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On polynomials whose roots are totally real

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Abstract Using circulant matrices, we establish characterization criteria for a class of algebraic equations of degree less than or equal to 5 whose roots are totally real, and we explicitly calculate these roots.

1 Introduction

Polynomials are significant in mathematics because they are relatively easy to study and can be utilized to model various real-world phenomena. The study of polynomial equations has been one of the oldest areas of focus in algebra for centuries. The challenge was to find formulas that could provide the roots of polynomials based on their coefficients. Early on, first order equations were solved, and centuries later, the Arabic mathematician Al-Khwarizmi (780-850) developed a method for solving quadratic equations, Omar Khayyam's studies on cubic equations inspired the 12th century Persian mathematician Sharaf al-Din Tusi to investigate the number of positive roots [8, 2]. In 1545, the solutions for cubic and quartic equations became well-known after the publication of Geronimo Cardano's Ars Magna. The hint for solving the cubic was obtained from Niccolo Tartaglia, and the solution to the quartic was first discovered by Ludovico Ferrari. The discovery of solutions for cubic and quartic equations using radicals was the most important accomplishment in mathematics of the sixteenth century. These discoveries inspired mathematicians to attempt solving quintic (fifth degree) and higher degree equations using radicals. In 1824, the Norwegian mathematician Neils Abel (1802-1829) proved that it is impossible to solve the general equation of the fifth degree in terms of radicals, closing the door on further exploration in this direction. Around the same time, a French mathematician, Evarist Galois (1811-1832), extended this proof to all degrees greater than five [4].

Polynomials with only real zeros arise often in combinatorics and other branches of mathematics (see [12] and the references therein). This subject has interested several mathematicians who have treated it by using different approaches. In this article, we propose a matrix approach for the real roots of polynomials based on the circulant matrices (see [7, 9, 11, 13]). First of all, we find the necessary and sufficient conditions that must be verified by coefficients of these polynomials of degree less than 4 so that its roots are totally real. After passing to extend their method and use circulant matrices to find expressions for the exact roots of many families of quintic polynomial equations and characterizing some classes of equations including their real roots.

2 Priliminary

In this section, we introduce some standard notations and definitions which will be useful to prove our main results.

Definition 2.1. [5] A $n \times n$ **circulant matrix** is a square matrix of the form:

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \ddots & & & c_{n-2} \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ c_2 & \cdots & & \ddots & c_0 & c_1 \\ c_1 & c_2 & & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

The whole circulant is evidently determined by the first row. We may also write a circulant in the form:

$$C = (c_{jk}) = (c_{k-j \bmod n}) = \text{circ}\,(c_0, c_1, \cdots, c_{n-1})\,.$$

Properties of circulant matrices can be studied using W matrix as defined in[5], a special permutation matrix of the form:

$$W = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Lemma 2.2. *We list some properties of permutation matrices which can be easily verified:*

(i) *The n^{th} exponent of W is the identity matrix I_n.*

(ii) $W^k = \text{circ}(0, \cdots, 0, 1, 0, \cdots, 0)$*, where 1 is in the $(k+1)^{th}$ position.*

Proof. For the proof we refer to [5]. □

Remark 2.3. We can represent a circulant matrix using W matrix as follows:

$$C = \text{circ}\,(c_0, c_1, \cdots, c_{n-1}) = c_0 I_n + c_1 W + \cdots + c_{n-1} W^{n-1} = q(W).$$

Where $q(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$.

Proposition 2.4. *[5] The eigenvalues of a circulant matrix $C = \text{circ}\,(c_0, c_1, \cdots, c_{n-1})$ are given by*

$$\lambda_m(C) = \sum_{k=0}^{n-1} c_k \ \exp\left(\frac{2\pi m k i}{n}\right), \quad 0 \le m < n.$$

Proof. The characteristic polynomial of W is:

$$P_W(\lambda) = \det(\lambda I_n - W) = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix} = \lambda^n - 1 = \prod_{k=0}^{n-1} \left(\lambda - \omega^k\right).$$

Where, $\omega = e^{i\frac{2\pi}{n}}$. Since $C = q(W)$, then the eigenvalues of C are $q(\omega^k)$, for $0 \le k < n$. □

Proposition 2.5. *[10] The circulant matrice C has all eigenvalues real if, and only if, it is self-adjoint or, equivalently, if its coefficients satisfy the* ***reflection property***

$$c_0 \in \mathbb{R}, \quad c_{n-k} = \overline{c_k}, \quad k = 1, \ldots, n.$$

Proof. It is clear that C is self-adjoint if, and only if, the reflection property holds. So, this property implies the reality of the spectrum of C. Conversely, if all the eigenvalues λ_j of C are real, then from Proposition 2.4 one has

$$\overline{\lambda_j} = \overline{c_0} + \overline{c_{n-1}}\omega_{j-1} + \overline{c_{n-2}}\omega_{j-1}^2 + \cdots + \overline{c_1}\omega_{j-1}^{n-1} = \lambda_j, \quad j = 1, \ldots, n.$$

□

3 Main Results

In this section, we characterize the polynomial equations with degree less than or equal than 5 of which all the roots are real, and we give the expressions of its roots.

