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New approach to third-order Jacobsthal sequence

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Abstract For any real nonzero number k. In this paper, we first define generalizations of third-order Jacobsthal and third-order Jacobsthal–Lucas sequences by recurrence relations

$$J_n^{(3)}(k) = (k-1)J_{n-1}^{(3)}(k) + (k-1)J_{n-2}^{(3)}(k) + kJ_{n-3}^{(3)}(k)$$

and

$$j_n^{(3)}(k) = (k-1)j_{n-1}^{(3)}(k) + (k-1)j_{n-2}^{(3)}(k) + kj_{n-3}^{(3)}(k),$$

with initial conditions $J_0^{(3)}(k) = 0$, $J_1^{(3)}(k) = 1$, $J_2^{(3)}(k) = k-1$, and $j_0^{(3)}(k) = 2$, $j_1^{(3)}(k) = k-1$, $j_2^{(3)}(k) = k^2+1$, respectively. We give generating functions and Binet's formulas for these sequences. Also, we obtain some new identities of these sequences.

1 Introduction

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, [1, 7, 9]). The Jacobsthal numbers $\{J_n^{(2)}\}_{n\geq 0}$ are defined by the recurrence relation

$$J_0^{(2)} = 0,\ J_1^{(2)} = 1,\ J_{n+2}^{(2)} = J_{n+1}^{(2)} + 2J_n^{(2)},\ \ n \geq 0. \tag{1.1}$$

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation $j_{n+2}^{(2)} = j_{n+1}^{(2)} + 2j_n^{(2)}$, where $j_0^{(2)} = 2$ and $j_1^{(2)} = 1$ (see, [8]).

In [6] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by Horadam [8] is expanded and extended to several identities for some of the higher order cases. Furthermore, Cook and Bacon defined the third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n\geq 0}$, and third-order Jacobsthal-Lucas numbers, $\{j_n^{(3)}\}_{n\geq 0}$, are defined by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1,\ n \geq 0, \tag{1.2}$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5,\ n \geq 0, \tag{1.3}$$

respectively.

Some of the following properties given for third-order Jacobsthal numbers and third-order Jacobsthal-Lucas numbers are used in this paper (for more details, see [2, 3, 4, 5, 6]). Note that Eqs. (1.7) and (1.12) have been corrected in this paper, since they have been wrongly described in [6].

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \tag{1.4}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)},\ n \geq 3, \tag{1.5}$$

$$J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \not\equiv 1 \pmod 3 \end{cases}, \tag{1.6}$$

$$j_n^{(3)} - 4J_n^{(3)} = \begin{cases} 2 & \text{if} \quad n \equiv 0 \pmod 3 \\ -3 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.7}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)}, \tag{1.8}$$

$$j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 3 \\ -1 & \text{if} \quad n \equiv 1 \pmod 3 \\ 0 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.9}$$

$$\left(j_{n-3}^{(3)}\right)^2 + 3J_n^{(3)} j_n^{(3)} = 4^n, \tag{1.10}$$

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if} \quad n \not\equiv 0 \pmod 3 \\ J_{n+1}^{(3)} - 1 & \text{if} \quad n \equiv 0 \pmod 3 \end{cases} \tag{1.11}$$

and

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2} j_{n-3}^{(3)}, \ n \geq 3. \tag{1.12}$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0; \ x = 2, \ \text{and} \ x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{1}{7}\left[2^{n+1} - \left(\frac{3+2i\sqrt{3}}{3}\right)\omega_1^n - \left(\frac{3-2i\sqrt{3}}{3}\right)\omega_2^n\right] \tag{1.13}$$

and

$$j_n^{(3)} = \frac{1}{7}\left[2^{n+3} + \left(3+2i\sqrt{3}\right)\omega_1^n + \left(3-2i\sqrt{3}\right)\omega_2^n\right], \tag{1.14}$$

respectively. Now, we use the notation

$$Z_n = \frac{A\omega_1^n - B\omega_2^n}{\omega_1 - \omega_2} = \begin{cases} 2 & \text{if} \quad n \equiv 0 \pmod 3 \\ -3 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.15}$$

where $A = -3 - 2\omega_2$ and $B = -3 - 2\omega_1$. Furthermore, note that for all $n \geq 0$ we have

$$Z_{n+2} + Z_{n+1} + Z_n = 0, \ Z_0 = 2 \text{ and } Z_1 = -3. \tag{1.16}$$

From the Binet formulas (1.13), (1.14) and Eq. (1.15), we have

$$J_n^{(3)} = \frac{1}{7}\left(2^{n+1} - Z_n\right) \text{ and } j_n^{(3)} = \frac{1}{7}\left(2^{n+3} + 3Z_n\right). \tag{1.17}$$

Motivated essentially by the works [2], [4] and [6], in this paper we introduce the generalized third-order Jacobsthal and third-order Jacobsthal–Lucas sequences and we give some properties, including the Binet-style formula and the generating functions for these sequences. Some new identities involving these sequences are also provided.

