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On FP-injectivity with respect to modules of projective dimension at most one

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Abstract. This paper introduces and investigates the notions of flatness and FP-injectivity with respect to the class $\mathcal{P}_1$ consisting of modules over a ring R of projective dimension at most one. This allows us to characterize, for bunch of specific rings, when the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. In particular, we prove that a ring R is semi-hereditary and $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if $\mathcal{P}_1^{\perp}$ coincides with the class FP-$\mathcal{I}(R)$ of FP-injective modules. Finally, we prove that, given a ring R and a flat ring extension Q of R, if $K = Q/R$, then $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ and $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$ if and only if $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$ and $K \in \varinjlim P_1$ extending [1, Corollary 6.8], where $\mathcal{F}(Q)$ stands for the class of flat right Q-modules.

1 Introduction

Throughout this paper, R denotes an associative ring with unit element and the R-modules are supposed to be unital. Given an R-module M, M^+ denotes the character R-module of M, that is, $M^+ := \mathrm{Hom}_{\mathbb{Z}}\left(M, \dfrac{\mathbb{Q}}{\mathbb{Z}}\right)$, $\mathrm{pd}_R(M)$ denotes the projective dimension of M, $\mathrm{id}_R(M)$ the injective dimension of M and $\mathrm{fd}_R(M)$ the flat dimension of M. As for the global dimensions, l-gl-dim(R) designates the left global dimension of R and wgl-dim(R) the weak global dimension of R. Also, FPD(R) denotes the finitistic projective dimension of R and $\mathrm{f.dim}(R)$ denotes the little finitistic dimension of R. Mod(R) stands for the class of all right R-modules, $\mathcal{P}(R)$ stands for the class of all projective right R-modules, $\mathcal{I}(R)$ the class of all injective right R-modules and $\mathcal{F}(R)$ the class of flat right modules. Also, we denote by $\mathcal{P}_1$ the class of right R-modules M such that $\mathrm{pd}_R(M) \leq 1$ and by $\mathcal{P}_1^{\mathrm{fp}}$ the subclass of $\mathcal{P}_1$ consisting of right R-modules which are finitely presented. Any unreferenced material is standard as in [5, 16, 19, 20].

In [1], one of the main goals of Bazzoni and Herbera is to characterize the rings R for which the equality $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ holds. In this context, via [1, Theorem 6.7], they proved the following key result towards such a characterization for the rings R with classical ring of quotients Q: *Let R be a ring with classical of quotients Q. Then the following assertions are equivalent:*

1) $\mathrm{f.dim}(Q) = 0$*;*

2) $\mathcal{P}_1(R) \otimes_R Q \subseteq \mathcal{F}(Q)$*;*

3) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$.

They deduced the following result which generalizes a theorem of Hügel and Trlifaj stating that if R is a domain, then $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ [11, Theorem 3.5]: *Let R be a ring with classical ring of quotients Q. Then $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ and* $\mathrm{f.dim}(Q) = 0$ *if and only if* $\mathrm{FFD}(Q) = 0$ [1, Corollary 6.8], where FFD(Q) denotes the finitistic flat dimension of Q. Furthermore, recall that the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is said to be of finite type if $\mathcal{P}_1^{\perp} = \mathcal{P}_1(\mathrm{mod}(R))^{\perp}$ (see [1]), where mod(R) stands for the class of right modules that admit projective resolutions consisting of finitely generated projective modules. Bazzoni and Herbera proved in [1] that if the ring is an order in an $\aleph_0$-Noetherian ring Q of little finitistic dimension 0, then the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if Q has finitistic projective dimension FPD$(Q) = 0$. This allows to prove that $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type for orders in semisimple artinian rings [1, Corollary 8.1] and then, in particular, for commutative domains. Their findings answered in the affirmative an open problem posed by L. Fuchs and L. Salce [8, Problem 6, p. 139] on the structure of one dimensional divisible modules over domains. Moreover, Bazzoni and Herbera were concerned in [1] by characterizing the commutative Noetherian rings for which $(P_1, P_1^{\perp})$ is of finite type and proved that these rings are the ones that are orders into artinian rings.

In this paper, we introduce the notions of $\mathcal{P}_1$-flat modules and $\mathcal{P}_1^{\mathrm{fp}}$-injective modules as the Tor-orthogonal class of $\mathcal{P}_1$ and Ext-orthogonal class of $\mathcal{P}_1^{\mathrm{fp}}$, respectively. We give numerous properties of such entities. First, we prove that an R-module M is $\mathcal{P}_1$-flat if and only if the character module M^+ is $\mathcal{P}_1^{\mathrm{fp}}$-injective. This fact is reminiscent of the flatness of a module M being equivalent to the (FP-) injectivity of the character module M^+. It is to be noted that, switching the location of the character module, the equivalence of the FP-injectivity of M and the flatness of M^+ holds for an arbitrary R-module M if and only if R is coherent. As to the context of $\mathcal{P}_1$-flatness and $\mathcal{P}_1^{\mathrm{fp}}$-injectivity, we prove that the equivalence M is $\mathcal{P}_1^{\mathrm{fp}}$-injective if and only if M^+ is $\mathcal{P}_1$-flat always holds. Also, we typify specific rings R by the notions of $\mathcal{P}_1^{\mathrm{fp}}$-injectivity and $\mathcal{P}_1$-flatness of R-modules. First, it is worth recalling that if R is a Prüfer domain, then $\varinjlim \mathcal{P}_1 = \mathrm{Mod}(R)$. More generally, Hügel and Trlifaj prove that the equality $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ is in fact an inherent property to integral domains [11, Theorem 3.5]. In connection with these two later results, we show that the equality $\varinjlim \mathcal{P}_1 = \mathrm{Mod}(R)$ totally characterizes the semi-herditary rings. Effectively, we prove that R is left semi-hereditary if and only if any $\mathcal{P}_1^{\mathrm{fp}}$-injective module is FP-injective if and only if $\varinjlim \mathcal{P}_1 = \mathrm{Mod}(R)$. Moreover, we characterize when the little finististic dimension of a ring R is zero via proving that $\mathrm{f.dim}(R) = 0$ if and only if any R-module M is $\mathcal{P}_1^{\mathrm{fp}}$-injective (Theorem 3.4). In Section 5, we describe totally when $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type in terms of $\mathcal{P}_1$-injectivity and $\mathcal{P}_1$-flatness. This allows us to recover the result of Bazzoni-Herbera that if R is a domain, then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. Also, we investigate, through studying bunch of kinds of rings R, when the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. In this context, we focus our attention on the hereditary rings, semi-hereditary rings, self injective rings and perfect rings. For instance we prove the following: *Given a ring R, then R is left semi-hereditary and $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if $(\mathcal{P}_1, \mathcal{P}_1^{\perp}) = (\text{FP-}\mathcal{P}(R), \text{FP-}\mathcal{I}(R))$, where FP-$\mathcal{P}(R)$ stands for the class of FP-projective modules over R.* Finally, we aim through Section 6 to extend the above-cited theorem [1, Theorem 6.7] of Bazzoni and Herbera to flat ring extensions. Our main theorem in this section reads the following: *Let R be a ring and Q a flat ring extension of R. Let $K := \frac{Q}{R}$. Assume that $K \in \varinjlim \mathcal{P}_1$. Then the following assertions are equivalent:*

1) $\mathcal{P}_1(R) \otimes_R Q \subseteq \mathcal{F}(Q)$;

2) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$.

We deduce from this result the next theorem which extends [1, Corollary 6.8]: *Let R be a ring and Q a flat ring extension of R. Let $K = \frac{Q}{R}$. Then the following assertions are equivalent.*

1) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ and $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$;

2) $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$ and $K \in \varinjlim P_1$.

2 $\mathcal{P}_1$-flat modules and $\mathcal{P}_1^{\mathrm{fp}}$-injective modules

This section introduces and studies the notions of $\mathcal{P}_1$-flat and $\mathcal{P}_1^{\mathrm{fp}}$-flat modules as well as the dual notion of $\mathcal{P}_1^{\mathrm{fp}}$-injective modules.

Let $\mathcal{C}$ be a class of right R-modules and $\mathcal{D}$ be a class of left R-modules. We put

$$\mathcal{C}^{\top} = \ker \mathrm{Tor}_1^R(\mathcal{C}, \text{-}) = \{\text{left } R\text{-modules } M : \mathrm{Tor}_1^R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$$

and

$${}^{\top}\mathcal{D} = \ker \mathrm{Tor}_1^R(\text{-}, \mathcal{D}) = \{\text{right } R\text{-modules } N : \mathrm{Tor}_1^R(N, D) = 0 \text{ for all } D \in \mathcal{D}\}.$$

A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-modules is called a Tor-torsion theory if $\mathcal{A} = {}^{\top}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\top}$. Let $\mathcal{C}$ be a class of right R-modules. Then it is easy to check that $({}^{\top}(\mathcal{C}^{\top}), \mathcal{C}^{\top})$ is a Tor-torsion theory. Also, we put $\widehat{\mathcal{C}} := {}^{\top}(\mathcal{C}^{\top})$. Note that $\varinjlim \mathcal{C} \subseteq \widehat{\mathcal{C}}$ as $\widehat{\mathcal{C}}$ is stable under direct limits. A Tor-torsion theory $(\mathcal{A}, \mathcal{B})$ is said to be generated by $\mathcal{C}$ if $A = \widehat{\mathcal{C}}$ (and thus $B = \mathcal{C}^{\top}$). Let $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ two Tor-torsion theories generated by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then the two pairs $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ coincide if and only if $\widehat{\mathcal{C}_1} = \widehat{\mathcal{C}_2}$.