Tschirnhaus gave transformations for the elimination of some of the intermediate terms in a polynomial [1], for example: the general result, for $p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, is that the substitution $y = x - \frac{a_{n-1}}{n}$ eliminates the term of degree $n-1$ (*reduced* form or *depressed*). Recall that a_{n-1} equals the sum of the roots of p. If p is the characteristic polynomial of a matrix C, then the sum of the roots is the sum of the eigenvalues, that is, the trace, of C. Accordingly, eliminating the degree $n-1$ term corresponds to making the trace vanish.

3.1 Quadratic equation:

Proposition 3.1. *Let α and β be two real numbers, the roots of the quadratic polynomial equation $x^2 - \alpha x - \beta = 0$ are all real if and only if $\alpha^2 + 4\beta \geq 0$, and are given by:* $x_1 = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2+4\beta}}{2}$ *and* $x_2 = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2+4\beta}}{2}$.

Proof. Consider a general quadratic polynomial : $p(x) = x^2 - \alpha x - \beta$. We must find a general 2×2 symetric circulant matrix of the form $C = \begin{pmatrix} b & c \\ c & b \end{pmatrix} = circ\,(a, b)$, such as his characteristic polynomial is p.
The characteristic polynomial of C is

$$P_C(x) = x^2 - 2bx + b^2 - c^2. \tag{3.1}$$

Identifying $p(x)$ with $P_C(x)$, results in the nonlinear system

$$b = \frac{\alpha}{2} \tag{3.2}$$

$$c^2 - b^2 = \beta. \tag{3.3}$$

Substituting for b in the equation (3.3), we obtain

$$c^2 = \frac{\alpha^2}{4} + \beta \geq 0. \tag{3.4}$$

Here, $b = \frac{\alpha}{2}$ and $c = \pm\sqrt{\frac{\alpha^2}{4} + \beta}$. For convenience we define c with the positive sign. Hence

$$C = \begin{pmatrix} \frac{\alpha}{2} & \frac{\sqrt{\alpha^2+4\beta}}{2} \\ \frac{\sqrt{\alpha^2+4\beta}}{2} & \frac{\alpha}{2} \end{pmatrix}$$

and

$$q(t) = \frac{\alpha}{2} + t\frac{\sqrt{\alpha^2 + 4\beta}}{2}.$$

The roots of the original quadratic equation are now found by applying q to the two square roots of unity:

$$x_1 = q(1) = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 + 4\beta}}{2},$$

$$x_2 = q(-1) = \frac{\alpha}{2} - \frac{\sqrt{\alpha^2 + 4\beta}}{2}.$$

□

3.2 Cubic equation:

Starting from the cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c, and d are real numbers, we will achieve the desired simplification by changing variables to obtain an equation with no degree-two term.

Lemma 3.2. *Let a, b, and c be three real numbers. Then the general cubic of the form $x^3 + bx^2 + cx + d$ can be reduced to a depressed form by a suitable substitution involving a new variable. Accodingly, the substitution takes the form of:* $x = y - \frac{b}{3}$.

The depressed cubic takes the form: $y^3 - ry - s$ *, where* r *and* s *are expressed in the forms:*

$$r = \frac{b^2}{3} - c \quad \textit{and} \quad s = \frac{bc}{3} - \frac{2b^3}{27} - d.$$

Proof. For the proof we refer to [6]. □

Lemma 3.3. *Let α and β be two real numbers, such that, $\beta^2 - \frac{4\alpha^3}{27} \geq 0$. The roots of the cubic polynomial equation $x^3 - \alpha x - \beta = 0$ are given by: $x_m = b\ \exp\left(\frac{2\pi mi}{3}\right) + c\ \exp\left(\frac{4\pi mi}{3}\right)$, $m = 0, 1, 2$, with*

$$b = \sqrt[3]{\frac{\beta + \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{2}},$$

$$c = \sqrt[3]{\frac{\beta - \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{2}}.$$

Proof. The roots of $p(x) = x^3 - \alpha x - \beta = 0$ as the eigenvalues of a traceless circulant matrix. Let $C = \begin{pmatrix} 0 & b & c \\ c & 0 & b \\ b & c & 0 \end{pmatrix}$, its characteristic polynomial is $x^3 - 3bcx - b^3 - c^3$. This equals $p(x)$ if

$$b^3 + c^3 = \beta,$$
$$3bc = \alpha,$$

which on solving gives

$$b = \sqrt[3]{\frac{\beta + \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{2}},$$

$$c = \sqrt[3]{\frac{\beta - \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{2}}.$$

By the Proposition 2.4, the roots of $p(x)$ are the eigenvalues of C with b and c so obtained. □

Theorem 3.4. *For any real numbers α and β, the roots of the cubic polynomial equation $x^3 - \alpha x - \beta = 0$ are all real, if and only if $\alpha > 0$ and $\beta^2 - \dfrac{4\alpha^3}{27} \geq 0$, and are given by:*

$$x_0 = 2b,$$

$$x_1 = b - \sqrt{\alpha - 3b^2},$$

$$x_2 = b + \sqrt{\alpha - 3b^2},$$

with $b = \sqrt[3]{\frac{\beta - \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{16}} + \sqrt[3]{\frac{\beta + \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{16}}$.