2 Generalized third-order Jacobsthal and third-order Jacobsthal–Lucas sequences

In this section, we define generalizations of third-order Jacobsthal and third-order Jacobsthal–Lucas sequences. Then, we give generating functions and Binet's formulas for these generalized sequences.

Definition 2.1. For any real nonzero number k, the third-order k-Jacobsthal sequence $\{J_n^{(3)}(k)\}$ and the third-order k-Jacobsthal–Lucas sequence $\{j_n^{(3)}(k)\}$ are defined recursively, for $n \geq 3$, by

$$J_n^{(3)}(k) = (k-1)J_{n-1}^{(3)}(k) + (k-1)J_{n-2}^{(3)}(k) + kJ_{n-3}^{(3)}(k), \tag{2.1}$$

$$j_n^{(3)}(k) = (k-1)j_{n-1}^{(3)}(k) + (k-1)j_{n-2}^{(3)}(k) + kj_{n-3}^{(3)}(k), \tag{2.2}$$

with initial conditions $J_0^{(3)}(k) = 0$, $J_1^{(3)}(k) = 1$, $J_2^{(3)}(k) = k-1$, and $j_0^{(3)}(k) = 2$, $j_1^{(3)}(k) = k-1$, $j_2^{(3)}(k) = k^2+1$, respectively.

It is obvious that, in Eqs. (2.1) and (2.2), if we take $k = 2$, we obtain classical third-order Jacobsthal and third-order Jacobsthal–Lucas sequences.

The following theorem gives us generating functions for generalized third-order Jacobsthal and third-order Jacobsthal–Lucas sequences:

Theorem 2.2. *For any real nonzero number k, the generating functions of the third-order k-Jacobsthal sequence $\{J_n^{(3)}(k)\}$ and the third-order Jacobsthal–Lucas sequence $\{j_n^{(3)}(k)\}$ are given, respectively, by*

$$J(k;x) = \frac{x}{1-(k-1)x-(k-1)x^2-kx^3} \tag{2.3}$$

and

$$j(k;x) = \frac{2-(k-1)x+2x^2}{1-(k-1)x-(k-1)x^2-kx^3}. \tag{2.4}$$

Proof. The generating functions $J(k;x)$ and $j(k;x)$ can be written as $J(k;x) = \sum_{n\geq 0} J_n^{(3)}(k)x^n$ and $j(k;x) = \sum_{n\geq 0} j_n^{(3)}(k)x^n$. Then, we write

$$\begin{aligned}
J(k;x) &= \sum_{n\geq 0} J_n^{(3)}(k)x^n = J_0^{(3)}(k) + J_1^{(3)}(k)x + J_2^{(3)}(k)x^2 + \sum_{n\geq 3} J_n^{(3)}(k)x^n \\
&= x + (k-1)x^2 \\
&\quad + (k-1)\sum_{n\geq 3} J_{n-1}^{(3)}(k)x^n + (k-1)\sum_{n\geq 3} J_{n-2}^{(3)}(k)x^n + k\sum_{n\geq 3} J_{n-3}^{(3)}(k)x^n \\
&= x + (k-1)x^2 - (k-1)x^2 \\
&\quad + (k-1)x\sum_{n\geq 0} J_n^{(3)}(k)x^n + (k-1)x^2\sum_{n\geq 0} J_n^{(3)}(k)x^n + kx^3\sum_{n\geq 0} J_n^{(3)}(k)x^n \\
&= x + (k-1)xJ(k;x) + (k-1)x^2J(k;x) + kx^3J(k;x).
\end{aligned}$$

Thus, we obtain

$$\left(1-(k-1)x-(k-1)x^2-kx^3\right)J(k;x) = x.$$

Hence, we have

$$J(k;x) = \frac{x}{1-(k-1)x-(k-1)x^2-kx^3}.$$

Similarly, we obtain the Eq. (2.4). □

We now give the Binet's formulas for the third-order k-Jacobsthal and third-order k-Jacobsthal–Lucas sequences by following:

