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Some new stability results of a generalization of Cauchy's and the quadratic functional equations in n-Banach spaces

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Abstract *Using the fixed point approach, we investigate the stability and hyperstability, in the sense of Găvruţă, of the following generalization of Cauchy's and the quadratic functional equations*

$$\sum_{k=0}^{n-1} f(x+b_k y) = nf(x) + nf(y),$$

where $n \in \mathbb{N}_2$, $b_k = e^{\frac{2i\pi k}{n}}$ *for* $0 \leq k \leq n-1$, *in* n*-Banach spaces.*
In addition, we prove the hyperstability of the considered equation and the inhomogeneous equation

$$\sum_{k=0}^{n-1} f(x+b_k y) = nf(x) + nf(y) + G(x,y).$$

1 Introduction

The famous talk of S. M. Ulam in 1940 [50] seems to be the starting point for studying the stability of functional equations, in which he discussed a number of important unsolved problems. Among these was the question of the stability of group homomorphisms.
Ulam problem:[50] *Given a group* G_1, *a metric group* G_2 *with metric* $d(\cdot\ ,\ \cdot)$ *and a positive number* ε, *does there exist a* $\delta > 0$ *such that if* $f : G_1 \to G_2$ *satisfies*

$$d\big(f(xy)\ ,\ f(x)f(y)\big) \leq \varepsilon$$

for all $x, y \in G_1$, *then a homomorphism* $\phi : G_1 \to G_2$ *exists with*

$$d\big(f(x)\ ,\ \phi(x)\big) \leq \delta$$

for all $x \in G_1$?
These kinds of questions serve as the foundation for the theory of stability. Under the assumption that G_1 *and* G_2 *are Banach spaces, the case of approximately additive mappings was solved by D. H. Hyers in 1941 [34].*
Hyers [34] and Ulam [50] referred to this property as the stability of the functional equation $f(x+y) = f(x) + f(y)$. *Hyers' work has initiated much of the current research in the theory of the stability of functional equations. In 1978, the theorem of Hyers was significantly generalized by Th. Rassias [46], taking into account cases where the relevant inequality is not bound. This property was called the Hyers-Ulam-Rassias stability of the additive Cauchy functional equation* $f(x+y) = f(x) + f(y)$.
This terminology also applies to other functional equations. The result of Rassias [46] has been

further generalized by Rassias [47], Th. Rassias and P. Šemrl [48], P. Găvruţă [31], and S. -M. Jung [36]. Simultaneously, a special kind of stability has emerged, which is called the hyperstability of functional equations. This kind states that if f satisfies a stability inequality related to the given equation, then it is also a solution to this equation. It seems that the first hyperstability result was published in [14] and concerned ring homomorphisms. The term "hyperstability", on the other hand, appeared for the first time in [40]. Hyperstability is frequently mistaken with superstability, which also admits bounded functions. Further, J. Brzdęk and K. Ciepliński [16] introduced the following definition which described the main ideas of such a hyperstability notion for equations in several variables ($\mathbb{R}^+$ stands for the set of all nonnegative reals and C^D denotes the family of all functions mapping a set $D \neq \phi$ into a set $C \neq \phi$).

Definition 1.1. [16] Let S be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_+^{S^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset Y^S$ into Y^{S^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \cdots, x_n) = \mathcal{F}_2\varphi(x_1, \cdots, x_n), \quad x_1, \cdots, x_n \in S, \tag{1.1}$$

is ε-hyperstable provided every $\varphi_0 \in \mathcal{D}$ that satisfies the inequality

$$d\big(\mathcal{F}_1\varphi_0(x_1, \cdots, x_n)\ ,\ \mathcal{F}_2\varphi_0(x_1, \cdots, x_n)\big) \leq \varepsilon(x_1, \cdots, x_n), \quad x_1, \cdots, x_n \in S, \tag{1.2}$$

fulfils the equation (1.1).

Brzdęk et al. [16] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:

Theorem 1.2. *[16] Let X be a nonempty set, (Y, d) be a complete metric space, and $\Lambda : Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0$$

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with

$$\lim_{n\to\infty} \delta_n = 0$$

Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \leq \Lambda\big(\Delta(\xi, \mu)\big)(x), \ \ \xi, \mu \in Y^X, \ x \in X, \tag{1.3}$$

where $\Delta : Y^X \times Y^X :\to \mathbb{R}_+^X$ is a mapping which is defined by

$$\Delta(\xi, \mu)(x) := d\big(\xi(x), \mu(x)\big), \ \ \xi, \mu \in Y^X, \ x \in X. \tag{1.4}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\big((\mathcal{T}\varphi)(x), \varphi(x)\big) \leq \varepsilon(x) \tag{1.5}$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} \big(\Lambda^n\varepsilon\big)(x) < \infty \tag{1.6}$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} \big(\mathcal{T}^n\varphi\big)(x) \tag{1.7}$$

exists for each $x \in X$. Moreover, the function $\psi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} \big(\mathcal{T}^n\varphi\big)(x) \tag{1.8}$$

is a fixed point of $\mathcal{T}$ with

$$d\big(\varphi(x)\ ,\ \psi(x)\big) \leq \varepsilon^*(x) \tag{1.9}$$

for all $x \in X$.