Proof. Consider a reduced cubic polynomial : $f(x) = x^3 - \alpha x - \beta$. We must find a general 3×3 self-adjoint circulant matrix of the form $C = \begin{pmatrix} 0 & b+ic & b-ic \\ b-ic & 0 & b+ic \\ b+ic & b-ic & 0 \end{pmatrix} = circ\,(0, b+ic, b-ic)$ such as his characteristic polynomial is f.
The characteristic polynomial of C is

$$P_C(x) = x^3 - 3(b^2 + c^2)x - 2b^3 + 6bc^2. \tag{3.5}$$

Identifying $f(x)$ with $P_C(x)$, results in the nonlinear system

$$b^2 + c^2 = \frac{\alpha}{3} > 0, \tag{3.6}$$

$$2b^3 - 6bc^2 = \beta. \tag{3.7}$$

Substituting for c^2 in the equation (3.7), we obtain

$$8b^3 - 2\alpha b - \beta = 0. \tag{3.8}$$

When the discriminant of equation (3.8) is $\Delta = \frac{1}{64}\left(\beta^2 - \frac{4\alpha^3}{27}\right) \geq 0$, and b is a real number, then (by Lemma 3.3)
$b = \sqrt[3]{\frac{\beta - \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{16}} + \sqrt[3]{\frac{\beta + \sqrt{\beta^2 - \frac{4\alpha^3}{27}}}{16}}$ and $c = \sqrt{\frac{\alpha}{3} - b^2}$.

Let be

$$q(t) = (b+ic)t + (b-ic)t^2.$$

The roots of the reduced cubic equation are now found by applying q to the three square roots of unity:

$$x_0 = q(1) = 2b,$$
$$x_1 = q(j) = (b+ic)e^{i\frac{2\pi}{3}} + (b-ic)e^{i\frac{4\pi}{3}},$$
$$x_2 = q(\bar{j}) = (b+ic)e^{i\frac{4\pi}{3}} + (b-ic)e^{i\frac{8\pi}{3}}.$$

where $j = e^{i\frac{2\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $\bar{j} = e^{i\frac{4\pi}{3}} = e^{i\frac{-2\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. It follows that

$$x_0 = 2b,$$
$$x_1 = b\left(e^{i\frac{2\pi}{3}} + e^{i\frac{-2\pi}{3}}\right) + ic\left(e^{i\frac{2\pi}{3}} - e^{i\frac{-2\pi}{3}}\right) = b - \sqrt{3}c = b - \sqrt{\alpha - 3b^2},$$
$$x_2 = b\left(e^{i\frac{4\pi}{3}} + e^{i\frac{2\pi}{3}}\right) + ic\left(e^{i\frac{4\pi}{3}} - e^{i\frac{-2\pi}{3}}\right) = b + \sqrt{3}c = b + \sqrt{\alpha - 3b^2}.$$

□

3.3 Quartic equation:

Lemma 3.5. *Let $f(x) = x^4 - \alpha x^2 - \beta x - \gamma$ be a polynomial with real coefficients, then $R_s(t) = t^3 - \frac{\alpha}{2}t^2 + \left(\left(\frac{\alpha}{4}\right)^2 + \frac{\gamma}{4}\right)t - \left(\frac{\beta}{8}\right)^2 = 0$, is a cubic resolvent of quartic equation $f(x) = 0$.*

Proof. Consider a reduced quartic polynomial : $f(x) = x^4 - \alpha x^2 - \beta x - \gamma$. We must find a general 4×4 self-adjoint circulant matrix of the form

$$C = \begin{pmatrix} 0 & p+iq & b & p-iq \\ p-iq & 0 & p+iq & b \\ b & p-iq & 0 & p+iq \\ p+iq & b & p-iq & 0 \end{pmatrix} = circ\,(0, p+iq, b, p-iq)$$

such as his characteristic polynomial is f.
The characteristic polynomial of C is

$$P_C(x) = x^4 - 2\left(2p^2 + 2q^2 + b^2\right)x^2 - 8\left(bp^2 - bq^2\right)x + b^4 - 4b^2p^2 - 4b^2q^2 + 16p^2q^2. \quad (3.9)$$

Identifying $f(x)$ with $P_C(x)$, results in the nonlinear system

$$2(p^2 + q^2) + b^2 = \frac{\alpha}{2}, \quad (3.10)$$

$$b(p^2 - q^2) = \frac{\beta}{8}, \quad (3.11)$$

$$b^4 - 4b^2(p^2 + q^2) + 16p^2q^2 = -\gamma. \quad (3.12)$$

If $b = 0$ then $\beta = 0$. We can regard $f(x) = x^4 - \alpha x^2 - \gamma$ as a quadratic polynomial in x^2, use the sign of $\alpha^2 + 4\gamma$ to determine whether the two values of x^2 occurring as roots are real.
If $b \neq 0$

$$p^2 + q^2 = \frac{\alpha}{4} - \frac{b^2}{2}, \quad (3.13)$$

$$p^2 - q^2 = \frac{\beta}{8b},$$

Then

$$p^2 = \frac{\alpha}{8} - \frac{b^2}{4} + \frac{\beta}{16b}, \quad (3.14)$$

$$q^2 = \frac{\alpha}{8} - \frac{b^2}{4} - \frac{\beta}{16b}, \quad (3.15)$$

and

$$16p^2q^2 = b^4 - \alpha b^2 + \frac{\alpha^2}{4} - \frac{\beta^2}{16b^2}. \quad (3.16)$$