Theorem 2.3. *For any real nonzero number k, the n-th terms of the third-order k-Jacobsthal and third-order k-Jacobsthal–Lucas sequences are given by*

$$J_n^{(3)}(k) = \frac{1}{k^2+k+1}\left[k^{n+1} - Z_n(k)\right], \tag{2.5}$$

$$j_n^{(3)}(k) = \frac{1}{k^2+k+1}\left[\left(k^2+k+2\right)k^n + (k+1)Z_n(k)\right] \tag{2.6}$$

and

$$Z_n(k) = \frac{(k\omega_1 - 1)\omega_1^n - (k\omega_2 - 1)\omega_2^n}{\omega_1 - \omega_2}, \tag{2.7}$$

where ω_1 and ω_2 are the roots of the equation $x^2+x+1=0$.

Proof. Using the partial fraction decomposition, $J(k;x)$ can be expressed as

$$J(k;x) = \frac{1}{k^2+k+1}\left[\frac{k}{1-kx} + \frac{x-k}{x^2+x+1}\right].$$

However, note that $\omega_1+\omega_2=-1$, $\omega_1\omega_2=1$ and $(1-\omega_1 x)(1-\omega_2 x) = x^2+x+1$. Then, we have

$$\begin{aligned} J(k;x) &= \frac{x}{1-(k-1)x-(k-1)x^2-kx^3} \\ &= \frac{1}{k^2+k+1}\left[\frac{k}{1-kx} - \frac{k-x}{x^2+x+1}\right] \\ &= \frac{1}{k^2+k+1}\sum_{n\geq 0}\left[k^{n+1} - \left(\frac{(k\omega_1-1)\omega_1^n - (k\omega_2-1)\omega_2^n}{\omega_1-\omega_2}\right)\right]x^n \\ &= \frac{1}{k^2+k+1}\sum_{n\geq 0}\left[k^{n+1} - Z_n(k)\right]x^n. \end{aligned}$$

Inspecting the above expressions, we get the following:

$$J_n^{(3)}(k) = \frac{1}{k^2+k+1}\left[k^{n+1} - Z_n(k)\right].$$

Similarly, we can obtain the result of the Eq. (2.6). □

Note that sequence $Z_n(k)$ is simple, since it satisfies relation $Z_n(k) = -Z_{n-1}(k) - Z_{n-2}(k)$, where $Z_0(k) = k$ and $Z_1(k) = -(k+1)$. Then,

$$Z_n(k) = \begin{cases} k & \text{if} \quad n \equiv 0 \pmod 3 \\ -(k+1) & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}.$$

3 Some identities on third-order k-Jacobsthal and third-order k-Jacobsthal–Lucas sequences

In this section, we give well-known identities Catalan's, Cassini's and d'Ocagne's for the third-order k-Jacobsthal and third-order k-Jacobsthal–Lucas sequences. Further, we investigate some nameless identities of these generalized sequences, and give some relations between these new sequences.

Theorem 3.1 (Catalan's Identity). *For any real nonzero number k, any integers m and r such that $m \geq r$, we have*

$$J_{m+r}^{(3)}(k)J_{m-r}^{(3)}(k) - \left[J_m^{(3)}(k)\right]^2 = \frac{1}{\sigma_k^2}\left\{\begin{array}{c} k^{m+1}\left(k^r - k^{-r}\right)X_r Z_{m+1}(k) \\ -k^{m+1}\left(k^r X_{r+1} - k^{-r}X_{r-1} - 2\right)Z_m(k) \\ -(k^2+k+1)X_r^2 \end{array}\right\}$$

and

$$j_{m+r}^{(3)}(k)j_{m-r}^{(3)}(k)-\left[j_{m}^{(3)}(k)\right]^2=\frac{k+1}{\sigma_k^2}\left\{\begin{array}{c}\rho_k k^m\left(k^{-r}-k^r\right)X_r Z_{m+1}(k)\\+\rho_k k^m\left(k^r X_{r+1}-k^{-r}X_{r-1}-2\right)Z_m(k)\\-(k^2+k+1)(k+1)X_r^2\end{array}\right\},$$

where $Z_n(k)$ *as in Eq. (2.7),* $X_r=\frac{\omega_1^r-\omega_2^r}{\omega_1-\omega_2}$, $\sigma_k=k^2+k+1$ *and* $\rho_k=\sigma_k+1$.