On the other hand, given a class $\mathcal{F}$ of right R-modules, consider the two associated classes:

$$\mathcal{F}^{\perp} = \{X \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(L, X) = 0, \forall L \in \mathcal{F}\}$$

and

$${}^{\perp}\mathcal{F} = \{X \in \mathrm{Mod}(R) : \mathrm{Ext}^1_R(X, L) = 0, \forall L \in \mathcal{F}\}.$$

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a cotorsion theory [6] provided that ${}^{\perp}\mathcal{C} = \mathcal{F}$ and $\mathcal{F}^{\perp} = \mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called complete [19] if every R-module has a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover. Note that for every class $\mathcal{L}$, ${}^{\perp}\mathcal{L}$ is a resolving class, that is, it is closed under extensions, kernels of epimorphisms and contains the projective modules. In particular, it is syzygy-closed. Dually, $\mathcal{L}^{\perp}$ is coresolving: it is closed under extensions, cokernels of monomorphisms and contains the injective modules. In particular, it is cosyzygy-closed. A pair $(\mathcal{F}, \mathcal{C})$ is called a hereditary cotorsion pair if ${}^{\perp\infty}\mathcal{C} = \mathcal{F}$ and $\mathcal{F}^{\perp\infty} = \mathcal{C}$. It is easy to see that $(\mathcal{F}, \mathcal{C})$ is a hereditary cotorsion pair if and only if $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair such that F is resolving, if and only if $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair such that C is coresolving. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called complete if every R-module has a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called perfect if $\mathcal{F}$ is a covering class and $\mathcal{C}$ is a an enveloping class. For a class $\mathcal{T}$ of right modules, the pair $({}^{\perp}\mathcal{T}, ({}^{\perp}\mathcal{T})^{\perp})$ is a cotorsion (hereditary) pair; it is called the cotorsion pair cogenerated by $\mathcal{T}$.

We begin by proving the following results of general interest.

Proposition 2.1. *Let R be a ring. Then*

$$\varinjlim \mathcal{P}_1^{\mathrm{fp}} = \varinjlim \mathcal{P}_1.$$

Proof. Let $\mathcal{P}_1^{\infty}$ denote the class of all elements of $\mathcal{P}_1$ which admit projective resolutions consisting of finitely generated projective modules. It is easy to see that $\mathcal{P}_1^{\infty} = \mathcal{P}_1^{\mathrm{fp}}$. Also, it is well known that $\mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1^{\infty}$ (see [1, page 12]). Hence $\mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1^{\mathrm{fp}}$. By [11, Lemma 1.2], $\varinjlim \mathcal{P}_1^{\mathrm{fp}}$ is closed under direct limit. Hence $\varinjlim \mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1^{\mathrm{fp}}$. Now, since $\varinjlim \mathcal{P}_1^{\mathrm{fp}} \subseteq \varinjlim \mathcal{P}_1$ (as $\mathcal{P}_1^{\mathrm{fp}} \subseteq \mathcal{P}_1$), it follows that $\varinjlim \mathcal{P}_1^{\mathrm{fp}} = \varinjlim \mathcal{P}_1$, as desired. □

Proposition 2.2. *Let R be a ring. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of right R-modules.*
1) $(\varinjlim \mathcal{C})^{\top} = \widehat{\mathcal{C}}^{\top} = \mathcal{C}^{\top}$.
2) If $\mathcal{C} \subseteq \mathcal{D} \subseteq \widehat{\mathcal{C}}$, then $\widehat{\mathcal{C}} = \widehat{\mathcal{D}}$.
3) If $\varinjlim \mathcal{C} = \varinjlim \mathcal{D}$, then $\mathcal{C}^{\top} = \mathcal{D}^{\top}$ and $\widehat{\mathcal{C}} = \widehat{\mathcal{D}}$.

Proof. 1) Note that $\widehat{\mathcal{C}}^{\top} = \mathcal{C}^{\top}$ and that $\widehat{\mathcal{C}}^{\top} \subseteq (\varinjlim \mathcal{C})^{\top} \subseteq \mathcal{C}^{\top}$ as $\mathcal{C} \subseteq \varinjlim \mathcal{C} \subseteq \widehat{\mathcal{C}}$. Then the result easily follows.
2) Assume that $\mathcal{C} \subseteq \mathcal{D} \subseteq \widehat{\mathcal{C}}$. Then $\widehat{\mathcal{C}} \subseteq \widehat{\mathcal{D}} \subseteq \widehat{\widehat{\mathcal{C}}}$. Now, as $\widehat{\widehat{\mathcal{C}}} = \widehat{\mathcal{C}}$, we get $\widehat{\mathcal{C}} = \widehat{\mathcal{D}}$, as desired.
3) It follows easily from (1). □

Next, we introduce the notions of $\mathcal{P}_1^{\mathrm{fp}}$-flat modules and $\mathcal{P}_1$-flat modules.

Definition 2.3. 1) A left R-module M is said to be $\mathcal{P}_1$-flat if $\mathrm{Tor}^1_R(H, M) = 0$ for each right module $H \in \mathcal{P}_1$, that is, $M \in \mathcal{P}_1^{\top}$. The class of all left $\mathcal{P}_1$-flat modules is denoted by $\mathcal{P}_1\mathcal{F}(R)$.
2) A left R-module M is said to be $\mathcal{P}_1^{\mathrm{fp}}$-flat if $\mathrm{Tor}^1_R(H, M) = 0$ for each right module $H \in \mathcal{P}_1^{\mathrm{fp}}$, that is, $M \in \mathcal{P}_1^{\mathrm{fp}\,\top}$. The class of all left $\mathcal{P}_1^{\mathrm{fp}}$-flat modules is denoted by $\mathcal{P}_1^{\mathrm{fp}}\mathcal{F}(R)$.

The following proposition lists some properties of $\mathcal{P}_1$-flat modules and $\mathcal{P}_1^{\mathrm{fp}}$-flat modules.

Proposition 2.4. *Let R be a ring. Then*
1) $\mathcal{P}_1\mathcal{F}(R) \subseteq \mathcal{P}_1^{\text{fp}}\mathcal{F}(R)$.
2) $\mathcal{P}_1\mathcal{F}(R)$ and $\mathcal{P}_1^{\text{fp}}\mathcal{F}(R)$ are stable under direct sums and direct limits.
3) $\mathcal{P}_1\mathcal{F}(R)$ and $\mathcal{P}_1^{\text{fp}}\mathcal{F}(R)$ are stable under submodules.
4) Any left ideal of R is $\mathcal{P}_1$-flat and $\mathcal{P}_1^{\text{fp}}$-flat.

Proof. 1) and 2) are clear as the functor $\text{Tor}_n^R(H, -)$ commutes with direct sums and direct limits for any right R-module H and each positive integer n.
3) Let N be a submodule of a left $\mathcal{P}_1$-flat module M. Let $H \in \mathcal{P}_1$be a right module and consider the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow \dfrac{M}{N} \longrightarrow 0$ of left modules. Then applying the functor $H \otimes_R -$, we get the exact sequence

$$\text{Tor}_2^R\left(H, \frac{M}{N}\right) \longrightarrow \text{Tor}_1^R(H, N) \longrightarrow \text{Tor}_1^R(H, M).$$

Now, as $\text{Tor}_1^R(H, M) = 0$ since M is $\mathcal{P}_1$-flat and $\text{Tor}_2^R\left(H, \dfrac{M}{N}\right) = 0$ as $\text{fd}_R(H) \leq 1$, we deduce that $\text{Tor}_1^R(H, N) = 0$. Therefore N is a $\mathcal{P}_1$-flat left R-module, as desired.
4) It follows from 3). □

The next proposition proves that the two notions of $\mathcal{P}_1$-flat modules and $\mathcal{P}_1^{\text{fp}}$-flat modules collapse.