In 2013, Brzdęk [18] gave the fixed point result by applying Theorem 1.2 as follows:

Theorem 1.3. *[18] Let X be a nonempty set, (Y, d) be a complete metric space, $f_1, \cdots, f_r : X \to X$ and $L_1, \ldots, L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the conditions*

$$d\big(\mathcal{T}\xi(x)\ ,\ \mathcal{T}\mu(x)\big) \leq \sum_{i=1}^{r} L_i(x) d\Big(\xi\big(f_i(x)\big), \mu\big(f_i(x)\big)\Big), \tag{1.10}$$

for all $\xi, \mu \in Y^X, x \in X$ and

$$\Lambda\delta(x) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x)\big), \quad \delta \in \mathbb{R}_+^X, x \in X. \tag{1.11}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\Big(\mathcal{T}\varphi(x)\ ,\ \varphi(x)\Big) \leq \varepsilon(x) \tag{1.12}$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \left(\Lambda^n \varepsilon\right)(x) < \infty \tag{1.13}$$

for all $x \in X$, then the limit (1.7) exists for each $x \in X$. Moreover, the function (1.8) is a fixed point of $\mathcal{T}$ with (1.9) for all $x \in X$.

Then, by using this theorem, Brzdęk [18] improved, extended, and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular, Theorem 1.2). Many papers on the stability and hyperstability of functional equations were published thanks to this important achievement, for example, we refer to [1]-[8], [21]-[23], and [45]. Another point worth noting is that there are other version of Theorem 3.1 in ultrametric space [5], in 2-Banach space [6], [21], and in n-Banach space [22] that helped to discuss many results on the stability of functional equations. For more details on the stability and hyperstability in 2-Banach spaces and n-Banach spaces, we refer the reader to seeing the survey [9].

2 Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_{m_0}$ the set of all integers greater than or equals m_0 $(m_0 \in \mathbb{N})$, $\mathbb{R}_+ = [0, \infty)$ and we use the notation X_0 for the set $X \backslash \{0\}$.
The concept of n-normed space was given by A. Misiak [42] as a generalization of the notions of classical normed space and of a 2-normed space introduced by S. Gähler [29], [30]. We need to recall some basic facts concerning n-normed spaces and some preliminary results.

Definition 2.1. [42] Let $n \in \mathbb{N}_2$, X be a real linear space with dim $X \geq n$. An n-norm on X is a real function $\|\cdot, \cdots, \cdot\| : X^n \to [0, \infty)$ satisfies the following conditions:

(i) $\|x_1, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,

(ii) $\|x_1, \cdots, x_n\| = \|x_{i_1}, \cdots, x_{i_n}\|$ for every permutaion $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$,

(iii) $\|\alpha x_1, \cdots, x_n\| = |\alpha|\ \|x_1, \cdots, x_n\|$,

(iv) $\|x_1 + y, x_2, \cdots, x_n\| \leq \|x_1, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

for all $\alpha \in \mathbb{R}$, and all $x, y, x_1, \cdots, x_n \in X$. The pair $\big(X, \|\cdot, \cdots, \cdot\|\big)$ is called *an n-normed space*.

We note that $\|x_1, \cdots, x_n\| \geq 0$ *for all* $x_1, \cdots, x_n \in X$ *because*

$$\begin{aligned} 2\|x_1, \cdots, x_n\| &\geq \|x_1 - x_1, \cdots, x_n\| \\ &= \|0, \cdots, x_n\| \\ &= 0. \end{aligned}$$

Example 2.2. $\mathbb{R}^n$ equipped with the function $\|\cdot, \cdots, \cdot\|_E$ defined by

$$\|x_1, \cdots, x_n\|_E = \left|\det(x_{ij})\right| = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots, x_{in}) \in \mathbb{R}^n$ for $i \in \{1, \cdots, n\}$, is n-normed space.

If $(X, \|\cdot, \cdots, \cdot\|)$ *is a* n*-normed space, then the function* $\|\cdot, \cdots, \cdot\|$ *is non-negative and*

$$\left\|\sum_{i=1}^{k} y_i, x_2, \cdots, x_n\right\| \leq \sum_{i=1}^{k} \|y_i, x_2, \cdots, x_n\|$$

for any $k \in \mathbb{N}_2, x_2, \cdots, x_n \in X$ *and all* $y_i \in X$ *for* $i \in \{1, \cdots, n\}$.

A sequence $\{y_k\}_{k\in\mathbb{N}}$ *of elements of an* n*-normed space* $(X, \|\cdot, \cdots, \cdot\|)$ *is called a Cauchy sequence if*

$$\lim_{k,\ell\to\infty} \|y_k - y_\ell, x_2, \cdots, x_n\| = 0 \quad \textit{for every } x_2, \cdots, x_n \in X,$$

where $\{y_k\}_{k\in\mathbb{N}}$ *is said to be convergent if there exists a* $y \in X$ *(called the limit of the sequence and denoted by* $\lim_{k\to\infty}$*) with*

$$\lim_{k\to\infty} \|y_k - y, x_2, \cdots, x_n\| = 0 \quad \textit{for every } x_2, \cdots, x_n \in X.$$

An n*-normed space in which every Cauchy sequence is convergent is called an* n*-Banach space. Noted that in* n*-normed spaces every convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product are valid.*

Lemma 2.3. *[22]* $(X, \|\cdot, \cdots, \cdot\|)$ *be an* n*-normed space. Then the following conditions hold:*

(i) if $x_1, \cdots, x_n \in X$, $\alpha \in \mathbb{R}$, $i, j \in \{1, \cdots, n\}$ $i < h$*, then*

$$\left\|x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n\right\| = \left\|x_1, \cdots, x_i, \cdots, x_j + \alpha x_i, \cdots, x_n\right\|;$$

(ii) if $x, y, y_2, y_3, \cdots, y_n \in X$*, then*

$$\left|\left\|x, y_2, y_3, \cdots, y_n\right\| - \left\|y, y_2, y_3, \cdots, y_n\right\|\right| \leq \left\|x -, y_2, y_3, \cdots, y_n\right\|;$$