Proof. Using the notation $\sigma_k=k^2+k+1$, $\rho_k=\sigma_k+1$ and Binet's formula of the third-order k-Jacobsthal–Lucas sequence in Theorem 2.3, we write

$$\begin{aligned}&\sigma_k^2\left(j_{m+r}^{(3)}(k)j_{m-r}^{(3)}(k)-\left[j_{m}^{(3)}(k)\right]^2\right)\\&=\left(\rho_k k^{m+r}+(k+1)Z_{m+r}(k)\right)\left(\rho_k k^{m-r}+(k+1)Z_{m-r}(k)\right)\\&\quad-\left(\rho_k k^m+(k+1)Z_m(k)\right)^2\\&=\rho_k(k+1)k^m\left(k^r Z_{m-r}(k)+k^{-r}Z_{m+r}(k)-2Z_m(k)\right)\\&\quad+(k+1)^2\left(Z_{m+r}(k)Z_{m-r}(k)-\left[Z_m(k)\right]^2\right).\end{aligned}$$

Using the following identity for the sequence $Z_m(k)$:

$$Z_{m+r}(k)=X_r Z_{m+1}(k)-X_{r-1}Z_m(k),$$

where $X_r=\frac{\omega_1^r-\omega_2^r}{\omega_1-\omega_2}$ and $X_{-r}=-X_r$. Thus, we obtain

$$j_{m+r}^{(3)}(k)j_{m-r}^{(3)}(k)-\left[j_{m}^{(3)}(k)\right]^2=\frac{k+1}{\sigma_k^2}\left\{\begin{array}{c}\rho_k k^m\left(k^{-r}-k^r\right)X_r Z_{m+1}(k)\\+\rho_k k^m\left(k^r X_{r+1}-k^{-r}X_{r-1}-2\right)Z_m(k)\\-(k^2+k+1)(k+1)X_r^2\end{array}\right\}.$$

Using Binet's formula of the third-order k-Jacobsthal sequence, the first identity can be proved in a similar manner. □

Taking $r=1$ in Theorem 3.1, we obtain the following:

Corollary 3.2 (Cassini's Identity). *Let* m *be any integer* $m\geq 1$. *Then, we have*

$$J_{m+1}^{(3)}(k)J_{m-1}^{(3)}(k)-\left[J_{m}^{(3)}(k)\right]^2=\frac{1}{\sigma_k^2}\left\{\begin{array}{c}\left(k^{m+2}-k^m\right)Z_{m+1}(k)\\+k^{m+1}\left(k+2\right)Z_m(k)-(k^2+k+1)\end{array}\right\}$$

and

$$j_{m+1}^{(3)}(k)j_{m-1}^{(3)}(k)-\left[j_{m}^{(3)}(k)\right]^2=\frac{k+1}{\sigma_k^2}\left\{\begin{array}{c}\rho_k k^m\left(k^{-1}-k\right)Z_{m+1}(k)\\-\rho_k k^m\left(k+2\right)Z_m(k)\\-(k^2+k+1)(k+1)\end{array}\right\}.$$

Theorem 3.3 (d'Ocagne's Identity). *For any real nonzero number* k. *Let* m *and* r *be any integers such that* $m\geq r$. *Then, we have*

$$J_{m}^{(3)}(k)J_{r+1}^{(3)}(k)-J_{m+1}^{(3)}(k)J_{r}^{(3)}(k)=\frac{1}{\sigma_k}\left[k^{m+1}X_r-k^{r+1}X_m+X_{m-r}\right]$$

and

$$j_{m}^{(3)}(k)j_{r+1}^{(3)}(k)-j_{m+1}^{(3)}(k)j_{r}^{(3)}(k)=\frac{k+1}{\sigma_k^2}\left[\rho_k k^{r+1}X_m-\rho_k k^{m+1}X_r+(k+1)\sigma_k X_{m-r}\right],$$

where $X_n=\frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$, $\sigma_k=k^2+k+1$ *and* $\rho_k=\sigma_k+1$.