By equations (3.10), (3.12) and (3.15), we obtain

$$\begin{aligned}
&b^4 - 2b^2\left(\frac{\alpha}{2} - b^2\right) + b^4 - \alpha b^2 + \frac{\alpha^2}{4} - \frac{\beta^2}{16b^2} = -\gamma, \\
&2b^4 - \alpha b^2 + 2b^4 - \alpha b^2 + \frac{\alpha^2}{4} - \frac{\beta^2}{16b^2} = -\gamma, \\
&4b^4 - 2\alpha b^2 - \frac{\beta^2}{16b^2} = -\gamma - \frac{\alpha^2}{4}, \quad (3.17) \\
&4b^6 - 2\alpha b^4 - \frac{\beta^2}{16} = -\left(\gamma + \frac{\alpha^2}{4}\right)b^2, \\
&4b^6 - 2\alpha b^4 + \left(\gamma + \frac{\alpha^2}{4}\right)b^2 - \frac{\beta^2}{16} = 0,
\end{aligned}$$

and

$$b^6 - \frac{\alpha}{2}b^4 + \left(\left(\frac{\alpha}{4}\right)^2 + \frac{\gamma}{4}\right)b^2 - \left(\frac{\beta}{8}\right)^2 = 0. \quad (3.18)$$

Notice that this equation can he regarded as a cubic polynomial equation in b^2. Let us therefore introduce a new variable for b^2, setting $b^2 = t$. Then the equation becomes

$$R_s(t) = t^3 - \frac{\alpha}{2}t^2 + \left(\left(\frac{\alpha}{4}\right)^2 + \frac{\gamma}{4}\right)t - \left(\frac{\beta}{8}\right)^2 = 0. \quad (3.19)$$

The cubic $R_s(t)$ called the *cubic resolvent* of quartic. By equation (3.10), the coefficient α must be s*trictly positive*, and by equation (3.13) the component b of the circulant matrice must satisfy the condition

$$0 < b^2 < \frac{\alpha}{2}. \quad (3.20)$$

Although our cubic equation in t, like all cubic equations, has a real solution, and since $0 < t = b^2 < \dfrac{\alpha}{2}$, we must therefore know that the equation (3.19) has both a real and a positive solution. By the intermediate value theorem of elementary calculus, we show that the cubic polynomial equation (3.19) has a positive real root. The sufficient condition that $0 < t < \dfrac{\alpha}{2}$ is $R_s(\frac{\alpha}{2}) > 0$. □

In order for all roots of f to be real, and hence for all corosponding circulants to be self-adjoint, it is necessary that all roots of equation (3.18) be real. Since (3.18) is cubic in b^2, the necessary condition reduces to this: the cubic equation (3.19) must have all real, nonnegative roots. Conversely, if the cubic equation (3.19) has all real, nonnegative roots, then f has all real roots. Now, what are the conditions on the coefficients of $R_s(t)$ to guarantee this goal? the derivative of R_s is

$$R'_s(t) = 3t^2 - \alpha t + \left(\frac{\alpha^2}{16} + \frac{\gamma}{4}\right). \tag{3.21}$$

Then, it is necessary that

$$\frac{\alpha^2}{12} > \gamma, \tag{3.22}$$

otherwise R_s will be increasing on $\mathbb{R}$ and thus admits only one real root.
If we can solve $R_s(t) = 0$. we can then take square roots to get a value of b, and use it to find the other two components of the circulant matrice associated with our quartic equation.

Solving the Resolvent Polynomial

Lemma 3.6. *Let α and β be two real numbers. Then the equation $t^3 - \frac{\alpha}{2}t^2 + \left(\frac{\alpha^2}{16} + \frac{\gamma}{4}\right)t - \left(\frac{\beta}{8}\right)^2 = 0$, can be reduced to a depressed equation $y^3 - ry - s = 0$, by a substitution involving a variable $t = y + \frac{\alpha}{6}$, where r and s are given by:*

$$r = \frac{1}{4}\left(\frac{\alpha^2}{12} - \gamma\right) \quad \textit{and} \quad s = \frac{1}{8}\left(\frac{\beta^2}{8} - \frac{\alpha^3}{108} - \frac{\alpha\gamma}{3}\right),$$

with $s^2 - \dfrac{4r^3}{27} > 0$.

Proof. It suffices to apply Lemma 3.2 and the reality of roots. □

Lemma 3.7. *The roots of the equation $t^3 - \frac{\alpha}{2}t^2 + \left(\frac{\alpha^2}{16} + \frac{\gamma}{4}\right)t - \left(\frac{\beta}{8}\right)^2 = 0$, are given by:*

$$t_0 = 2\rho + \frac{\alpha}{6},$$

$$t_1 = \rho - \sqrt{r - 3\rho^2} + \frac{\alpha}{6},$$

and

$$t_2 = \rho + \sqrt{r - 3\rho^2} + \frac{\alpha}{6}.$$

where

$$\rho = \sqrt[3]{\frac{s + \sqrt{s^2 - \frac{4r^3}{27}}}{2}} + \sqrt[3]{\frac{s - \sqrt{s^2 - \frac{4r^3}{27}}}{2}}.$$