Proposition 2.5. *Let R be a ring. Then*
1) The pair $(\widehat{\mathcal{P}_1}, \mathcal{P}_1\mathcal{F}(R))$ is a Tor*-torsion theory.*
2) $(\widehat{\mathcal{P}_1^{\text{fp}}}, \mathcal{P}_1^{\text{fp}}\mathcal{F}(R))$ is a Tor*-torsion theory with*

$$\varinjlim \mathcal{P}_1 = \widehat{\mathcal{P}_1^{\text{fp}}} = \widehat{\mathcal{P}_1}.$$

3) $(\widehat{\mathcal{P}_1}, \mathcal{P}_1\mathcal{F}(R)) = (\widehat{\mathcal{P}_1^{\text{fp}}}, \mathcal{P}_1^{\text{fp}}\mathcal{F}(R))$ and thus $\mathcal{P}_1\mathcal{F}(R) = \mathcal{P}_1^{\text{fp}}\mathcal{F}(R)$.

Proof. 1) It is direct.
2) Note that

$$\mathcal{P}_1^{\text{fp}} \subseteq \mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1 = \varinjlim \mathcal{P}_1^{\text{fp}} \subseteq \widehat{\mathcal{P}_1^{\text{fp}}} \subseteq \widehat{\mathcal{P}_1}.$$

Then, by Proposition 2.1 and Proposition 2.2, $\widehat{\mathcal{P}_1^{\text{fp}}} = \widehat{\mathcal{P}_1}$. Moreover, by [11, Theorem 2.3], $\varinjlim \mathcal{P}_1^{\text{fp}} = \widehat{\mathcal{P}_1^{\text{fp}}}$. Then we are done.
3) It is direct using (2). □

Dually, we next introduce the concept of $\mathcal{P}_1^{\text{fp}}$-injective modules.

Definition 2.6. 1) A left R-module M is said to be $\mathcal{P}_1^{\text{fp}}$-injective if $\text{Ext}_1^R(H, M) = 0$ for each left module $H \in \mathcal{P}_1^{\text{fp}}$, that is, $M \in \mathcal{P}_1^{\text{fp}\perp}$. The class of all $\mathcal{P}_1^{\text{fp}}$-injective modules is denoted by $\mathcal{P}_1^{\text{fp}}\mathcal{I}(R)$.
2) The ring R is said to be a self $\mathcal{P}_1^{\text{fp}}$-injective ring if it is a $\mathcal{P}_1^{\text{fp}}$-injective left R-module.

We next recall the following lemmas which will be useful in the sequel.

Lemma 2.7. *[17, Proposition 2.2] Let A be a finitely presented left R-module and $(M_i)_{i \in I}$ a direct system of submodules of some module. Then*

$$\varinjlim \text{Ext}_R^1(A, M_i) \cong \text{Ext}_R^1(A, \varinjlim M_i).$$

Lemma 2.8. *[3, Lemma 2.10(2)] Let A be a 2-presented left R-module and $(M_i)_{i \in I}$ a family of right R-modules. Then*

$$\prod_i \text{Tor}_1^R(M_i, A) \cong \text{Tor}_1^R\left(\prod_i M_i, A\right).$$

Lemma 2.9. *[3, Lemma 2.9(2)] Let A be a 2-presented left R-module and $(M_i)_{i\in I}$ a direct system of left R-modules. Then*

$$\varinjlim \mathrm{Ext}^1_R(A, M_i) \cong \mathrm{Ext}^1_R(A, \varinjlim M_i).$$

Next, we list some properties of $\mathcal{P}_1^{\mathrm{fp}}$-injective modules. We denote by $\mathcal{I}(R)$ the class of injective left R-modules and by FP-$\mathcal{I}(R)$ the class of FP-injective left R-modules.

Proposition 2.10. *Let R be a ring. Then*
1) $\mathcal{I}(R) \subseteq$ FP-$\mathcal{I}(R) \subseteq \mathcal{P}_1^{\mathrm{fp}}\mathcal{I}(R)$.
2) $\mathcal{P}_1^{\mathrm{fp}}\mathcal{F}(R)$ is closed under extensions, direct products and direct summands.
3) $\mathcal{P}_1^{\mathrm{fp}}\mathcal{I}(R)$ is closed under pure submodules.
4) Any quotient of a $\mathcal{P}_1^{\mathrm{fp}}$-injective module is $\mathcal{P}_1^{\mathrm{fp}}$-injective.

Proof. 1) and 2) are clear as the functor $\mathrm{Ext}^n_R(H, -)$ commutes with direct products for any left R-module H and each positive integer n.
3) Let A be a pure submodule of a $\mathcal{P}_1^{\mathrm{fp}}$-injective left R-module B. For any $H \in \mathcal{P}_1^{\mathrm{fp}}$, we have the exact sequence

$$\mathrm{Hom}_R(H, B) \longrightarrow \mathrm{Hom}_R\left(H, \frac{B}{A}\right) \longrightarrow \mathrm{Ext}^1_R(H, A) \longrightarrow 0$$

But the sequence $\mathrm{Hom}_R(H, B) \longrightarrow \mathrm{Hom}_R\left(H, \frac{B}{A}\right) \longrightarrow 0$ is exact since H is finitly presented and A is a pure submodule of B, so $\mathrm{Ext}^1_R(H, A) = 0$. Therefore, A is $\mathcal{P}_1^{\mathrm{fp}}$-injective.
4) Let M be a $\mathcal{P}_1^{\mathrm{fp}}$-injective left R-module and let N be a submodule of M. Consider the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow \frac{M}{N} \longrightarrow 0$. Let $K \in \mathcal{P}_1^{\mathrm{fp}}$. Applying the functor $\mathrm{Hom}_R(K, -)$ to the considered sequence, we get the following exact sequence

$$\mathrm{Ext}^1_R(K, N) \longrightarrow \mathrm{Ext}^1_R(K, M) = 0 \longrightarrow \mathrm{Ext}^1_R\left(K, \frac{M}{N}\right) \longrightarrow \mathrm{Ext}^2_R(K, N).$$

As $K \in \mathcal{P}_1$, $\mathrm{Ext}^2_R(K, N) = 0$. Hence $\mathrm{Ext}^1_R\left(K, \frac{M}{N}\right) = 0$. It follows that $\frac{M}{N}$ is $\mathcal{P}_1^{\mathrm{fp}}$-injective, as desired.
□

It is known the direct limit of injective modules over a ring R is not injective, in general. The following proposition shows that the $\mathcal{P}_1^{\mathrm{fp}}$-injective modules well behave with respect to direct limits, in other words, any direct limit of injective modules is $\mathcal{P}_1^{\mathrm{fp}}$-injective.

Proposition 2.11. *Let R be a ring. Then any direct limit of $\mathcal{P}_1^{\mathrm{fp}}$-injective modules is $\mathcal{P}_1^{\mathrm{fp}}$-injective.*

Proof. It suffices to observe that any element $M \in \mathcal{P}_1^{\mathrm{fp}}$ is 2-presented and then to apply Lemma 2.9. □

Corollary 2.12. *Let R be a ring. Then any direct limit of injective modules is $\mathcal{P}_1^{\mathrm{fp}}$-injective.*

Proof. It follows from Proposition 2.10. □

It is well known that a right R-module M is flat if and only if M^+ is a left injective module. The next proposition provides the analog version of this result for the $\mathcal{P}_1^{\mathrm{fp}}$-flatness and $\mathcal{P}_1^{\mathrm{fp}}$-injectivity.

Proposition 2.13. *Let R be a ring and M a right R-module. Then the following assertions are equivalent:*
1) M is $\mathcal{P}_1$-flat;
2) M is $\mathcal{P}_1^{\mathrm{fp}}$-flat;
3) M^+ is $\mathcal{P}_1$-injective;
4) M^+ is $\mathcal{P}_1^{\mathrm{fp}}$-injective.
If R is a ring with classical ring of quotients Q satisfying $\mathrm{f.dim}(Q) = 0$, *then the above assertions are equivalent to the following one:*
5) M is torsion-free.

Proof. 1) ⇔ 2) It suffices to apply Proposition 2.5(3).
1) ⇔ 3) and 2) ⇔ 4) It follow easily from the standard isomorphism

$$\mathrm{Ext}^1_R(N, M^+) \cong \mathrm{Tor}^R_1(M, N)^+$$

for any left R-module N.

□

Recall that the direct product of flat right R-modules needs not be flat unless the base ring R is left coherent. Moreover, if M is an injective left R-module, the character module M^+ need not be flat unless R is left coherent. Next, we prove that the $\mathcal{P}_1$-flat modules behave well with respect to direct products and that the $\mathcal{P}_1^{\mathrm{fp}}$-injectivity of a module M is well characterized by the $\mathcal{P}_1$-flatness of the character module M^+.

Theorem 2.14. *Let R be a ring. Then*
1) Any direct product of $\mathcal{P}_1$-flat right R-modules is $\mathcal{P}_1$-flat.
2) Let M be an R-module. Then the following assertions are equivalent:
  a) M is $\mathcal{P}_1^{\mathrm{fp}}$-injective;
  b) M^+ is $\mathcal{P}_1$-flat;
  c) M^{++} is $\mathcal{P}_1^{\mathrm{fp}}$-injective.
3) A right R-module M is $\mathcal{P}_1$-flat if and only if M^{++} is $\mathcal{P}_1$-flat.