(iii) if $x \in X$ *and*

$$\left\|x, y_2, y_3, \cdots, y_n\right\| = 0, \quad y_2, y_3, \cdots, y_n \in X,$$

then $x = 0$*;*

(iv) if $\{x_k\}_{k\in\mathbb{N}}$ *is a convergent sequence of elements of* X*, then*

$$\lim_{k\to\infty} \|x_k, y_2, \cdots, y_n\| = \left\|\lim_{k\to\infty} x_k, y_2, \cdots, y_n\right\|, \quad \textit{for every } \ y_2, \cdots, y_n \in X.$$

In 2018, Brzdęk and Ciepliński [22] proved a new version of fixed point theorem in n-Banach spaces. Before present this theorem, we take the following three hypotheses.
(H1): *X is a nonempty set, $\big(Y, \|\cdot, \cdots, \cdot\|\big)$ be an $(m+1)$-Banach space, $f_1, \ldots, f_r : X \to X$ and $L_1, \ldots, L_r : X \to \mathbb{R}_+$ are given.*
(H2): *$\mathcal{T} : Y^X \to Y^X$ is an operator defined by*

$$\mathcal{T}\xi(x) := \sum_{i=1}^{r} L_i(x)\xi\big(f_i(x)\big), \quad \xi \in Y^X,$$

and satisfies the inequality

$$\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y_1, \cdots, y_m\Big\| \leq \sum_{i=1}^{r} L_i(x)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), y_1, \cdots, y_m\Big\|$$

for all $\xi, \mu \in Y^X$, $x \in X$, and all $y_1, \cdots, y_m \in Y$.
(H3): *$\Lambda : \mathbb{R}_+^{X \times Y^m} \to \mathbb{R}_+^{X \times Y^m}$ is an operator defined by*

$$\Lambda\delta(x, y_1, \cdots, y_m) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x)\big)$$

for all $\delta \in \mathbb{R}_+^{X \times Y^m}$, $x \in X$, and all $y_1, \cdots, y_m \in Y$.
The mentioned fixed point theorem is stated as follows.

Theorem 2.4. *Let hypotheses **(H1)** – **(H3)** be valid and functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ fulfill the two conditions:*

$$\big\|\mathcal{T}\varphi(x) - \varphi(x), y_1, \cdots, y_m\big\| \leq \varepsilon(x, y_1, \cdots, y_m), \tag{2.1}$$

and

$$\varepsilon^*(x, y_1, \cdots, y_m) := \sum_{n=0}^{\infty} \Gamma^n \varepsilon(x, y_1, \cdots, y_m) \tag{2.2}$$

for all $x \in X$ and all $y_1, \cdots, y_m \in Y$. Then, for every $x \in X$ and every $y_1, \cdots, y_m \in Y$, the limit

$$\psi(x) := \lim_{n \to \infty} \mathcal{T}^n \varphi(x) \tag{2.3}$$

exists and the function $\psi : X \to Y$, defined in this way, is the unique fixed point of $\mathcal{T}$ such that

$$\big\|\varphi(x) - \psi(x), y_1, \cdots, y_m\big\| \leq \varepsilon^*(x, y_1, \cdots, y_m) \tag{2.4}$$

for all $x \in X$ and all $y_1, \cdots, y_m \in Y$.

There are many results on the stability and hyperstability using the fixed point theorem, see for example, [3], [13], [23], and [43].
Our aim in this paper is to prove the stability and hyperstability results for the generalization of Cauchy's and the quadratic functional equations

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + nf(y), \tag{2.5}$$

where $n \in \mathbb{N}_2$ and $b_k = \exp(\frac{2i\pi k}{n})$ for $0 \leq k \leq n-1$, in n-Banach space according to Theorem 2.4 and using the type of fixed point approach proposed for the first time in [20]. The general solution and stability of this equation and its generalizations were studied by numerous researchers (see for example [12], [25], [39], and [49]).

3 Main Results

Suppose that X *is a complex normed space and* $\big(Y, \|\cdot,\cdots,\cdot\|\big)$ *is an* $(m+1)$*-Banach space with* $m \in \mathbb{N}_1$. *We will denote by* $\mathrm{Aut}(X)$ *the family of all automorphisms of* X. *Moreover, for each* $u : X \to X$, *we write* $ux := u(x)$ *for all* $x \in X$ *and define* u' *by* $u'x := x - ux$ *for all* $x \in X$. *According to Theorem 2.4 and using the type of fixed point approach proposed for the first time in [20], we will present and prove our hyperstability results for Eq. (2.5) on* X_0.

Theorem 3.1. *Let* X *by a complex normed space,* $\big(Y, \|\cdot,\cdots,\cdot\|\big)$ *be an* $(m+1)$*-Banach space with* $m \in \mathbb{N}_1$, $\varepsilon\colon X_0^{m+2} \to \mathbb{R}_+$, *and*

$$l(X) := \Big\{u \in Aut(X) : u', (u'+b_k u) \in Aut(X),\ \alpha_u := n\lambda(u')+n\lambda(u)+\sum_{k=1}^{n-1}\lambda(u'+b_k u) < 1\Big\} \neq \varnothing, \tag{3.1}$$

where

$$\lambda(u) := \inf\Big\{t \in \mathbb{R}_+ : \varepsilon(ux, uy, z_1, \cdots, z_m) \le t\,\varepsilon(x, y, z_1, \cdots, z_m), \quad x, y, z_1, \cdots, z_m \in X_0\Big\} \tag{3.2}$$

for all $u \in Aut(X)$. *Assume that* $f : X \longrightarrow Y$ *satisfies the inequality*

$$\left\|f(x+y) - nf(x) - nf(y) + \sum_{k=1}^{n-1} f(x+b_k y), g(z_1), \cdots, g(z_m)\right\| \le \varepsilon(x, y, z_1, \cdots, z_m) \tag{3.3}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ *such that* $x + b_k y \neq 0$ *for* $1 \le k \le n-1$, *where* $g : X \to Y$ *is a surjective mapping. Then, for each nonempty subset* $\mathcal{U} \subset l(X)$ *such that*