Here:

$$r = \frac{1}{4}\left(\frac{\alpha^2}{12} - \gamma\right) \quad \textit{and} \quad s = \frac{1}{8}\left(\frac{\beta^2}{8} - \frac{\alpha^3}{108} - \frac{\alpha\gamma}{3}\right).$$

Proof. From Lemma 3.6 and Theorem 3.4. □

Quartic whose Roots are totally real

We gather the conditions so that a polynomial quartic equation has only real roots in the following theorem:

Theorem 3.8. *For any real numbers α, β and γ , the roots of the quartic polynomial equation $x^4 - \alpha x^2 - \beta x - \gamma = 0$ are all real, if and only if , $\alpha > 0$, $\frac{\alpha^2}{12} > \gamma$ and*

$$\left(\frac{\beta^2}{8} - \frac{\alpha^3}{108} - \frac{\alpha\gamma}{3}\right)^2 - \left(\frac{\alpha^2}{12} - \gamma\right)^3 > 0.$$

Proof. The proof follows immediatley from Lemma 3.6, equations (3.10) and (3.22). □

Theorem 3.9. *If the roots of the quartic polynomial equation $x^4 - \alpha x^2 - \beta x - \gamma = 0$ are all real then they are given by:*

$$x_0 = 2p + b,$$
$$x_1 = -2p + b,$$
$$x_2 = -2q - b,$$
$$x_3 = 2q - b.$$

Here

$$b = \sqrt{t_{m_0}},$$

$$p = \sqrt{\frac{\alpha}{8} - \frac{b^2}{4} + \frac{\beta}{16b}},$$

and

$$q = \sqrt{\frac{\alpha}{8} - \frac{b^2}{4} - \frac{\beta}{16b}},$$

where t_{m_0} is a root of R_s, the cubic resolvent of quartic such that $t_{m_0} < \dfrac{\alpha}{2}$.

Proof. From Lemma 3.7 and equations (3.14), (3.15) and (3.20). □

Corollary 3.10. *Let α be a root of polynomial $f(x) = x^4 - 6x^2 - 8x - \dfrac{2}{3}$. Then $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a totally real algebraic extension of degree 4.*

Proof. We apply Eisenstein's Criteria with $p = 2$, the polynomial $f(x)$ is irreducible over $\mathbb{Q}$. Then by the Theorem 3.8, the roots of polynomial $f(x)$ are all real. □

3.4 Quintic equation:

This part is intended to study certain particular forms of equations of degree 5 with real coefficients whose roots are totally real and we give these roots explicitly.
The quintic equation has been written in the following forms [1], [3] :

$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	$= 0,$	General quintic form
$x^5 + b_3x^3 + b_2x^2 + b_1x + b_0$	$= 0,$	Reduced quintic form
$x^5 - 5c_3x^3 + 5c_3^2x - c_0$	$= 0,$	DeMoivre's quintic form
$x^5 + d_2x^2 + d_1x + d_0$	$= 0,$	Principal quintic form
$x^5 + e_1x + e_0$	$= 0.$	Bring-Jerrard quintic form

Remark 3.11. The roots of Principal quintic and Bring-Jerrard quintic are not all real.

Theorem 3.12. *For two non-zero complex number a and β , the roots of DeMoivre's quintic equation:*

$$x^5 - 5 \mid a \mid^2 x^3 + 5 \mid a \mid^4 x - \beta = 0,$$

are all real, if and only if $\beta = 2Re(a^5)$*, and are given by:*

$$x_0 = a + \overline{a} = 2Re(a),$$

$$x_1 = \frac{\sqrt{5}-1}{2}Re(a) + \frac{\sqrt{10+2\sqrt{5}}}{2}Im(a),$$

$$x_2 = -\frac{\sqrt{5}+1}{2}Re(a) + \frac{\sqrt{10-2\sqrt{5}}}{2}Im(a),$$

$$x_3 = -\frac{\sqrt{5}+1}{2}Re(a) - \frac{\sqrt{10-2\sqrt{5}}}{2}Im(a),$$

and

$$x_4 = \frac{\sqrt{5}-1}{2}Re(a) - \frac{\sqrt{10+2\sqrt{5}}}{2}Im(a).$$

Proof. We consider the polynomial $f(x) = x^5 - 5\alpha x^3 + 5\alpha^2 x - \beta$ with $\alpha, \beta \in \mathbb{R}$. We must first find the circulant matrix $C = \begin{pmatrix} 0 & a & c & \overline{c} & \overline{a} \\ \overline{a} & 0 & a & c & \overline{c} \\ \overline{c} & \overline{a} & 0 & a & c \\ c & \overline{c} & \overline{a} & 0 & a \\ a & c & \overline{c} & \overline{a} & 0 \end{pmatrix}$, such as his characteristic polynomial is

$$P_C(x) = x^5 - 5\left(|\, a\,|^2 + |\, c\,|^2\right)x^3 - 5\left(a^2\overline{c} + ac^2 + \overline{a}^2c + \overline{ac}^2\right)x^2 + 5\left(-a^3c + |\, a\,|^4 - |\, a\,|^2|\, c\,|^2\right.$$

$$\left.-a\overline{c}^3 - \overline{a}^3\overline{c} - \overline{a}c^3 + |\, c\,|^4\right)x - \left(a^5 + \overline{a}^5 - 5\left(|\, a\,|^2 - |\, c\,|^2\right)\left(a^2\overline{c} - ac^2 + \overline{a}^2c - \overline{ac}^2\right) + c^5 + \overline{c}^5\right).$$