Proof. 1) Note that, by Lemma 2.8, $\prod \mathrm{Tor}^R_1(M_i, A) \cong \mathrm{Tor}^R_1(\prod M_i, A)$ for any $A \in \mathcal{P}_1^{\mathrm{fp}}$ and any family $(M_i)_{i\in I}$ of right R-modules. Then any direct product of $\mathcal{P}_1^{\mathrm{fp}}$-flat right R-modules is $\mathcal{P}_1^{\mathrm{fp}}$-flat. It follows, by Proposition 2.5(3), that any direct product of $\mathcal{P}_1$-flat right R-modules is $\mathcal{P}_1$-flat
2) Observe that $\mathrm{Tor}^R_1(M^+, N) \cong \mathrm{Ext}^1_R(N, M)^+$ for any left R-module M and any $N \in \mathcal{P}_1^{\mathrm{fp}}$ by [3, Lemma 2.7(2)]. Hence M is $\mathcal{P}_1^{\mathrm{fp}}$-injective if and only if M^+ is $\mathcal{P}_1^{\mathrm{fp}}$-flat if and only if M^+ is $\mathcal{P}_1$-flat establishing the equivalence a) ⇔ b). Now, Proposition 2.13 guarantees the equivalence b) ⇔ c).
3) It follows from a combination of (2) and Proposition 2.13. □

3 $\mathcal{P}_1^{\mathrm{fp}}$-injectivity and specific rings

In this section, we characterize several kind of rings by homological properties of P_1^{fp}-injective modules.

Our first theorem characterizes rings in which the class of $\mathcal{P}_1^{\mathrm{fp}}$-injective R-modules coincides with the class of FP-injective ones. Moreover, recall that if R is a Prüfer domain, then $\varinjlim \mathcal{P}_1 = \mathrm{Mod}(R)$. More generally, Hügel and Trlifaj prove that the equality $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ is in fact an inherent property to integral domains [11, Theorem 3.5]. The next theorem shows that the equality $\varinjlim \mathcal{P}_1 = \mathrm{Mod}(R)$ characterizes the semi-herditary rings. We denote by $\mathrm{Mod}^{\mathrm{fp}}(R)$ the class of finitely presented R-modules.

Theorem 3.1. *Let R be a ring. Then the following assertions are equivalent.*
  1) Any $\mathcal{P}_1^{\mathrm{fp}}$-injective left R-module is FP*-injective;*
  2) Any $\mathcal{P}_1$-flat right R-module is flat;
  3) R is left semi-hereditary;
  4) $\mathrm{Mod}^{\mathrm{fp}}(R) = \mathcal{P}_1^{\mathrm{fp}}$*;*
  5) $\varinjlim \mathcal{P}_1 = \mathrm{Mod}(R)$.

Proof. 1) ⇒ 2) Let M be a $\mathcal{P}_1$-flat right module. Then, by Proposition 2.13, M^+ is a $\mathcal{P}_1^{\mathrm{fp}}$-injective left module. Hence, by (1), M^+ is FP-injective right R-module and thus M is flat, as desired.
2) ⇒ 3) Assume that (2) holds. Then any right ideal of R is flat. Hence wgl-dim$(R) \leq 1$. Also, as any flat right module is $\mathcal{P}_1$-flat, we get, by Theorem 2.14(1), any direct product of flat right R-modules is flat and thus R is left coherent. It follows that R is semi-hereditary, as desired.

3) $\Rightarrow$ 4) Note first that $\mathcal{P}_1^{\text{fp}} \subseteq \text{Mod}^{\text{fp}}(R)$. Assume that R is left semi-hereditary. Let M be a finitely presented module. Then there exists an exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ such that L is a finitely generated free R-module and K is a finitely generated module. As R is left semi-hereditary, we get that K is a finitely projective module and thus $M \in \mathcal{P}_1^{\text{fp}}$. This ensures that $\text{Mod}^{\text{fp}}(R) = \mathcal{P}_1^{\text{fp}}$, as desired.
4) $\Rightarrow$ 5) Assume that $\text{Mod}^{\text{fp}}(R) = \mathcal{P}_1^{\text{fp}}$. It is known that any R-module is a direct limit of finitely presented modules. Then any R-module is a direct limit of elements of $\mathcal{P}_1^{\text{fp}}$, that is, $\varinjlim \mathcal{P}_1^{\text{fp}} = \text{Mod}(R)$. Now, since by Proposition 2.1, $\varinjlim \mathcal{P}_1 = \varinjlim \mathcal{P}_1^{\text{fp}}$, it follows that $\varinjlim \mathcal{P}_1 = \text{Mod}(R)$, as contended.
5) $\Rightarrow$ 2) Assume that $\varinjlim \mathcal{P}_1 = \text{Mod}(R)$. Then, by Proposition 2.1, $\varinjlim \mathcal{P}_1^{\text{fp}} = \text{Mod}(R)$. Let M be a $\mathcal{P}_1$-flat module. Let $N \in \text{Mod}(R)$. Then there exists a direct system $(N_i)_i$ of elements of $\mathcal{P}_1^{\text{fp}}$ such that $N = \varinjlim N_i$. Therefore, as M is $\mathcal{P}_1$-flat,

$$\begin{aligned} \text{Tor}_1^R(M, N) &= \text{Tor}_1^R(M, \varinjlim N_i) \\ &\cong \varinjlim \text{Tor}_1^R(M, N_i) \\ &= 0. \end{aligned}$$

It follows that M is flat, as desired.
4) $\Rightarrow$ 1) It is direct completing the proof of the theorem.

□

It is well known that R is a left Noetherian ring if and only if any FP-injective R-module is injective [15, Theorem 3]. The following corollary characterizes rings in which any $\mathcal{P}_1^{\text{fp}}$-injective module is injective.

Corollary 3.2. *Let R be a ring. The following are equivalent:*
1) R is a left Noetherian hereditary ring;
2) Any $\mathcal{P}_1^{\text{fp}}$-injective left R-module is injective.

Proof. 1) $\Rightarrow$ 2) Assume that R is a left Noetherian hereditary ring. Let M be a $\mathcal{P}_1^{\text{fp}}$-injective left R-module. Since R is left hereditary, by Theorem 3.1, M is FP-injective. Now, since R is left Noetherian, we get M is injective, as desired.
2) $\Rightarrow$ 1) Assume that any $\mathcal{P}_1^{\text{fp}}$-injective R-module is injective. First, by Theorem 3.1, R is left semi-hereditary. Also, as any FP-injective module is $\mathcal{P}_1^{\text{fp}}$-injective, we get that any FP-injective left R-module is injective. Hence, by [15, Theorem 3], R is left Noetherian. This completes the proof. □

Corollary 3.3. *Let R be an integral domain. The following are equivalent:*
1) R is a Dedekind ring;
2) Any $\mathcal{P}_1^{\text{fp}}$-injective R-module is injective.

Proof. It is clear as any integral domain R is a Dedekind ring if R is hereditary. Also, by [16, Corollary 4.26], any Dedekind ring is Noetherian.

□

Recall that the (left) little finitistic dimension, denoted by $\text{f.dim}(R)$, is the supremum of the projective dimension of the left R-modules of finite projective dimension in $\text{mod}(R)$. The next theorem characterizes rings in which any R-module is $\mathcal{P}_1^{\text{fp}}$-injective.

Theorem 3.4. *Let R be a ring. Then the following assertions are equivalent.*
1) ${}_R R$ is self $\mathcal{P}_1^{\text{fp}}$-injective;
2) Any free R-module is $\mathcal{P}_1^{\text{fp}}$-injective;
3) Any projective R-module is $\mathcal{P}_1^{\text{fp}}$-injective;
4) Any left R-module is $\mathcal{P}_1^{\text{fp}}$-injective;
5) Every submodule of a $\mathcal{P}_1^{\text{fp}}$-injective is $\mathcal{P}_1^{\text{fp}}$-injective;
6) Any right R-module is $\mathcal{P}_1$-flat;

7) $\mathcal{P}_1 \subseteq \mathcal{F}(R)$.
8) Any quotient of a $\mathcal{P}_1$*-flat right* R*-module is* $\mathcal{P}_1$*-flat;*
9) $\mathcal{P}_1^{\rm fp} \subseteq \mathcal{P}(R)$*;*
10) $\mathcal{P}_1 \cap \mathrm{mod}(R) = \mathcal{P}(R) \cap \mathrm{mod}(R)$*;*
11) $\mathrm{f.dim}(R) = 0$.