$$u \circ v = v \circ u, \quad \forall u, v \in \mathcal{U}, \tag{3.4}$$

there exists a unique function $Q : X \longrightarrow Y$ *satisfies the equation* (2.5) *and*

$$\big\|f(x) - Q(x), g(z_1), \cdots, g(z_m)\big\| \le \tilde{\varepsilon}(x, z_1, \cdots, z_m), \quad x, z_1, \cdots, z_m \in X_0, \tag{3.5}$$

where $\tilde{\varepsilon} : X_0^{m+1} \to \mathbb{R}_+$ *and*

$$\tilde{\varepsilon}(x, z_1, \cdots, z_m) := \inf\Big\{\frac{\varepsilon(u'x, ux, z_1, \cdots, z_m)}{1-\alpha_u} : u \in \mathcal{U}\Big\}, \quad x, z_1, \cdots, z_m \in X_0.$$

Proof. First, we fix a nonempty and commutative $\mathcal{U} \subset \ell(X)$. By replace x by $u'x$ and y by ux in (3.3), we obtain that

$$\left\|f(x) - nf(u'x) - nf(uy) + \sum_{k=1}^{n-1} f\big((u'+b_k u)x\big), g(z_1), \cdots, g(z_m)\right\| \le \varepsilon(u'x, uy, z_1, \cdots, z_m) \tag{3.6}$$

for all $x, z_1, \cdots, z_m \in X_0$. So, we can define, for each $u \in \mathcal{U}$, the operators $\mathcal{T}_u : Y^{X_0} \to Y^{X_0}$ and $\Lambda_u : \mathbb{R}_+^{X_0^{m+1}} \to \mathbb{R}_+^{X_0^{m+1}}$ by

$$\mathcal{T}_u\xi(x) := n\xi(u'x) + n\xi(ux) - \sum_{k=1}^{n-1}\xi\big((u'+b_k u)x\big), \tag{3.7}$$

and

$$\Lambda_u\delta\,(x, z_1, \cdots, z_m) := n\delta(u'x, z_1, \cdots, z_m)+n\delta(ux, z_1, \cdots, z_m)+\sum_{k=1}^{n-1}\delta\big((u'+b_k u)x, z_1, \cdots, z_m\big), \tag{3.8}$$

for all $x, z_1, \cdots, z_m \in X_0$, $\xi \in Y^{X_0}$, and $\delta \in \mathbb{R}_+^{X_0^{m+1}}$. Note that, for every $u \in \mathcal{U}$, the operator $\Lambda := \Lambda_u$ has the form given in **(H3)** with $X := X_0$, $r = n+1$, $L_1(x) = L_2(x) = n$ and

$L_k(x) = 1$ for $k = 1, \cdots, n-1$, $f_1(x) = u'x$, $f_2(x) = ux$, and $f_{k+2}(x) = (b_k u' + u)x$ for $k = 1, \cdots, n-1$.
When we put

$$\varepsilon_u(x, z_1, \cdots, z_m) := \varepsilon(u'x, uy, z_1, \cdots, z_m),$$

then the inequality (3.6) becomes

$$\left\| \mathcal{T}_u f(x) - f(x), g(z_1), \cdots, g(z_m) \right\|_* \leq \varepsilon_u (x, z_1, \cdots, z_m), \quad x, z_1, \cdots, z_m \in X_0, \quad u \in \mathcal{U}. \tag{3.9}$$

From here till the end of the paper, we denote by f the restriction of $f : X \to Y$ to the set $X_0 \subset X$ unless we mention otherwise. Moreover, for every $\xi, \mu \in Y^{X_0}$, we have

$$\begin{aligned}
\left\| \mathcal{T}_u \xi(x) - \mathcal{T}_u \mu(x), g(z_1), \cdots, g(z_m) \right\| &= \Big\| n\xi(u'x) + n\xi(ux) - \sum_{k=1}^{n-1} \xi\big((u' + b_k u)x\big) \\
&\quad - n\mu(u'x) - n\mu(ux) + \sum_{k=1}^{n-1} \mu\big((u' + b_k u)x\big), g(z_1), \cdots, g(z_m) \Big\| \\
&\leq n \left\| \xi(u'x) - \mu(u'x), g(z_1), \cdots, g(z_m) \right\| + n \left\| \xi(ux) - \mu(ux), g(z_1), \cdots, g(z_m) \right\| \\
&\quad + \sum_{k=1}^{n-1} \left\| \xi\big((u' + b_k u)x\big) - \mu\big((u' + b_k u)x\big), g(z_1), \cdots, g(z_m) \right\| \\
&= \sum_{i=1}^{n+1} L_i(x) \left\| \xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big), g(z_1), \cdots, g(z_m) \right\|
\end{aligned}$$

for all $x, z_1, \cdots, z_m \in X_0$ and all $u \in \mathcal{U}$. This means that the inequality **(H2)** holds for $\mathcal{T} := \mathcal{T}_u$ for any $u \in \mathcal{U}$. In view of the definition of $\lambda(u)$, we note that

$$\varepsilon(ux, uy, z_1, \cdots, z_m) \leq \lambda(u) \varepsilon(x, y, z_1, \cdots, z_m), \qquad x, y, z_1, \cdots, z_m \in X_0.$$

So, it is easy to show that

$$\Lambda_u^s \varepsilon_u(x, z_1, \cdots, z_m) \leq \alpha_u^s \varepsilon(u'x, ux, z_1, \cdots, z_m),$$

for all $x, z_1, \cdots, z_m \in X_0$ and $s \in \mathbb{N}_0$, where

$$\alpha_u = n\lambda(u') + n\lambda(u) + \sum_{k=1}^{n-1} \lambda(u' + b_k u).$$

