \equiv 1-x_2(\text{mod}\,(M_2 : A)), \ldots, 1-x_l \equiv 1-x_l(\text{mod}\,(M_l : A)).$$

Now, by a similar argument as above, it is not difficult to show that $a_{n-1} + (1 - x_i)a_n \notin M_i$ for all $1 \leq i \leq k$. □

Example 4.2. It follows from this theorem that every finitely generated module with a finite number of maximal submodules is B_J-stable. Actually, any finitely generated module in which every (proper) submodule has a unique decomposition of a finite intersection of maximal submodules is B_J-stable.

Theorem 4.3. *Let $A = Rx$ be a cyclic module over a ring R which satisfies the condition that for every a and c in A with $a \notin J(A)$, there is an $m \in A$ such that $\mathrm{Z}(m) = \mathrm{Z}(a) \setminus \mathrm{Z}(c)$. Then A is a B_J-stable module.*

Proof. The proof is essentially the same as the proof of Theorem 2.3 of [14]. Let $A = \langle a, b, c\rangle$ with $a \notin J(A)$. By the hypothesis on A, there exists $m \in A$ such that $\mathrm{Z}(m) = \mathrm{Z}(a)\setminus\mathrm{Z}(c)$. Hence, $\langle c, m\rangle = A$ since every proper submodule of a module is contained in a maximal submodule (Proposition 2.1(i)). Thus, $x - b \in \langle m, c\rangle$ implies that for some r in R, $x \in \langle m, b + rc\rangle$. Therefore, $A = \langle m, b + rc\rangle$. Now, we claim $\langle a, b + rc\rangle = A$. Otherwise, by Proposition 2.1(i), there exists a maximal submodule M of A such that $\langle a, b + rc\rangle \subseteqq M$. Hence, $M \in \mathrm{Z}(a)$ and $M \in \mathrm{Z}(b + rc)$. Since $A = \langle m, b + rc\rangle$, it follows that $M \notin \mathrm{Z}(m)$, so $M \in \mathrm{Z}(c)$. But we now have $M \in \mathrm{Z}(b)$, contrary to $\langle a, b, c\rangle = A$. Therefore, $\langle a, b + rc\rangle = A$. Now, Theorem 3.6 completes the proof. □

For the proof of the next theorem, we need the following lemma.

Lemma 4.4. (cf. [11, Theorem 2.4]) *Let A be a finitely generated R-module and B a submodule of A. Then the following results are equivalent:*

(i) *$B \subseteq J(A)$;*

(ii) *if $A = \langle\{a_i\}_{i\in I}\rangle$, then $A = \langle\{a_k - rb, \{a_i\}_{i\neq k}\rangle$ for any $k \in I$, $r \in R$, and $b \in B$;*

(iii) *given a minimal generating set $\{a_1, \ldots, a_n\}$ of A, then for each $i = 1, \ldots, n$ and any $r \in R$, $\{a_1, \ldots, a_i - rb, \ldots, a_n\}$ is a minimal generating set of A for all $b \in B$.*

Proof. (i) ⇒ (ii): This result follows from Proposition 2.1(iii).

(ii) ⇒ (iii): Let $\{a_1, \ldots, a_n\}$ be a minimal generating set of A. In view of our assumption, we have that for any $k = 1, \ldots, n$, r in R, and b in B, then $A = \langle a_1, \ldots, a_k - rb, \ldots, a_n\rangle$. It suffices then to show that $\{a_1, \ldots, a_k - rb, \ldots, a_n\}$ is a minimal generating set. Suppose otherwise; then it may be "reduced" to a minimal generating set (i.e., some proper subset of $\{a_1, \ldots, a_k - rb, \ldots, a_n\}$). Note that $a_k - rb$ must belong to any such set–otherwise the minimality of $\{a_1, \ldots, a_n\}$ would be contradicted. However, we could replace $a_k - rb$ in the "reduced" set with $(a_k - rb) - (-r)b = a_k$ and still have a generating set for A. This also would contradict the minimality of $\{a_1, \ldots, a_n\}$. Thus $\{a_1, \ldots, a_k - rb, \ldots, a_n\}$ is a minimal generating set of A.

(iii) ⇒ (i): Suppose there is $b \in B \setminus J(A)$ and let $\{a_1, \ldots, a_n\}$ be a minimal generating set of A. Since $b \notin J(A)$, there is a maximal submodule M of A such that $b \notin M$. Thus $\langle b, M\rangle = A$. Clearly, for some $i = 1, \ldots, n$, we must also have $a_i \notin M$. For every such a_i, we have $a_i = r_i b + m_i$ for some r_i in R and m_i in M. Replacing these a_i's in the set $\{a_1, \ldots, a_n\}$ with $a_i - r_i b$ by hypothesis would produce a minimal generating set of A. However, the resulting submodule generated by the new set is clearly contained in M, a contradiction. Thus $B \subseteq J(A)$. □

Note that from the above lemma, it is clear that no element of a minimal generating set of A belongs to the Jacobson radical of A. Suppose that $(a_1, a_2, \ldots, a_n)$ is a minimal generating set of A and for instance, $a_1 \in J(A)$. Thus, $\langle a_1 - a_1 = 0, a_2, \ldots, a_n\rangle$ is a minimal generating set of A, which contradicts the minimality of $\langle a_1, a_2, \ldots, a_n\rangle$.

Finally, we end the paper by showing that any module of rank greater than or equal to 3 is never B_J-stable and consequently, not a B-stable module.

Theorem 4.5. *Let A be a finitely generated R-module of rank $m \geq 3$ a fixed integer. Then A is not a B_J-stable module.*

Proof. Suppose to the contrary that A is a B_J-stable module. Without loss of generality, assume rank of A is 3. Let $X = \{a_1, a_2, a_3\}$ be a minimal generating set of A. By the above lemma, $a_1 \notin J(A)$. Thus, $A = \langle a_1, a_2 + ra_3\rangle$ for some r in R. Hence, $Y = \{a_1, a_2 + ra_3\}$ is a minimal generating set of A since no proper subset of Y can generate A. Otherwise, for instance, $A = (a_2 + ra_3) \subseteq (a_2, a_3) \subseteq A$ contradicts the minimality of X. Therefore, rank$(A) \leq 2$, which is a contradiction. □

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Received: 2024-02-24
Accepted: 2024-06-27