Proof. 1) ⇒ 2) It holds easily as $\mathcal{P}_1^{\rm fp}\mathcal{I}(R)$ is closed under direct sums.
2) ⇒ 3) It suffices to note that $\mathcal{P}_1^{\rm fp}\mathcal{I}(R)$ is closed under direct summand.
3) ⇒ 4) It follows from the fact any left R-module M is a quotient of a projective module and that $\mathcal{P}_1^{\rm fp}\mathcal{I}(R)$ is stable under quotients.
4) ⇒ 1) It is straightforward.
4) ⇒ 9) First, note that $\mathcal{P}_1^{\rm fp} \subseteq^{\perp} \mathcal{P}_1^{\rm fp}\mathcal{I}(R)$. Then, using (4), we get $\mathcal{P}_1^{\rm fp} \subseteq \mathcal{P}(R) = {}^{\perp}\,\mathrm{Mod}(R)$.
9) ⇒ 4) It is straightforward.
4) ⇔ 5) It holds easily as any R-module is a submodule of an injective module which is a $\mathcal{P}_1^{\rm fp}$-injective R-module.
4) ⇔ 6) It is straightforward using Proposition 2.13 and Theorem 2.14.
6) ⇔ 8) It follows from the fact that any R-module M is a quotient of a projective module which is $\mathcal{P}_1$-flat.
6) ⇔ 7) It is direct as $(\widehat{\mathcal{P}_1}, \mathcal{P}_1\mathcal{F}(R))$ is a torsion theory and $\mathcal{P}_1 \subseteq \widehat{\mathcal{P}_1} = \varinjlim \mathcal{P}_1$.
9) ⇒ 10) Let M be any element of $\mathcal{P}_1 \cap \mathrm{mod}(R)$. Then by [1, Lemma 6.4], there is a finitely generated projective module P and a short exact sequence $0 \longrightarrow R^n \longrightarrow R^m \longrightarrow M \oplus P \longrightarrow 0$. Then $M \oplus P \in \mathcal{P}_1^{\rm fp}$, and by 8) M is projective.
10) ⇒ 11) it is direct.
11) ⇒ 9) is clear as $\mathcal{P}_1^{\rm fp} \subset \mathcal{P} \cap \mathrm{mod}(R)$, as desired completing the proof. □

Corollary 3.5. *Let* R *be a ring. If* R *is self-injective, then any* R*-module is* $\mathcal{P}_1^{\rm fp}$*-injective and* $\mathrm{f.dim}(R) = 0$.

Corollary 3.6. *Let* R *be a ring. Then the following assertions are equivalent.*
1) R *is von Neumann regular;*
2) R *is left semi-hereditary and* R *is a (left) self* $\mathcal{P}_1^{\rm fp}$*-injective ring.*

Proof. Combine Theorem 3.4 and Theorem 3.1. □

Corollary 3.7. *Let* R *be an Artinian ring. Then* $\mathrm{f.dim}(R) = 0$.

Proof. Let $M \in \mathcal{P}_1^{\rm fp}$. Let m be a maximal ideal of R. Note that $\mathrm{depth}(R_m) = 0$. Then, $M_m \in \mathcal{P}_1^{\rm fp}(R_m)$ and thus, by Auslander-Buchsbaum formula, we get $\mathrm{pd}_{R_m}(M_m) = 0$. Hence M_m is a projective R_m-module for each maximal ideal m of R. Therefore M is a projective R-module. It follows that $\mathcal{P}_1^{\rm fp} \subseteq \mathcal{P}(R)$ and thus, by Theorem 3.4, we get $\mathrm{f.dim}(R) = 0$, as desired. □

Corollary 3.8. *Let* R *be a Noetherian commutative ring with classical ring of quotients* Q*. Then,* Q *is a self* $\mathcal{P}_1^{\rm fp}$*-injective ring.*

Proof. It follows from Theorem 3.4 and [1, Lemma 8.3]. □

4 $\mathcal{P}_1^{\rm fp}$-injectivity and homological dimensions

The aim of this section is to characterize the homological dimension of modules over a ring R via the vanishing of the functors Ext and Tor by the class of $\mathcal{P}_1^{\rm fp}$-injective modules.

Proposition 4.1. *Let* R *be a ring. Let* M *be a left* R*-module and* n *a positive integer. Then the following statements are equivalent.*
1) $\mathrm{id}_R(M) \leq n$*;*
2) $\mathrm{Ext}_R^{n+1}(N, M) = 0$ *for each* $\mathcal{P}_1^{\rm fp}$*-injective left* R*-module* N.

The proof requires the following lemma.

Lemma 4.2. *Let R be a ring. Then $(\mathcal{P}_1^{\text{fp}}\mathcal{I}(R))^{\perp} = \mathcal{I}(R)$.*

Proof. We only need to prove that if $M \in (\mathcal{P}_1^{\text{fp}}\mathcal{I}(R))^{\perp}$, then M is injective. In fact, let $M \in (\mathcal{P}_1^{\text{fp}}\mathcal{I}(R))^{\perp}$. There exists a short exact sequence of left R-modules $0 \longrightarrow M \longrightarrow I \longrightarrow G \longrightarrow 0$ with I injective. Then G is $\mathcal{P}_1^{\text{fp}}$-injective, by Proposition 2.10(4). Hence, $\text{Ext}_R^1(G, M) = 0$, and thus the sequence $0 \longrightarrow M \longrightarrow I \longrightarrow G \longrightarrow 0$ splits. It follows that M is injective, as desired. □

Proof of Proposition 4.1. 1) ⇒ 2) is straightforward.

2) ⇒ 1) Let $0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} E_0 \overset{d_1}{\longrightarrow} E_1 \overset{d_2}{\longrightarrow} E_2 \longrightarrow \cdots$ be an injective resolution of M. Let $L_0 = \text{Im}(\varepsilon)$ and $L_i = \text{Im}(d_i)$ for each integer $i \geq 1$. Then, for any $\mathcal{P}_1^{\text{fp}}$-injective module N, by [16, Corollary 6.16], $\text{Ext}_R^1(N, L_n) \cong \text{Ext}_R^{n+1}(N, M) = 0$. Hence $L_n \in (\mathcal{P}_1^{\text{fp}}\mathcal{I}(R))^{\perp}$, so that, by Lemma 4.2, L_n is injective. It follows that $\text{id}_R(M) \leq n$. □

Proposition 4.3. *Let R be a ring. Then*

$$\text{l-gl-dim}(R) = \sup\{\text{pd}_R(M) : M \in \mathcal{P}_1^{\text{fp}}\mathcal{I}(R)\}.$$

Proof. First, note that $\text{l-gl-dim}(R) \geq \sup\{\text{pd}_R(M) : M \text{ is a } \mathcal{P}_1^{\text{fp}}\text{-injective left } R\text{-module}\}$. If $\sup\{\text{pd}_R(M) : M \text{ is a } \mathcal{P}_1^{\text{fp}}\text{-injective left } R\text{-module}\} = +\infty$, then we are done. Now, assume that there exists an integer $n \geq 0$ such that $\text{pd}_R(M) \leq n$ for any $\mathcal{P}_1^{\text{fp}}$-injective R-module M. Then $\text{Ext}_R^{n+1}(M, N) = 0$ for any $\mathcal{P}_1$-injective R-module M and any R-module N. Hence, by Proposition 4.1, $\text{id}_R(N) \leq n$ for any R-module N. It follows that $\text{l-gl-dim}(R) \leq n$ and thus the desired equality follows. □

We deduce the following characterization of semisimple rings.

Corollary 4.4. *Let R be a ring. Then the following assertions are equivalent.*
1) R is semisimple;
2) Any $\mathcal{P}_1^{\text{fp}}$-injective module is projective.

Proposition 4.5. *Let R be a ring. Let M be a right R-module and n a positive integer. Then the following assertions are equivalent:*
1) $\text{fd}_R(M) \leq n$;
2) $\text{Tor}_{n+1}^R(M, N) = 0$ for any $\mathcal{P}_1^{\text{fp}}$-injective left R-module N.

First, we establish the following lemma.

Lemma 4.6. *Let R be a ring and M a right R-module. Then the following assertions are equivalent:*
1) M is a flat right R-module;
2) $\text{Tor}_1^R(M, N) = 0$ for any $\mathcal{P}_1^{\text{fp}}$-injective left R-module N.