Hence, we obtain

$$\begin{aligned}
\varepsilon^*(x, z_1, \cdots, z_m) := \sum_{r=0}^{\infty} \Lambda_u^r \varepsilon_u(x, z_1, \cdots, z_m) \\
\leq \varepsilon(u'x, ux, z_1, \cdots, z_m) \sum_{r=0}^{\infty} \alpha_u^r \\
= \frac{\varepsilon(u'x, ux, z_1, \cdots, z_m)}{1 - \alpha_u} < \infty
\end{aligned} \tag{3.10}$$

for all $x, z_1, \cdots, z_m \in X_0$. According to Theorem 2.4, there exists a unique solution $Q_u : X \to Y$ of the equation

$$Q_u(x) = nQ_u(u'x) + nQ_u(ux) - \sum_{k=1}^{n-1} Q_u\big((u' + b_k u)x\big) \tag{3.11}$$

for all $x \in X_0$, which is the fixed point of $\mathcal{T}_u$ such that

$$\Big\|Q_u(x) - f(x), g(z_1), \cdots, g(z_m)\Big\| \le \frac{\varepsilon(u'x, ux, z_1, \cdots, z_m)}{1-\alpha_u}, \quad x, z_1, \cdots, z_m \in X_0. \tag{3.12}$$

Moreover,

$$Q_u(x) = \lim_{r\to\infty} \mathcal{T}_u^r f(x).$$

To prove that Q_u satisfies the functional equation (2.5) on X_0, we just prove the following inequality

$$\left\|\mathcal{T}_u^r f(x+y) - n\mathcal{T}_u^r f(x) - n\mathcal{T}_u^r f(y) + \sum_{k=1}^{n-1} \mathcal{T}_u^r f(x+b_k y), g(z_1), \cdots, g(z_m)\right\| \le \alpha_u^r \varepsilon(x, y, z_1, \cdots, z_m) \tag{3.13}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $r \in \mathbb{N}_0$ such that $x + b_k y \neq 0$ for $1 \le k \le n-1$.

It is clear that if $r = 0$, then (3.13) holds by (3.3). Fix an $n \in \mathbb{N}_0$ and assume that (3.13) holds for any $u \in \mathcal{U}$ and $x, y, z_1, \cdots, z_m \in X_0$ such that $x + b_k y \neq 0$. Then, in view of (3.13), we get

$$\begin{aligned}
&\left\|\mathcal{T}_u^r f(x+y) - n\mathcal{T}_u^r f(x) - n\mathcal{T}_u^r f(y) + \sum_{k=1}^{n-1} \mathcal{T}_u^r f(x+b_k y), g(z_1), \cdots, g(z_m)\right\| = \Big\|n\mathcal{T}_u^r f\Big(u'(x+y)\Big) \\
&+ n\mathcal{T}_u^r f\Big(u(x+y)\Big) - \sum_{k=1}^{n-1} \mathcal{T}_u^r f\Big((u'+b_k u)(x+y)\Big) - n^2\mathcal{T}_u^r f(u'x) - n^2\mathcal{T}_u^r f(ux) \\
&+ n\sum_{k=1}^{n-1} \mathcal{T}_u^r f((u'+b_k u)x) \\
&- n^2\mathcal{T}_u^r f(u'y) - n^2\mathcal{T}_u^r f(uy) + n\sum_{k=1}^{n-1} \mathcal{T}_u^r f((u'+b_k u)y) + \sum_{k=1}^{n-1} \Big\{n\mathcal{T}_u^r f\Big(u'(x+b_k y)\Big) \\
&+ n\mathcal{T}_u^r f\Big(u(x+b_k y)\Big) \\
&- \sum_{k=1}^{n-1} \mathcal{T}_u^r f\Big((u'+b_k u)(x+b_k y)\Big)\Big\}, g(z_1), \cdots, g(z_m)\Big\| \\
&\le n\left\|\mathcal{T}_u^r f\Big(u'(x+y)\Big) - n\mathcal{T}_u^r f(u'x) - n\mathcal{T}_u^r f(u'y) + \sum_{k=1}^{n-1} \mathcal{T}_u^r f\Big(u'(x+b_k y)\Big), g(z_1), \cdots, g(z_m)\right\| \\
&+ n\left\|\mathcal{T}_u^r f\Big(u(x+y)\Big) - n\mathcal{T}_u^r f(ux) - n\mathcal{T}_u^r f(uy) + \sum_{k=1}^{n-1} \mathcal{T}_u^r f\Big(u(x+b_k y)\Big), g(z_1), \cdots, g(z_m)\right\| \\
&+ \sum_{k=1}^{n-1} \Big\|\mathcal{T}_u^r f\Big((u'+b_k u)(x+y)\Big) - n\mathcal{T}_u^r f\Big((u'+b_k u)x\Big) - n\mathcal{T}_u^r f\Big((u'+b_k u)y\Big) \\
&+ \sum_{k=1}^{n-1} \mathcal{T}_u^r f\Big((u'+b_k u)(x+b_k y)\Big), g(z_1), \cdots, g(z_m)\Big\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_u^r \Big(n\varepsilon(u'x, u'y, z_1, \cdots, z_m) + n\varepsilon(ux, uy, z_1, \cdots, z_m) \\
&\quad + \sum_{k=1}^{n-1} \varepsilon\Big((u' + b_k u)x, (u' + b_k u)y, z_1, \cdots, z_m)\Big)\Big) \\
&\leq \alpha_u^r \left(n\lambda(u') + n\lambda(u) + \sum_{k=1}^{n-1} \lambda(u' + b_k u) \right) \varepsilon(x, y, z_1, \cdots, z_m) \\
&= \alpha_u^{r+1} \varepsilon(x, y, z_1, \cdots, z_m).
\end{aligned}
$$