Identifying $f(x)$ with $P_C(x)$, results in the nonlinear system:

$$|\, a\,|^2 + |\, c\,|^2 = \alpha, \tag{3.23}$$

$$a^2\overline{c} + ac^2 + \overline{a}^2c + \overline{ac}^2 = 0, \tag{3.24}$$

$$-a^3c + |\, a\,|^4 - |\, a\,|^2|\, c\,|^2 - a\overline{c}^3 - \overline{a}^3\overline{c} - \overline{a}c^3 + |\, c\,|^4 = \alpha^2, \tag{3.25}$$

$$a^5 + \overline{a}^5 - 5\left(|\, a\,|^2 - |\, c\,|^2\right)\left(a^2\overline{c} - ac^2 + \overline{a}^2c - \overline{ac}^2\right) + c^5 + \overline{c}^5 = \beta, \tag{3.26}$$

Using the equation (3.24) , we have

$$a^5 + \overline{a}^5 - 10\left(|\, a\,|^2 - |\, c\,|^2\right)\left(a^2\overline{c} + \overline{a}^2c\right) + c^5 + \overline{c}^5 = \beta. \tag{3.27}$$

It follows that $a^2\overline{c} + \overline{a}^2c = 0$.
Since a and c play the same role, we assume that $c = 0$, and using the equations (3.23), and (3.25), we get $\alpha = |\, a\,|^2$ and $\beta = 2\mathrm{Re}(a^5)$. Using Poposition 1, the roots are given by:

$$x_m = a \quad \exp\left(\frac{2\pi mi}{5}\right) + \overline{a} \quad \exp\left(\frac{8\pi mi}{5}\right), \quad 0 \le m < 5.$$

We use the following formulas:
$a.e^{i\theta} + \overline{a}.e^{-i\theta} = 2Re(a)cos(\theta) - 2Im(a)sin(\theta)$, $\cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4}$,
$\sin\frac{4\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$, $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4}$.
We get

$$x_0 = a + \overline{a} = 2Re(a),$$

$$x_1 = \frac{\sqrt{5}-1}{2}Re(a) + \frac{\sqrt{10+2\sqrt{5}}}{2}Im(a),$$

$$x_2 = -\frac{\sqrt{5}+1}{2}Re(a) + \frac{\sqrt{10-2\sqrt{5}}}{2}Im(a),$$

$$x_3 = -\frac{\sqrt{5}+1}{2} Re(a) - \frac{\sqrt{10-2\sqrt{5}}}{2} Im(a),$$

and

$$x_4 = \frac{\sqrt{5}-1}{2} Re(a) - \frac{\sqrt{10+2\sqrt{5}}}{2} Im(a),$$

□

Corollary 3.13. *The roots of polynomial* $f(x) = x^5 - \frac{20}{9}x^3 + \frac{80}{81}x - \frac{16}{243}$ *are all real and are given by:*
$x_0 = \frac{4}{3}$, $x_1 = \frac{\sqrt{5}-1}{3} + \sqrt{\frac{10+2\sqrt{5}}{3}}$, $x_2 = -\frac{\sqrt{5}+1}{3} + \sqrt{\frac{10-2\sqrt{5}}{3}}$, $x_3 = -\frac{\sqrt{5}+1}{3} - \sqrt{\frac{10-2\sqrt{5}}{3}}$, *and* $x_4 = \frac{\sqrt{5}-1}{3} - \sqrt{\frac{10-2\sqrt{5}}{3}}$.

Proof. We apply the previous theorem with $a = \frac{1}{3} + i\frac{1}{\sqrt{3}} = \frac{2}{3}e^{i\frac{\pi}{3}}$. □

Corollary 3.14. *For any strictly positive number* α*, the roots of DeMoivre's quintic equation:*

$$x^5 - 5\alpha x^3 + 5\alpha^2 x - 2\alpha^2\sqrt{\alpha} = 0,$$

are all real and are given by:

$$x_0 = 2\sqrt{\alpha},$$

$$x_1 = x_4 = \frac{\sqrt{5}-1}{2}\sqrt{\alpha},$$

and

$$x_2 = x_3 = -\frac{\sqrt{5}+1}{2}\sqrt{\alpha}.$$

Proof. It suffices to use the previous theorem by taking $a = \sqrt{\alpha}$ for α strictly positive number. □

Corollary 3.15. *The general solution of the ordinary differential equation (ODE):*

$$y^{(5)} - 10y''' + 20y' - 8\sqrt{2}y = 0 \tag{3.28}$$

is

$$y(t) = C_1 e^{2\sqrt{2}t} + (C_2 + C_3 t)\, e^{\frac{(1-\sqrt{5})}{2}t} + (C_4 + C_5 t)\, e^{-\frac{(1+\sqrt{5})}{2}t}.$$

where C_j*, j=1,2,...,5, are arbitrary real numbers.*

Proof. The characteristic polynomial of ODE (3.28) is

$$f(r) = r^5 - 10r^3 + 2r - 8\sqrt{2}$$

Hence, by using the corollary 3.14, the roots are $r_0 = 2\sqrt{2}$, $r_1 = r_4 = -(1+\sqrt{5})/\sqrt{2}$ and $r_2 = r_3 = (1+\sqrt{5})/\sqrt{2}$. We conclude that the general solution is $y(t) = C_1 e^{2\sqrt{2}t} + (C_2 + C_3 t)\, e^{\frac{1-\sqrt{5}}{2}t} + (C_4 + C_5 t)\, e^{-\frac{1+\sqrt{5}}{2}t}$. □