Proof. We only need to prove that 2) ⇒ 1) Assume that $\text{Tor}_1^R(M, N) = 0$ for every $\mathcal{P}_1^{\text{fp}}$-injective left R-module N. Consider a short exact sequence of left R-modules $0 \longrightarrow M^+ \longrightarrow E \longrightarrow G \longrightarrow 0$ with E an injective left module. Then E is $\mathcal{P}_1^{\text{fp}}$-injective and thus G is $\mathcal{P}_1^{\text{fp}}$-injective by Proposition 2.10. Hence, $\text{Ext}_R^1(G, M^+) = \text{Tor}_1^R(M, G)^+ = 0$. Therefore, the considered exact sequence $0 \longrightarrow M^+ \longrightarrow E \longrightarrow G \longrightarrow 0$ splits, and thus M^+ is injective left R-module. Hence, M is a flat right R-module completing the proof. □

Proof of Proposition 4.5. It suffices to prove that 2) ⇒ 1) Assume that 2) holds. Let F_{n-1} be the $(n-1)$th yoke of a flat resolution of M and let N be any $\mathcal{P}_1^{\text{fp}}$-injective left R-module. By [16, Corollary 6.13], $\text{Tor}_{n+1}^R(M, N) \cong \text{Tor}_1^R(F_{n-1}, N)$. Then, using (2), we get $\text{Tor}_1^R(F_{n-1}, N) = 0$, and thus by Lemma 4.6, F_{n-1} is flat. Hence $\text{fd}_R(M) \leq n$, as desired. □

Proposition 4.7. *Let R be a ring. Then*

$$\text{wgl-dim}(R) = \sup\{\text{fd}_R(M) : M \in \mathcal{P}_1^{\text{fp}}\mathcal{I}(R)\}.$$

Proof. If $\sup\{\mathrm{fd}_R(M) : M \text{ is a } \mathcal{P}_1^{\mathrm{fp}}\text{-injective left } R\text{-module }\} = +\infty$, then we are done. Assume that there exists a positive integer n such that $\mathrm{fd}_R(M) \leq n$ for any $\mathcal{P}_1^{\mathrm{fp}}$-injective left module M. Then $\mathrm{Tor}_{n+1}^R(A, M) = 0$ for any right R-module A. Then, by Proposition 4.3, $\mathrm{fd}_R(A) \leq n$ for each right R-module A. Therefore wgl-dim$(R) \leq n$. This establishes the desired equality. □

Our last result of this section provides a characterization of von Neumann regular rings via the flatness of $\mathcal{P}_1^{\mathrm{fp}}$-injective modules.

Corollary 4.8. *Let R be a ring. Then the following statements are equivalent:*
1) R is von Neumann regular;
2) Any $\mathcal{P}_1^{\mathrm{fp}}$-injective left R-module is flat.

5 When is the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ of finite type?

Recall that the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is said to be of finite type if $\mathcal{P}_1^{\perp} = \mathcal{P}_1(\mathrm{mod}(R))^{\perp}$ (see [1]), where $\mathrm{mod}(R)$ stands for the class of modules that admit projective resolutions consisting of finitely generated projective modules. In [1], Bazzoni and Herbera proved that if the ring is an order in an $\aleph_0$-Noetherian ring Q of little finitistic dimension 0, then the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if Q has finitistic projective dimension $\mathrm{FPD}(Q) = 0$. They deduced from this that $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type for orders in semisimple artinian rings [1, Corollary 8.1] so, in particular, for commutative domains. Their findings answered in the affirmative an open problem posed by L. Fuchs and L. Salce [23, Problem 6, p. 139] on the structure of one dimensional divisible modules over domains. Moreover, Bazzoni and Herbera were concerned by characterizing the commutative Noetherian rings for which $(P_1, P_1^{\perp})$ is of finite type. Our concern in this section is to investigate, through studying bunch of kinds of rings R, when the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. In this context, we focus our attention on the hereditary rings, semi-hereditary rings, self injective rings and perfect rings.

Our first main result of this section characterizes when $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type in terms of $\mathcal{P}_1$-injectivity and $\mathcal{P}_1$-flatness. This allows us to recover the result of Bazzoni-Herbera that if R is a domain, then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.

Theorem 5.1. *Let R be a ring. Then the following assertions are equivalent:*
1) $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type;
2) $\mathcal{P}_1^{\perp} = (\mathcal{P}_1^{\mathrm{fp}})^{\perp}$;
3) Any direct sum of $\mathcal{P}_1$-injective modules is $\mathcal{P}_1$-injective;
4) Any direct limit of $\mathcal{P}_1$-injective modules is $\mathcal{P}_1$-injective;
5) An R-module M is $\mathcal{P}_1$-injective module if and only if M is $\mathcal{P}_1^{\mathrm{fp}}$-injective;
6) An R-module M is $\mathcal{P}_1$-injective if and only if M^+ is $\mathcal{P}_1$-flat;
7) Any pure submodule of a $\mathcal{P}_1$-injective module is $\mathcal{P}_1$-injective.
Moreover, if R is a ring with classical ring of quotients Q such that $\mathrm{f.dim}(Q) = 0$, then the above assertions are equivalent to the following one:
8) $\mathcal{P}_1^{\perp} = \mathcal{D}$.

Proof. First, note that $\mathcal{P}_1(\mathrm{mod}(R)) = \mathcal{P}_1^{\mathrm{fp}}$. Then 1) ⇔ 2) ⇔ 5) hold. Also, the equivalence 1) ⇔ 3) holds by [1, Proposition 4.1]. For 3) ⇔ 4) use [2, Proposition 2.8].
5) ⇔ 6) It is direct by Theorem 2.14.
5) ⇒ 7) Let M be a $\mathcal{P}_1$-injective module and N a pure submodule of M. Then M is $\mathcal{P}_1^{\mathrm{fp}}$-injective and N is a pure submodule of M. Hence, by Proposition 2.10(3), N is $\mathcal{P}_1^{\mathrm{fp}}$-injective and thus N is $\mathcal{P}_1$-injective, as desired.
7) ⇒ 5) Let M be a $\mathcal{P}_1^{\mathrm{fp}}$-injective module. Then, by Theorem 2.14(2), M^{++} is $\mathcal{P}_1^{\mathrm{fp}}$-injective. Therefore, by Proposition 2.13, M^{++} is $\mathcal{P}_1$-injective. Now, since M is a pure submodule of M^{++}, we get, by (7), that M is $\mathcal{P}_1$-injective, as contended.
Assume that R is a ring with classical ring of quotients Q such that $\mathrm{f.dim}(Q) = 0$. Then, by [1, Theorem 6.7], $\mathcal{P}_1(\mathrm{mod}(R))^{\perp} = \mathcal{D}$, that is, $\mathcal{P}_1^{\mathrm{fp}\perp} = \mathcal{D}$. Hence 2) ⇔ 7) holds easily.

□

Corollary 5.2. *Let R be an integral domain. Then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.*

Proof. It follows easily from Theorem 5.1 as, if K denotes the quotient field of R, $\mathrm{f.dim}(K) = 0$ and it is well known that $\mathcal{P}_1^{\perp} = \mathcal{D}$ in the case of integral domains. □

The following corollary proves that, if $\mathrm{f.dim}(R) = 0$, then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ coincides with the cotorsion pair $(\mathcal{P}(R), \mathrm{Mod}(R))$.

Corollary 5.3. *Let R be a ring. Then the following assertions are equivalent.*
1) $\mathrm{f.dim}(R) = 0$ *and* $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ *is of finite type;*
2) $\mathrm{FPD}(R) = 0$.

Proof. 1) ⇒ 2) Assume that $\mathrm{f.dim}(R) = 0$ and $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. Then, as $\mathrm{f.dim}(R) = 0$, we get, by Theorem 3.4, that $\mathcal{P}_1^{\mathrm{fp}\perp} = \mathrm{Mod}(R)$. Also, since $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type, then $\mathcal{P}_1^{\perp} = \mathcal{P}_1^{\mathrm{fp}\perp}$. Hence $\mathcal{P}_1^{\perp} = \mathrm{Mod}(R)$. It follows, by [2, Proposition 3.5], that $\mathrm{FPD}(R) = 0$.
2) ⇒ 1) Assume that $\mathrm{FPD}(R) = 0$. Then, in particular, $\mathrm{f.dim}(R) = 0$. Also, by [2, Proposition 3.5], we get $\mathcal{P}_1^{\perp} = \mathrm{Mod}(R)$. Now, since $\mathcal{P}_1^{\perp} \subseteq \mathcal{P}_1^{\mathrm{fp}\perp}$, it follows that $\mathcal{P}_1^{\perp} = \mathcal{P}_1^{\mathrm{fp}\perp} = \mathrm{Mod}(R)$ and thus $(P_1, \mathcal{P}_1^{\perp})$ is of finite type completing the proof. □

Corollary 5.4. *Let R be a perfect commutative ring. Then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.*

Proof. It is direct as $\mathrm{FPD}(R) = 0$. □

Our next result characterizes when the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type for self-injective rings.

Proposition 5.5. *Let R be a self-injective ring. Then the following assertions are equivalent:*
1) $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ *is of finite type;*
2) $\mathrm{FPD}(R) = 0$.
Moreover, if R is commutative, then the above assertions are equivalent to the following one:
3) R is a perfect ring.