By mathematical induction, we deduce that (3.13) holds for any $n \in \mathbb{N}_0$. Letting $n \longrightarrow \infty$ in (3.13) and using the surjectivity of g in view of Lemma 2.3, we obtain the equality

$$
\sum_{k=0}^{n-1} Q_u(x + b_k y) = nQ_u(x) + nQ_u(y),
$$

for all $x, y \in X_0$ such that $x + b_k y \neq 0$ for $0 \leq k \leq n - 1$. Thus, we have proved that for each $u \in \mathcal{U}$, there exists a function $Q_u : X_0 \to Y$ which is a solution of the functional equation (2.5) on X_0 and satisfies

$$
\Big\| f(x) - Q_u(x), g(z_1), \cdots, g(z_m) \Big\| \leq \frac{\varepsilon(u'x, ux, z_1, \cdots, z_m)}{1 - \alpha_u}
$$

for all $x, z_1, \cdots, z_m \in X_0$. Next, we prove that each solution $Q : X_0 \to Y$ of (2.5) satisfying the inequality

$$
\Big\| f(x) - Q(x), g(z_1), \cdots, g(z_m) \Big\| \leq L\, \varepsilon(v'x, vx, z_1, \cdots, z_m) \quad x, z_1, \cdots, z_m \in X_0 \tag{3.14}
$$

with some $L > 0$ and $v \in \mathcal{U}$, is equal to J_w for each $w \in \mathcal{U}$. So, fix $v, w \in \mathcal{U}$, $L > 0$ and $Q : X \to Y$ a solution of (2.5) satisfying (3.14). Note that, by (3.12) and (3.14), there is $L_0 > 0$ such that

$$
\begin{aligned}
\Big\| Q(x) - Q_w(x), g(z_1), \cdots, g(z_m) \Big\| &\leq \Big\| Q(x) - f(x), g(z_1), \cdots, g(z_m) \Big\| \\
&\quad + \Big\| f(x) - Q_w(x), g(z_1), \cdots, g(z_m) \Big\| \\
&\leq L_0 \Big(\varepsilon(v'x, vx, z_1, \cdots, z_m) \\
&\quad + \varepsilon(w'x, wx, z_1, \cdots, z_m) \Big) \cdot \sum_{r=0}^{\infty} \alpha_w^r
\end{aligned} \tag{3.15}
$$

for $x, z_1, \cdots, z_m \in X_0$. In other side, Q and Q_w are solutions of (3.11) because they are satisfy (2.5). For each $j \in \mathbb{N}_0$ and each $x, z_1, \cdots, z_m \in X_0$ we show that

$$
\Big\| Q(x) - Q_w(x), g(z_1), \cdots, g(z_m) \Big\| \leq L_0 \Big(\varepsilon(v'x, vx, z_1, \cdots, z_m) + \varepsilon(w'x, wx, z_1, \cdots, z_m) \Big) \cdot \sum_{r=j}^{\infty} \alpha_w^r. \tag{3.16}
$$

The case $j = 0$ is exactly (3.15). So fix $\gamma \in \mathbb{N}_0$ and assume that (3.16) holds for $j = \gamma$. Then, in view of the definition of $\lambda(u)$, we obtain

$$\begin{aligned}
&\Big\|Q(x)-Q_w(x),g(z_1),\cdots,g(z_m)\Big\| = \Big\|nQ(w'x)+nQ(wx)-\sum_{k=1}^{n-1}Q\big((w'+b_kw)x\big)\\
&-nQ_w(w'x)-nQ_w(wx)+\sum_{k=1}^{n-1}Q_w\big((w'-b_kw)x\big),g(z_1),\cdots,g(z_m)\Big\|\\
&\le n\Big\|Q(w'x)-Q_w(w'x),g(z_1),\cdots,g(z_m)\Big\|+n\Big\|Q(wx)-Q_w(wx),g(z_1),\cdots,g(z_m)\Big\|\\
&+\sum_{k=1}^{n-1}\Big\|Q\big((w'+b_kw)x\big)-Q_w\big((w'+b_kw)x\big),g(z_1),\cdots,g(z_m)\Big\|\\
&\le n\ L_0\left(\varepsilon(v'w'x,vw'x,z_1,\cdots,z_m)+\varepsilon(w'w'x,ww'x,z_1,\cdots,z_m)\right)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\
&+n\ L_0\left(\varepsilon(v'wx,vwx,z_1,\cdots,z_m)+\varepsilon(w'wx,wwx,z_1,\cdots,z_m)\right)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\
&+\ L_0\sum_{k=1}^{n-1}\Bigg(\varepsilon\big(v'(w'+b_kw)x,v(w'+b_kw)x,z_1,\cdots,z_m\big)\\
&\quad+\varepsilon\big(w'(w'+b_kw)x,w(w'+b_kw)x,z_1,\cdots,z_m\big)\Bigg)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\
&\le L_0\left(\varepsilon(v'x,vx,z_1,\cdots,z_m)+\varepsilon(w'x,wx,z_1,\cdots,z_m)\right)\left(n\lambda(w')+n\lambda(w)+\sum_{k=1}^{n-1}\lambda(w'+b_kw)\right)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\
&=L_0\Big(\varepsilon(v'x,vx,z_1,\cdots,z_m)+\varepsilon(w'x,wx,z_1,\cdots,z_m)\Big)\cdot\sum_{r=\gamma+1}^{\infty}\alpha_w^r.
\end{aligned}$$

Now, letting $j\to\infty$ in (3.16), we get

$$Q(x)=Q_w(x)\quad\forall x\in X_0. \tag{3.17}$$

In this way, we also have proved that $Q_u=Q_w$ for each $u\in\mathcal{U}$, which yields

$$\Big\|f(x)-Q_w(x),g(z_1),\cdots,g(z_m)\Big\|\le\frac{\varepsilon(u'x,ux,z_1,\cdots,z_m)}{1-\alpha_u}\quad x,z_1,\cdots,z_m\in X_0,\ u\in\mathcal{U}.$$

This implies (3.5) with $Q:=Q_w$ and the uniqueness of Q is given by (3.17). □

In the following theorem, we prove the hyperstability of equation (2.5) in n-Banach spaces.