Proof. Using Binet's formula of the third-order k-Jacobsthal sequence in Theorem 2.3, we write

$$\begin{aligned}
&\sigma_k^2\left(J_m^{(3)}(k)J_{r+1}^{(3)}(k)-J_{m+1}^{(3)}(k)J_r^{(3)}(k)\right)\\
&=\left(k^{m+1}-Z_m(k)\right)\left(k^{r+2}-Z_{r+1}(k)\right)-\left(k^{m+2}-Z_{m+1}(k)\right)\left(k^{r+1}-Z_r(k)\right)\\
&=k^{m+1}\left(kZ_r(k)-Z_{r+1}(k)\right)-k^{r+1}\left(kZ_m(k)-Z_{m+1}(k)\right)\\
&\quad+Z_m(k)Z_{r+1}(k)-Z_{m+1}(k)Z_r(k)\\
&=\sigma_k\left[k^{m+1}X_r-k^{r+1}X_m+X_{m-r}\right],
\end{aligned}$$

where $X_n=\frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$. The second statement of the theorem can be proved similarly. □

As a consequence of the Binet formulae of Theorem 2.3, we get, for the sequence of third-order k-Jacobsthal numbers, the following interesting identities.

Theorem 3.4. *Let r be any integer such that $r\geq 2$. Then, we obtain*

$$j_r^{(3)}(k)=(k-1)J_r^{(3)}(k)+2kJ_{r-1}^{(3)}(k)+2kJ_{r-2}^{(3)}(k),$$
$$(\sigma_k+1)(k+1)J_r^{(3)}(k)=(k^2-k-2)j_r^{(3)}(k)+2k^2j_{r-1}^{(3)}(k)+2k^2j_{r-2}^{(3)}(k),$$

where $\sigma_k=k^2+k+1$.

Proof. Using $\sigma_k=k^2+k+1$ and Binet's formula of the third-order k-Jacobsthal sequence, we write

$$\begin{aligned}
&\sigma_k\left[(k-1)J_r^{(3)}(k)+2kJ_{r-1}^{(3)}(k)+2kJ_{r-2}^{(3)}(k)\right]\\
&=(k-1)k^{r+1}-(k-1)Z_r(k)+2k\cdot k^r-2kZ_{r-1}(k)\\
&\quad+2k\cdot k^{r-1}-2kZ_{r-2}(k)\\
&=\left(k^2+k+2\right)k^r+(k+1)Z_r(k).
\end{aligned}$$

This proves the first identity. the second identity can be proved in a similar manner. □

Theorem 3.5. *for every integer $r\geq 0$, we have*

$$j_r^{(3)}(k)-k^2J_r^{(3)}(k)+(k-2)k^r=\begin{cases}k & \textit{if}\quad r\equiv 0 \pmod 3\\ -(k+1) & \textit{if}\quad r\equiv 1 \pmod 3\\ 1 & \textit{if}\quad r\equiv 2 \pmod 3\end{cases}. \tag{3.1}$$

Proof. Using $\sigma_k=k^2+k+1$ and the Binet's formulas of the third-order k-Jacobsthal and third-order k-Jacobsthal–Lucas sequences, we write

$$\begin{aligned}
j_r^{(3)}(k)-k^2J_r^{(3)}(k)&=\frac{1}{\sigma_k}\left[(k^2+k+1)k^r+(k+1)Z_r(k)\right]\\
&\quad-\frac{k^2}{\sigma_k}\left[k^{r+1}-Z_r(k)\right]\\
&=\frac{1}{\sigma_k}\left[-\sigma_k(k-2)\cdot k^r+\sigma_kZ_r(k)\right]\\
&=-(k-2)\cdot k^r+Z_r(k).
\end{aligned}$$

Considering $\omega_1+\omega_2=-1$, $\omega_1\omega_2=1$ and

$$Z_r(k)=\begin{cases}k & \text{if}\quad r\equiv 0 \pmod 3\\ -(k+1) & \text{if}\quad r\equiv 1 \pmod 3\\ 1 & \text{if}\quad r\equiv 2 \pmod 3\end{cases},$$

we obtain desired result. □

Some identities for third-order k-Jacobsthal and third-order k-Jacobsthal–Lucas numbers are given without proof in the next theorem.