Proof. Since R is self-injective, we get that R is self-$\mathcal{P}_1$-injective and thus, by Theorem 3.4, $\mathrm{f.dim}(R) = 0$. Now, Corollary 5.3 establishes the equivalence 1) ⇔ 2). Also, it is well known that, if R is a commutative ring, then $\mathrm{FPD}(R) = 0$ if and only if R is a perfect ring. This fact allows to get the desired equivalences completing the proof.
□

The last results of this sections discuss the finite type notion of the pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ for von Neumann regular rings, hereditary rings and semi-hereditary rings.

Proposition 5.6. *Let R be a von Neumann regular ring. Then the following assertions are equivalent:*
1) $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ *is of finite type;*
2) $\mathrm{FPD}(R) = 0$.

Proof. First, since R is von Neumann regular, $\mathcal{F}(R) = \mathcal{P}_1\mathcal{F}(R) = \mathrm{Mod}(R)$. By Theorem 3.4, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type ⇔ a module M is $\mathcal{P}_1$-injective if and only if M^+ is $\mathcal{P}_1$-flat ⇔ a module M is $\mathcal{P}_1$-injective if and only if M^+ is an R-module ⇔ $\mathcal{P}_1^{\perp} = \mathrm{Mod}(R)$ ⇔ $\mathcal{P}_1 = \mathcal{P}(R)$ ⇔ $\mathrm{FPD}(R) = 0$, as desired. □

Proposition 5.7. *Let R be a hereditary ring. Then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if R is Noetherian.*

Proof. As R is hereditary, $\mathcal{P}_1 = \mathrm{Mod}(R)$. Then $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type ⇔ $\mathcal{P}_1^{\perp} = \mathcal{I}(R)$ is stable under direct sum ⇔ R is Noetherian completing the proof. □

Recall that a (left) module M over a ring R is said to be FP-projective if $\mathrm{Ext}_R^1(M, N) = 0$ for any FP-injective (left) R-module N. In this context, note that any finitely presented module is FP-projective and, more precisely, $(\mathrm{FP}\text{-}\mathcal{P}(R), \mathrm{FP}\text{-}\mathcal{I}(R))$ is a cotorsion pair cogenerated by the class of all finitely presented modules. Also, it is known the class $\mathrm{FP}\text{-}\mathcal{I}(R)$ of FP-injective modules is stable under direct sum.

Proposition 5.8. *Let R be a ring. Then the following assertions are equivalent:*
1) R is left semi-hereditary and $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type;
2) $(\mathcal{P}_1, \mathcal{P}_1^{\perp}) = (\text{FP-}\mathcal{P}(R), \text{FP-}\mathcal{I}(R))$.

Proof. 1) $\Rightarrow$ 2) Assume that R is left semi-hereditary and that $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. As R is semi-hereditary, then, by Theorem 3.1, $\mathcal{P}_1^{\text{fp}} = \text{Mod}(R)^{\text{fp}}$. Hence $\mathcal{P}_1^{\perp} = \text{Mod}(R)^{\text{fp}\perp} = \text{FP-}\mathcal{I}(R)$ and thus $\mathcal{P}_1 = \text{FP-}\mathcal{P}(R)$. It follows that the two cotorsion pairs $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ and $(\text{FP-}\mathcal{P}(R), \text{FP-}\mathcal{I}(R))$ coincide.
2) $\Rightarrow$ 1) Assume that $(\mathcal{P}_1, \mathcal{P}_1^{\perp}) = (\text{FP-}\mathcal{P}(R), \text{FP-}\mathcal{I}(R))$. Then $\mathcal{P}_1^{\perp} = \text{FP-}\mathcal{I}(R)$ and thus as the class $\text{FP-}\mathcal{I}(R)$ is stable under direct sum, we get that $\mathcal{P}_1^{\perp}$ is stable under direct sum. Hence, by Theorem 5.1, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type. Moreover, Let I be a finitely generated ideal of R. Then, considering the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \dfrac{R}{I} \longrightarrow 0$, we get that $\dfrac{R}{I}$ is finitely presented and thus an FP-projective module. Hence, by our assumptions, $\text{pd}_R\left(\dfrac{R}{I}\right) \leq 1$ so that I is projective. It follows that R is semi-hereditary completing the proof.

□

6 Finitistic dimensions of flat ring extensions

Let S denote the multiplicative set of all regular elements of a ring R and assume that S satisfies the left and right Ore condition. Denote by the localization $Q := S^{-1}R$ the classical ring of quotients of R. Note that the classical ring of quotients of a ring R does not always exist (see [12]). It is worth recalling that if the classical ring of quotients Q of a ring R exists, then Q is a flat ring extension of R and that $K := \dfrac{Q}{R} = \varinjlim_{r \in S} \dfrac{R}{rR}$ which is means, in particular, that $K \in \varinjlim \mathcal{P}_1$.

In [1, Theorem 6.7], Bazzoni and Herbera aims particularly at characterizing rings R for which $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ holds. In this context, they proved the following result which also characterizes the rings R with classical ring of quotients Q of little finististic dimension 0: *Let R be a ring with classical of quotients Q. Then the following assertions are equivalent:*
1) $\text{f.dim}(Q) = 0$*;*
2) $\mathcal{P}_1(R) \otimes_R Q \subseteq \mathcal{F}(Q)$*;*
3) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\text{Mod}(Q)$.
They deduced from this theorem the following result which generalizes a theorem of Hügel and Trlifaj stating that if R is a domain, then $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ [11, Theorem 3.5]: *Let R be a ring with classical ring of quotient Q. Then $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ and* $\text{f.dim}(Q) = 0$ *if and only if* $\text{FFD}(Q) = 0$ *[1, Corollary 6.8],* where $\text{FFD}(Q)$ stands for the finitistic flat dimension of Q.
The aim of this section is to extend the above-cited theorem of Bazzoni and Herbera to flat ring extensions. Thereby, we get a general version of the above corollary [1, Corollary 6.8] for flat ring extensions.

Next, we announce the main theorem of this section. It extends Bazzoni-Herbera theorem [1, Theorem 6.7]. In fact, we show that Bazzoni-Herbera theorem holds for any flat ring extension Q of R such that $K = \dfrac{Q}{R} \in \varinjlim \mathcal{P}_1$ and we recall, as mentioned above, that any classical ring of quotients, when it exists, satisfies this property. For easiness, put $\mathcal{P}_1 \otimes_R Q := \{M \otimes_R Q : M \in \mathcal{P}_1\}$ and $\mathcal{F}_1 \otimes_R Q := \{M \otimes_R Q : M \in \mathcal{F}_1\}$.

Theorem 6.1. *Let R be a ring and Q a (left) flat ring extension of R. Let $K := \dfrac{Q}{R}$. Assume that $K \in \varinjlim \mathcal{P}_1$. Then the following assertions are equivalent.*
1) $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$*;*
2) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\text{Mod}(Q)$.

The proof of Theorem 6.1 follows from the combination of Proposition 6.2 and Proposition 6.4.

Proposition 6.2. *Let R be a ring and Q a (left) flat ring extension of R. Then the following assertions are equivalent:*

1) $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$;

2) $\varinjlim \mathcal{P}_1 \subseteq \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$.

The proof of Proposition 6.2 requires the following lemma.

Lemma 6.3. *Let R be a ring and Q a (left) flat ring extension of R. Then*
1) $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{P}_1(Q)$.
2) $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}_1(Q)$.

Proof. 1) Let $C \in \mathcal{P}_1$ and let $0 \longrightarrow H \longrightarrow K \longrightarrow C \longrightarrow 0$ be an exact sequence with H and K are projective R-modules. Then, as Q is (left) flat over R, we get the exact sequence

$$0 \longrightarrow H \otimes_R Q \longrightarrow K \otimes_R Q \longrightarrow C \otimes_R Q \longrightarrow 0$$

such that $H \otimes_R Q$ and $K \otimes_R Q$ are projective Q-modules. Hence $C \otimes_R Q \in \mathcal{P}_1(Q)$. Therefore $\mathcal{P}_1(R) \otimes_R Q \subseteq \mathcal{P}_1(Q)$.
2) It is similar to (1).

□

Proof of Proposition 6.2. 1) ⇒ 2) Assume that $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$. Let $C \in \varinjlim \mathcal{P}_1$. Then there exists a direct system $\{C_i : i \in I\} \subseteq \mathcal{P}_1$ such that $C = \varinjlim C_i$. Thus

$$\begin{aligned} \mathrm{Tor}_1^R(C, H) &\cong \mathrm{Tor}_1^Q(\varinjlim C_i, H) \\ &= \varinjlim \mathrm{Tor}_1^R(C_i, H) \\ &\cong \varinjlim \mathrm{Tor}_1^Q(C_i \otimes_R Q, H) \\ &= 0 \end{aligned}$$

for any Q-module H. Hence $C \in {}^{\top}\mathrm{Mod}(Q)$. It follows that $\varinjlim \mathcal{P}_1 \subseteq \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$, as desired.
2) ⇒ 1) Assume that $\varinjlim \mathcal{P}_1 \subseteq \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$. Then, as $\mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1$, we get $\mathcal{P}_1 \subseteq \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$. Then $\mathcal{P}_1 \subseteq {}^{\top}\mathrm{Mod}(Q)$ and thus $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$ completing the proof.