Theorem 3.2. *Let X, Y and ε be as in Theorem 3.1. Suppose That there exists a nonempty set $\mathcal{U}\in l(X)$ such that $u\circ v=v\circ u,\quad\forall u,v\in\mathcal{U}$ and*

$$\begin{cases} inf_{u\in\mathcal{U}}\ \ \varepsilon(u'x,ux,z_1,\cdots,z_m)=0, & \forall x,z_1,\cdots,z_m\in X_0,\\ sup_{u\in\mathcal{U}}\ \alpha_u<1. \end{cases} \tag{3.18}$$

Then every $f:X\to Y$ satisfying (3.3) is a solution of (2.5).

Proof. Suppose that $f:X\to Y$ satisfies (3.3). According to Theorem 3.1, there exists a mapping $Q:X\to Y$ satisfies (2.5) and

$$\Big\|f(x)-Q(x),g(z_1),\cdots,g(z_m)\Big\|\le\tilde{\varepsilon}(x,z_1,\cdots,z_m),\quad\forall x,z_1,\cdots,z_m\in X_0.$$

In view of (3.18), we get $\tilde{\varepsilon}(x, z_1, \cdots, z_m) = 0, \ \ \forall x, z_1, \cdots, z_m \in X_0$ which means that $f(x) = Q(x), \ \ \forall x \in X_0$. Hence,

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + nf(y),$$

for all $x, y \in X_0$ such that $x + b_k y \neq 0$, for $0 \leq k \leq n-1$, which implies that f satisfies the functional equation (2.5) on X_0. □

from Theorems 3.1 and 3.2, we can obtain the following corollaries as natural results.

Corollary 3.3. *Let X and Y be a $\mathbb{C}$-normed space and a $(m+1)$-Banach space, respectively. Assume that $p, q \in \mathbb{R}$, $p < 0$, $q < 0$ and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| \sum_{k=0}^{n-1} f(x + b_k y) - nf(x) - nf(y), g(z_1), \cdots, g(z_m) \right\| \leq \theta \Big(\|x\|^p + \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \tag{3.19}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $x + b_k y \neq 0$ for $0 \leq k \leq n-1$, then f satisfies the functional equation (2.5) on X_0.

Proof. The proof follows from Theorem 3.2 by taking

$$\varepsilon(x, y, z_1, \cdots, z_m) := \theta \Big(\|x\|^p + \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r, \quad x, y, z_1, \cdots, z_m \in X_0,$$

with some real numbers θ, $r \geq 0$, $p < 0$ and $q < 0$. For each $\ell \in \mathbb{N}$, we define $u_\ell : X \to X$ by $u_\ell x := -\ell x$ and $u'_\ell : X \to X$ by $u'_\ell x := (1+\ell)x$. Then

$$\begin{aligned} \varepsilon(u_\ell x, u_\ell y, z_1, \cdots, z_m) &= \varepsilon(-\ell x, -\ell y, z_1, \cdots, z_m) \\ &= \theta \Big(\| -\ell x\|^p + \| -\ell y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &= \theta \Big(\ell^p \|x\|^p + \ell^q \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &\leq (\ell^p + \ell^q) \varepsilon(x, y, z_1, \cdots, z_m), \end{aligned}$$

$$\begin{aligned} \varepsilon(u'_\ell x, u'_\ell y, z_1, \cdots, z_m) &= \varepsilon\Big((1+\ell)x, (1+\ell)y, z_1, \cdots, z_m \Big) \\ &= \theta \Big(\|(1+\ell)x\|^p + \|(1+\ell)y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &= \theta \Big((1+\ell)^p \|x\|^p + (1+\ell)^q \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &\leq \Big((1+\ell)^p + (1+\ell)^q \Big) \varepsilon(x, y, z_1, \cdots, z_m), \end{aligned}$$

and

$$\begin{aligned} \varepsilon\Big((u'_\ell + b_k u_\ell)x, (u'_\ell + b_k u_\ell)y, z_1, \cdots, z_m \Big) &= \varepsilon\Big((1+\ell - b_k \ell)x, (1+\ell - b_k \ell)y, z_1, \cdots, z_m \Big) \\ &= \theta \Big(\|(1+\ell - b_k \ell)x\|^p + \|(1+\ell - b_k \ell)y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &= \theta \Big(|1+\ell - b_k \ell|^p \|x\|^p + |1+\ell - b_k \ell|^q \|y\|^q \Big) \prod_{i=1}^{m} \|z_i\|^r \\ &\leq \big(|1+\ell - b_k \ell|^p + |1+\ell - b_k \ell|^q \big) \varepsilon(x, y, z_1, \cdots, z_m), \end{aligned}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ and all $\ell \in \mathbb{N}$ where $0 \leq k \leq n-1$. Therefore, we deduce that $\lambda(u_\ell) = \ell^p + \ell^q$, $\lambda(u'_\ell) = (1+\ell)^p + (1+\ell)^q$, and $\lambda\big((u'_\ell + b_k u_\ell)\big) = \big|1+\ell-b_k\ell\big|^p + \big|1+\ell-b_k\ell\big|^q$ for $\ell \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $\ell \geq n_0$ and

$$\alpha_{u_\ell} = n\Big((1+\ell)^p + (1+\ell)^q + \ell^p + \ell^q\Big) + \sum_{k=1}^{n-1}\Big(\big|1+\ell-b_k\ell\big|^p + \big|1+\ell-b_k\ell\big|^q\Big) < 1.$$