Theorem 3.6. *For any integer $r \geq 0$, we have*

$$j_r^{(3)}(k) + (k+1)J_r^{(3)}(k) = 2k^r, \tag{3.2}$$

$$j_r^{(3)}(k) - (k+1)J_r^{(3)}(k) = 2\left[j_{r-3}^{(3)}(k) + (k-2)k^{r-3}\right],\ r \geq 3, \tag{3.3}$$

$$\left[j_r^{(3)}(k)\right]^2 - (k+1)^2\left[J_r^{(3)}(k)\right]^2 = 4k^r\left[j_{r-3}^{(3)}(k) + (k-2)k^{r-3}\right], \tag{3.4}$$

$$J_{r+2}^{(3)}(k) - k^2 J_r^{(3)}(k) = \begin{cases} k-1 & \text{if } r \equiv 0 \pmod 3 \\ -k & \text{if } r \equiv 1 \pmod 3 \\ 1 & \text{if } r \equiv 2 \pmod 3 \end{cases}, \tag{3.5}$$

$$(k-2)k^r + j_r^{(3)}(k) - J_{r+2}^{(3)}(k) = \begin{cases} 1 & \text{if } r \equiv 0 \pmod 3 \\ -1 & \text{if } r \equiv 1 \pmod 3 \\ 0 & \text{if } r \equiv 2 \pmod 3 \end{cases}. \tag{3.6}$$

The next result will be also used in the statement of the new identities for the third-order k-Jacobsthal number sequence.

Theorem 3.7. *Let m and r be any integers. Then, we have*

$$\begin{aligned} J_{m+r+1}^{(3)}(k) &= J_{m+1}^{(3)}(k)J_{r+1}^{(3)}(k) + \left((k-1)J_m^{(3)}(k) + kJ_{m-1}^{(3)}(k)\right)J_r^{(3)}(k) \\ &\quad + kJ_m^{(3)}(k)J_{r-1}^{(3)}(k). \end{aligned} \tag{3.7}$$

Proof. Using $\sigma_k = k^2 + k + 1$ and Binet's formula of the third-order k-Jacobsthal sequence, we write

$$\begin{aligned} &\sigma_k^2\left[J_{m+1}^{(3)}(k)J_{r+1}^{(3)}(k) + \left((k-1)J_m^{(3)}(k) + kJ_{m-1}^{(3)}(k)\right)J_r^{(3)}(k) + kJ_m^{(3)}(k)J_{r-1}^{(3)}(k).\right] \\ &= k^{m+r+4} - k^{m+2}Z_{r+1}(k) - k^{r+2}Z_{m+1}(k) + Z_{m+1}(k)Z_{r+1}(k) \\ &\quad + (k-1)\left[k^{m+r+2} - k^{m+1}Z_r(k) - k^{r+1}Z_m(k) + Z_m(k)Z_r(k)\right] \\ &\quad + k\left[k^{m+r+1} - k^m Z_r(k) - k^{r+1}Z_{m-1}(k) + Z_{m-1}(k)Z_r(k)\right] \\ &\quad + k\left[k^{m+r+1} - k^{m+1}Z_{r-1}(k) - k^r Z_m(k) + Z_m(k)Z_{r-1}(k)\right] \\ &= \sigma_k\left[k^{m+r+2} - Z_{m+r+1}(k)\right] \\ &= \sigma_k^2 J_{m+r+1}^{(3)}(k). \end{aligned}$$

Using $Z_n(k) = -Z_{n-1}(k) - Z_{n-2}(k)$, the theorem can be proved easily. □

Taking $m = r$ in Theorem 3.7, we obtain the following:

Corollary 3.8. *Let r be any integer. Then,*

$$J_{2r+1}^{(3)}(k) = \left[J_{r+1}^{(3)}(k)\right]^2 + (k-1)\left[J_r^{(3)}(k)\right]^2 + 2kJ_{r-1}^{(3)}J_r^{(3)}(k).$$

Taking $m = r - 1$ in Theorem 3.7 and Using Eq. (2.1), we obtain the following:

Corollary 3.9. *Let r be any integer. Then,*

$$J_{2r}^{(3)}(k) = k\left[J_{r-1}^{(3)}(k)\right]^2 - (k-1)\left[J_r^{(3)}(k)\right]^2 + 2J_r^{(3)}(k)J_{r+1}^{(3)}(k).$$

4 Conclusion

Sequences of numbers have been studied over several years, including the well-known Tribonacci sequence and, consequently, on the Tribonacci-Lucas sequence. In this paper we have also contributed for the study of generalized third-order Jacobsthal sequence, deducing some identities of these numbers, presenting the generating functions and their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences using binomials form and their combinatorial properties in the spirit of Wani et al. [10].

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