Theorem 3.16. *Let* α *be a real number, the roots of the reduced quintic equation:*

$$x^5 - 10\alpha^2 x^3 + 10\alpha^3 x^2 + 5\alpha^4 x - b = 0,$$

are all real if and only if $b = 2\alpha^5$*, and are given by:*

$$x_0 = 2\alpha,$$

$$x_1 = \frac{\alpha}{2}\left(\sqrt{5} - 1 - \sqrt{10+2\sqrt{5}}\right),$$

$$x_2 = \frac{\alpha}{2}\left(\sqrt{5} + 1 + \sqrt{10+2\sqrt{5}}\right),$$

$$x_3 = \frac{\alpha}{2}\left(\sqrt{5} + 1 - \sqrt{10+2\sqrt{5}}\right),$$

$$x_4 = \frac{\alpha}{2}\left(\sqrt{5} - 1 + \sqrt{10+2\sqrt{5}}\right).$$

Proof. We consider the reduced form of quintic:

$$f(x) = x^5 - 10\alpha^2 x^3 + 10\beta x^2 + 5\alpha^4 x - b, \quad \alpha, b \in \mathbb{R}. \tag{3.29}$$

We must first find the circulant matrix

$$C = \begin{pmatrix} 0 & p & iq & -iq & p \\ p & 0 & p & iq & -iq \\ -iq & p & 0 & p & iq \\ iq & -iq & p & 0 & p \\ p & iq & -iq & p & 0 \end{pmatrix} = circ\,(0, p, iq, -iq, p)$$

such as his characteristic polynomial is f. We have

$$\begin{aligned} P_C(x) =& -2p^5 + 5p^4x + 10p^3q^2 - 5p^2q^2x - 5p^2x^3 - 10pq^4 + 10pq^2x^2 + 5q^4x - 5q^2x^3 + x^5 \\ =& \, x^5 - 5\left(p^2+q^2\right)x^3 + 10pq^2x^2 + 5\left(p^4 - p^2q^2 + q^4\right)x - 2p^5 + 10p^3q^2 - 10pq^4 \end{aligned}$$

Identifying $f(x)$ with $p_C(x)$, results in the nonlinear system

$$p^2 + q^2 = 2\alpha^2, \tag{3.30}$$

$$pq^2 = \beta, \tag{3.31}$$

$$p^4 - p^2q^2 + q^4 = \alpha^4, \tag{3.32}$$

$$2p^5 - 10p^3q^2 + 10pq^4 = b. \tag{3.33}$$

By equations (3.30) and (3.32), we obtain q^2 and p^2 are the roots of the quadratic equation

$$u^2 - 2\alpha^2 u + \alpha^2 = 0, \tag{3.34}$$

Then $p^2 = q^2 = \alpha^2$ and by equation (3.31) the sign of p is that of β and $p = \alpha$, it follows that $\beta = \alpha^3$.
By applying the Proposition 1, we have:

$$\begin{aligned} x_0 &= 2\alpha, \\ x_1 &= 2\alpha\left(\cos\left(\tfrac{2\pi}{5}\right) - \sin\left(\tfrac{4\pi}{5}\right)\right) = \tfrac{\alpha}{2}\left(\sqrt{5} - 1 - \sqrt{10 + 2\sqrt{5}}\right), \\ x_2 &= 2\alpha\left(\cos\left(\tfrac{4\pi}{5}\right) + \sin\left(\tfrac{2\pi}{5}\right)\right) = \tfrac{\alpha}{2}\left(\sqrt{5} + 1 + \sqrt{10 + 2\sqrt{5}}\right), \\ x_3 &= 2\alpha\left(\cos\left(-\tfrac{4\pi}{5}\right) - \sin\left(\tfrac{2\pi}{5}\right) = \tfrac{\alpha}{2}\left(\sqrt{5} + 1 - \sqrt{10 + 2\sqrt{5}}\right)\right), \\ x_4 &= 2\alpha\left(\cos\left(-\tfrac{2\pi}{5}\right) - 2\sin\left(-\tfrac{4\pi}{5}\right)\right) = \tfrac{\alpha}{2}\left(\sqrt{5} - 1 + \sqrt{10 + 2\sqrt{5}}\right). \end{aligned}$$

□

Corollary 3.17. *The roots of polynomial* $f(x) = x^5 - \frac{5}{2}x^3 + \frac{5}{4}x^2 + \frac{5}{16}x - \frac{1}{16}$ *are all real, and are given by:*
$x_0 = 1, \quad x_1 = \frac{1}{4}\left(\sqrt{5} - 1 - \sqrt{10 + 2\sqrt{5}}\right), \quad x_2 = \frac{1}{4}\left(\sqrt{5} + 1 + \sqrt{10 + 2\sqrt{5}}\right),$
$x_3 = \frac{1}{4}\left(\sqrt{5} + 1 - \sqrt{10 + 2\sqrt{5}}\right)$, *and* $\quad x_4 = \frac{1}{4}\left(\sqrt{5} - 1 + \sqrt{10 + 2\sqrt{5}}\right)$.