So, it's easily seen that (2.5) is fulfilled with

$$\mathcal{U} := \Big\{u_\ell \in \mathrm{Aut}(X) : \ell \in \mathbb{N}_{n_0}\Big\} \neq \phi.$$

In addition, we have

$$\begin{aligned}\lim_{\ell\to\infty} \varepsilon(u'_\ell x, u_\ell y, z_1, \cdots, z_m) &\leq \lim_{\ell\to\infty} \left((1+\ell)^p + \ell^q\right)\varepsilon(x, y, z_1, \cdots, z_m)\\ &= 0,\end{aligned}$$

for all $x, y, z_1, \cdots, z_m \in X_0$ which means that (3.18) is valid. Therefore, by Theorem 3.2, every $f : X \to Y$ satisfying (3.19) is a solution of the functional equation (2.5) on X_0. □

By similar method, we can prove the following corollaries.

Corollary 3.4. *Let X and Y be a $\mathbb{C}$-normed space and a $(m+1)$-Banach space, respectively. Assume that $p, q \in \mathbb{R}$, $p + q < 0$ and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\|\sum_{k=0}^{n-1} f(x + b_k y) - nf(x) - nf(y), g(z_1), \cdots, g(z_m)\right\| \leq \theta\, \|x\|^p \|y\|^q \prod_{i=1}^{m} \|z_i\|^r$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $x + b_k y \neq 0$ for $0 \leq k \leq n-1$, then f satisfies the functional equation (2.5) on X_0.

Corollary 3.5. *Let X and Y be a $\mathbb{C}$-normed space and a $(m+1)$-Banach space, respectively. Assume that $p < 0$ and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\|\sum_{k=0}^{n-1} f(x + b_k y) - nf(x) - nf(y), g(z_1), \cdots, g(z_m)\right\| \leq \theta\Big(\|x\|^p + \|y\|^p\Big) \prod_{i=1}^{m} \|z_i\|^r$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $x + b_k y \neq 0$ for $0 \leq k \leq n-1$, then f satisfies the functional equation (2.5) on X_0.

Corollary 3.6. *Let X and Y be a $\mathbb{C}$-normed space and a $(m+1)$-Banach space, respectively. Assume that $p, q \in \mathbb{R}$, $p + q < 0$ and $\theta, r \geq 0$. If $f : X \to Y$ satisfies*

$$\left\|\sum_{k=0}^{n-1} f(x + b_k y) - nf(x) - nf(y), g(z_1), \cdots, g(z_m)\right\| \leq \theta\left(\|x\|^p + \|y\|^q + \|x\|^p \|y\|^q\right) \prod_{i=1}^{m} \|z_i\|^r$$

for all $x, y, z_1, \cdots, z_m \in X_0$ such that $x + b_k y \neq 0$ for $0 \leq k \leq n-1$, then f satisfies the functional equation (2.5) on X_0.

The next corollary corresponds to the results for the following inhomogeneous functional equation

$$\sum_{k=0}^{n-1} f(x + b_k y) = nf(x) + nf(y) + G(x, y), \tag{3.20}$$

for all $x, y \in X_0$ such that $x + b_k y \neq 0$ with $0 \leq k \leq n-1$.

Corollary 3.7. *Let X, Y and ε be as in Theorem 3.1 and $G : X^2 \to Y$. Suppose that*

$$\Big\|G(x,y), g(z_1), \cdots, g(z_m)\Big\| \leq \varepsilon(x,y,z_1,\cdots,z_m), \quad x,y,z_1,\cdots,z_m \in X_0, \tag{3.21}$$

where $G(x_0,y_0) \neq 0$ for some $x_0, y_0 \in X_0$, and there exists a nonempty $\mathcal{U} \subset l(X)$ such that $u \circ v = v \circ u$, $\forall u,v \in \mathcal{U}$ and (3.18) hold. Then the inhomogeneous equation (3.20) has no solutions in the class of functions $f : X \to Y$.

Proof. Suppose that $f : X \to Y$ is a solution to (3.20). Then

$$\begin{aligned}\Big\|\sum_{k=0}^{n-1} f(x+b_k y) - nf(x) - nf(y), g(z_1),\cdots,g(z_m)\Big\| &= \Big\|nf(x)+nf(y)+G(x,y) \\ &\quad - nf(x) - nf(y), g(z_1),\cdots,g(z_m)\Big\| \\ &= \Big\|G(x,y), g(z_1),\cdots,g(z_m)\Big\| \\ &\leq \varepsilon(x,y,z_1,\cdots,z_m),\end{aligned}$$

for all $x,y,z_1,\cdots,z_m \in X_0$ such that $x+b_k y \neq 0$ for $0 \leq k \leq n-1$. Consequently, by Theorem 3.2, f is a solution of (2.5). Thus,

$$G(x_0,y_0) = \sum_{k=0}^{n-1} f(x_0+b_k y_0) - nf(x_0) - nf(y_0) = 0,$$

which is a contradiction. □

Conclusions:

Throughout this paper, we demonstrated the stability and hyperstability of a generalization of Cauchy's and the quadratic functional equations. Additionally, we derived some colollaries as special cases and direct consequences of the main results presented in this paper. This work perhaps open new horizons for the study of this type of equations in n-Banach spaces.

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