□

Proposition 6.4. *Let R be a ring and Q a (left) flat ring extension of R such that $K = \dfrac{Q}{R} \in \varinjlim \mathcal{P}_1$. Then*

$$\mathcal{F}_1 \cap^{\top} \mathrm{Mod}(Q) \subseteq \varinjlim \mathcal{P}_1.$$

Proof. Let $M \in \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$. Let C be a $\mathcal{P}_1$-flat module. Tensoring the exact sequence $0 \longrightarrow R \longrightarrow Q \longrightarrow K \longrightarrow 0$ with C yields the following exat sequence

$$0 = \mathrm{Tor}_1^R(K, C) \longrightarrow C \longrightarrow Q \otimes_R C \longrightarrow K \otimes_R C \longrightarrow 0$$

since $K \in \varinjlim \mathcal{P}_1$ and C is $\mathcal{P}_1$-flat. Now, tensoring this later exact sequence with M yields the next exact sequence

$$\mathrm{Tor}_2^R(M, K \otimes_R C) \longrightarrow \mathrm{Tor}_1^R(M, C) \longrightarrow \mathrm{Tor}_1^R(M, Q \otimes_R C).$$

Since $M \in \mathcal{F}_1$, we get $\mathrm{Tor}_2^R(M, K \otimes_R C) = 0$. Also, since $M \in {}^{\top}\mathrm{Mod}(Q)$ and $Q \otimes_R C \in \mathrm{Mod}(Q)$, we get $\mathrm{Tor}_1^R(M, Q \otimes_R C) = 0$. Therefore $\mathrm{Tor}_1^R(M, C) = 0$. It follows that $M \in {}^{\top}\mathcal{P}_1\mathcal{F}(R) = \widehat{\mathcal{P}_1}$ and thus, by Proposition 2.5(2), we get $M \in \varinjlim \mathcal{P}_1$. This completes the proof. □

Let us recall the above cited theorem, [1, Theorem 6.7], of Bazzoni and Herbera: If R is a ring with classical of quotients Q, then $\mathrm{f.dim}(Q) = 0$ if and only if $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$. We deduce from Proposition 6.4 that the following inequality always holds.

Corollary 6.5. *Let R be a ring with classical ring of quotients Q. Then*

$$\mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q) \subseteq \varinjlim \mathcal{P}_1.$$

Proof. It follows from Proposition 6.4 since, as mentioned above, $K = \dfrac{Q}{R} \in \varinjlim \mathcal{P}_1$. □

Corollary 6.6. *Let R be a ring and Q a (left) flat ring extension of R such that $\mathrm{f.dim}(Q) = 0$. Let $K := \dfrac{Q}{R}$. Then the following assertions are equivalent:*

1) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$*;*

2) $K \in \varinjlim \mathcal{P}_1$.

Proof. 1) ⇒ 2) It is direct as $K \in \mathcal{F}_1$.
2) ⇒ 1) Assume that $K \in \varinjlim \mathcal{P}_1$. As $\mathrm{f.dim}(Q) = 0$, we get, Theorem 3.4, $\mathcal{P}_1(Q) \subseteq \mathcal{F}(Q)$. Then, by Lemma 6.3, $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$. Therefore, by Theorem 6.1, $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$, as desired. □

The next corollary recovers [1, Theorem 6.7] of Bazzoni-Herbera.

Corollary 6.7. *Let R be a ring with classical ring of quotients Q. Then the following assertions are equivalent.*

1) $\mathrm{f.dim}(Q) = 0$*;*

2) $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$*;*

3) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$.

Proof. Let $K := \dfrac{Q}{R}$. Note that $K \in \varinjlim \mathcal{P}_1$. Also, by [1, Lemma 6.2], $\mathcal{P}_1 \otimes_R Q = \mathcal{P}_1(Q)$. Now, 1) ⇔ 2) holds by Theorem 3.4 completing the proof. □

Our next theorem extends the result of Bazzoni and Herbera which proves that given a ring R with classical ring of quotient Q, then $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ and $\mathrm{f.dim}(Q) = 0$ if and only if $\mathrm{FFD}(Q) = 0$ [1, Corollary 6.8].

Theorem 6.8. *Let R be a ring and Q a (left) flat ring extension of R. Let $K = \dfrac{Q}{R}$. Then the following assertions are equivalent.*

1) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ *and* $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$*;*

2) $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$ *and* $K \in \varinjlim P_1$.

Proof. 1) ⇒ 2) Note that $K = \dfrac{Q}{R} \in \mathcal{F}_1$. Then, by (1), $K \in \varinjlim \mathcal{P}_1$. Hence, since $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(R)$, we get, by Theorem 6.1, that $\mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q) = \varinjlim \mathcal{P}_1$. Hence, by (1), $\mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q) = \mathcal{F}_1$ and thus $\mathcal{F}_1 \subseteq {}^{\top}\mathrm{Mod}(Q)$. Then $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$, as desired.
2) ⇒ 1) Assume that $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$ and $K \in \varinjlim P_1$. Then, as Q is (left) flat R-module, $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$. Also, we get $\mathcal{F}_1 \subseteq {}^{\top}\mathrm{Mod}(Q)$. Hence, by Theorem 6.1, $\varinjlim \mathcal{P}_1 = \mathcal{F}_1 \cap {}^{\top}\mathrm{Mod}(Q)$. It follows, as $\mathcal{F}_1 \subseteq {}^{\top}\mathrm{Mod}(Q)$, that $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$, as desired. □

Next, we recover the result of Bazzoni and Herbera [1, Corollary 6.8].

Corollary 6.9. *Let R be a ring with classical ring of quotients Q. The the following assertions are equivalent:*

1) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ *and* $\mathrm{f.dim}(Q) = 0$*;*

2) $\mathrm{FFD}(Q) = 0$.

To prove Corollary 6.9, we need the following lemma. First, it is worth recalling that if R is a ring with classical ring of quotients Q, then $K := \dfrac{Q}{R} \in \varinjlim \mathcal{P}_1$ and $\mathcal{P}_1 \otimes_R Q = \mathcal{P}_1(Q)$ [1, Lemma 6.2]. The next lemma proves that the equality $\mathcal{F}_1 \otimes_R Q = \mathcal{F}_1(Q)$ holds as well.

Lemma 6.10. *Let R be a ring with classical ring of quotients Q. Then*
1) $\mathcal{F}_1 \otimes_R Q = \mathcal{F}_1(Q)$.
2) $\mathrm{FFD}(Q) = 0$ *if and only if* $\mathcal{F}_1 \otimes_R Q \subseteq F(Q)$.

Proof. 1) Let Σ denote the set of non zero-divisors of R. Then $Q = \Sigma^{-1}R$. As Q is flat over R, then it is easy to see that $\mathcal{F}_1(Q) \subseteq \mathcal{F}_1$. Also, let M be a Q-module. Then

$$M \otimes_R Q = M \otimes_R \Sigma^{-1}R \cong \Sigma^{-1}M = M.$$

Hence, since $\mathcal{F}_1(Q) \subseteq \mathcal{F}_1$ and Q is a left flat R-module, we get

$$\mathcal{F}_1(Q) \subseteq \mathcal{F}_1(Q) \otimes_R Q \subseteq \mathcal{F}_1 \otimes_R Q.$$

It follows, using Lemma 6.3, that $\mathcal{F}_1(Q) = \mathcal{F}_1 \otimes_R Q$.
2) Note that $\mathrm{FFD}(Q) = 0$ if and only if $\mathcal{F}_1(Q) \subseteq \mathcal{F}(Q)$. Then, using (1), we get the desired equivalence.

□

Proof of Corollary 6.9. By Lemma 6.9, $\mathrm{FFD}(Q) = 0$ if and only if $\mathcal{F}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$. Also, by Theorem 3.4, $\mathrm{f.dim}(Q) = 0$ if and only if $\mathcal{P}_1(Q) \subseteq \mathcal{F}(Q)$. Since $\mathcal{P}_1(Q) = \mathcal{P}_1 \otimes_R Q$, we get $\mathrm{f.dim}(Q) = 0$ if and only if $\mathcal{P}_1 \otimes_R Q \subseteq \mathcal{F}(Q)$. Therefore, via applying Theorem 6.8, we get the desired equivalence.

□

Corollary 6.11. *Let R be a ring. Then the following assertions are equivalent:*
1) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ *and* $\mathrm{f.dim}(R) = 0$*;*
2) $\mathrm{FFD}(R) = 0$.

Corollary 6.12. *Let R be a self-injective ring. Then the following assertions are equivalent:*
1) $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$*;*
2) $\mathrm{FFD}(R) = 0$.

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