Proof. We apply the previous theorem with $\alpha = \frac{1}{2}$. □

Theorem 3.18. *Let* α, β, *be a non-zero real numbers, the roots of the reduced quintic equation:*

$$x^5 - \frac{15}{2}\alpha^2 x^3 - 10\beta x^2 + 15\alpha^4 x + 36\alpha^2\beta = 0,$$

are all real if and only if $\beta^2 = \frac{\alpha^6}{8}$, *and are given by:*

$$x_0 = 2(b + c),$$

$$x_1 = x_4 = \frac{b}{2}\left(\sqrt{5}-1\right) - \frac{c}{2}\left(\sqrt{5}+1\right),$$

$$x_2 = x_3 = -\frac{b}{2}\left(\sqrt{5}+1\right) + \frac{c}{2}\left(\sqrt{5}-1\right).$$

With $b = \frac{-\beta}{\alpha^2} - \alpha\sqrt{\frac{5}{8}}, \quad c = \frac{-\beta}{\alpha^2} + \alpha\sqrt{\frac{5}{8}}$.

Proof. We consider the polynomial of the form $p(x) = x^5 - \frac{15}{2}\alpha^2 x^3 - \beta x^2 + 15\alpha^4 x - \varepsilon$, and a symetric circulant matrix

$$C = \begin{pmatrix} 0 & b & c & c & b \\ b & 0 & b & c & c \\ c & b & 0 & b & c \\ c & c & b & 0 & b \\ b & c & c & b & 0 \end{pmatrix}.$$

On the other hand, the characteristic polynomial of C is given by:

$$P_c(x) = x^5 - 5\left(b^2+c^2\right)x^3 - 10bc(b+c)x^2 + 5\left(b^4 - 2b^3c - b^2c^2 - 2bc^3 + c^4\right)x - 2b^5 + 10b^4c - 10b^3c^2 - 10b^2c^3 + 10bc^4 - 2c^5.$$

Identifying $p(x)$ with $P_C(x)$, results in the nonlinear system:

$$b^2 + c^2 = \frac{3\alpha^2}{2}, \tag{3.35}$$

$$bc(b+c) = \beta, \tag{3.36}$$

$$b^4 - 2bc\left(b^2+c^2\right) - b^2c^2 + c^4 = 3\alpha^4, \tag{3.37}$$

and

$$2\left(b^5+c^5\right) - 10bc\left(b^3+c^3\right) + 10b^2c^2(b+c) = \varepsilon. \tag{3.38}$$

We replace $b^4 + c^4$ in the equation, by $\left(b^2+c^2\right)^2 - 2b^2c^2$, we obtain

$$\frac{9\alpha^4}{4} - 3b^2c^2 - 3\alpha^2 bc = 3\alpha^4.$$

It follows that

$$b^2c^2 + \alpha^2 bc + \frac{\alpha^4}{4} = 0, \tag{3.39}$$

then $bc = \frac{-\alpha^2}{2}$.

By equation (3.36) we have

$$b + c = \frac{-2\beta}{\alpha^2}. \tag{3.40}$$

Then b and c are the roots of quadratic equation: $t^2 + \frac{2\beta}{\alpha^2}t - \frac{\alpha^2}{2} = 0$, whose discriminant is $\Delta = \frac{4\beta^2}{\alpha^4} + 2\alpha^2$.

Let $b = \frac{-\beta - \sqrt{\beta^2 + \frac{\alpha^6}{2}}}{\alpha^2}$ and $c = \frac{-\beta + \sqrt{\beta^2 + \frac{\alpha^6}{2}}}{\alpha^2}$.

By equation (3.35) we have

$$\beta^2 = \frac{\alpha^6}{8}. \tag{3.41}$$

It follows that

$$b = \frac{-\beta}{\alpha^2} - \alpha\sqrt{\frac{5}{8}}, \quad c = \frac{-\beta}{\alpha^2} + \alpha\sqrt{\frac{5}{8}}. \tag{3.42}$$

On the other hand
$c^5+b^5=(b+c)^5-10b^2c^2(b+c)-5bc\left(b^3+c^3\right)$, $\quad b^3+c^3=(b+c)^3-3bc\,(b+c)$ and $bc(b+c)=\beta$, then

$$\varepsilon=-36\alpha^2\beta. \tag{3.43}$$

By applying the Proposition 2.4, we have:

$$x_m=2b\cos\left(\frac{2\pi m}{5}\right)+2c\cos\left(\frac{4\pi m}{5}\right),\quad 0\leq m<5,$$

then

$$\begin{aligned} x_0 &= 2(b+c),\\ x_1 &= x_4=2b\cos\left(\frac{2\pi}{5}\right)+2c\cos\left(\frac{4\pi}{5}\right)=\frac{b}{2}\left(\sqrt{5}-1\right)-\frac{c}{2}\left(\sqrt{5}+1\right),\\ x_2 &= x_3=2b\cos\left(\frac{4\pi}{5}\right)+2c\cos\left(\frac{-2\pi}{5}\right)=-\frac{b}{2}\left(\sqrt{5}+1\right)+\frac{c}{2}\left(\sqrt{5}-1\right). \end{aligned}$$

□

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