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A New extension of Hermite Hadamard inequalities associating ψ-Hilfer fractional integral

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Abstract: *Fractional inequalities have been an essential topic in mathematics and have found applications in various domains. In this article, we established some new bounds (below and above) for mid-point type inequality and trapezoidal-type inequality for ψ-Hilfer- fractional integral by utilizing functions whose second derivatives are bounded. We also investigate some new generalization and extension of Hermite-Hadamard type inequalities for ψ-Hilfer fractional integrals for the functions having the condition: $\Omega'(\dot{p}+\ddot{q}-\delta)-\Omega'(\delta)\geq 0$, $\delta\in\left[\dot{p},\frac{\dot{p}+\ddot{q}}{2}\right]$.*

1 Introduction

Convexity theory is indeed a fundamental and versatile concept in mathematical analysis that finds applications in various disciplines. The framework of convexity provides a strong set of tools and properties that facilitate the study and analysis of functions. The characteristics of convex functions, including their optimality properties, efficient algorithms, and stability under perturbations, make them a powerful and versatile tool across a wide range of applications in mathematics, optimization, economics, finance, and machine learning.

Hermite-Hadamard inequality (H-H-I) is a fundamental result in the field of convexity theory. Named after Charles Hermite and Jacques Hadamard, this inequality establishes a relationship between the average value of a convex function over an interval and its endpoint values. The Hermite-Hadamard inequality (H-H-I) is a generalization of the midpoint rule for integrals and plays a significant role in the study of convex functions. Here is a basic statement of the Hermite-Hadamard inequality (H-H-I):

$$\Omega\left(\frac{\Upsilon+\Phi}{2}\right)\leq\frac{1}{\Phi-\Upsilon}\int_{\Upsilon}^{\Phi}\Omega(\vartheta)\,d\vartheta\leq\frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}.$$

Hermite-Hadamard inequality (H-H-I) has indeed been widely investigated and extended to various contexts, making it a versatile and essential tool in mathematical analysis. Here are some notable aspects of the contributions made by researchers in the evolution of the Hermite-Hadamard inequality (H-H-I). In particular, studies on the Trapezoid inequality and the Midpoint inequality, that is the right and left sides of the Hermite-Hadamard inequality (H-H-I) constitute the majority of the studies on this subject. Researchers have indeed made significant contributions to the development, generalizations and improvements were obtained see for example [14] and [10].

Fractional calculus continues to gain prominence in various scientific and engineering disciplines, the exploration of inequalities like the fractional Hermite-Hadamard inequality (H-H-I) is expected to grow further. Fractional calculus deals with the generalization of derivatives and integrals to non-integer orders, and it has emerged as a powerful mathematical tool for modeling and analyzing complex phenomena in physics, engineering, and other fields. Fractional inequalities have an important place in the discipline of mathematics. Fractional integrals solved many integrals and inequalities in mathematics, Hermite-Hadamard inequality (H-H-I) is one of them. For instance, Chen [4], Awan et al. [1] and Budak et al. [2] obtained extensions of the Hermite-Hadamard inequality (H-H-I) for convex functions having Riemann Liouvile fractional integral (RLFI). In [5], the extensions of the RLFI Hermite-Hadamard inequality (H-H-I) are produced for harmonic-convex function. Some Trapezoid and Midpoint type inequalities are presented for generalized fractional integrals in [3]. Also You et al. [13] developed some Hermite-Hadamard inequalities (H-H-I) for harmonic convex functions via generalized fractional integral. Mumcu et al. [9] constructed Hermite-Hadamard inequalities (H-H-I) via generalized proportional fractional integrals. In addition, some more researchers worked on fractional integral inequalities such as [15]- [19]. For further inequalities that can be solved using fractional integrals, and the references therein [20–27].

Some basic ideas of fractional calculus are presented as follows:

Definition 1.1. [8,11] Let $\Omega\in L_1[\Upsilon,\Phi]$. The RLFI $J_{\Upsilon+}^{\tilde{u}}\Omega$ and $J_{\Phi-}^{\tilde{u}}\Omega$ of order $\tilde{u}>0$ with $\Upsilon\geq 0$ are introduced by

$$J_{\Upsilon+}^{\tilde{u}}\Omega(\vartheta)=\frac{1}{\Gamma(\tilde{u})}\int_{\Upsilon}^{\vartheta}(\vartheta-\delta)^{\tilde{u}-1}\,\Omega(\delta)d\delta,\ \ \vartheta>\Upsilon$$

and

$$J_{\Phi-}^{\tilde{u}}\Omega(\vartheta)=\frac{1}{\Gamma(\tilde{u})}\int_{\vartheta}^{\Phi}(\delta-\vartheta)^{\tilde{u}-1}\,\Omega(\delta)d\delta,\ \ \vartheta<\Phi,$$

respectively.
Here, $\Gamma(\tilde{u})$ is the Gamma function and

$$J_{\Upsilon+}^{0}\Omega(\vartheta)=J_{\Phi-}^{0}\Omega(\vartheta)=\Omega(\vartheta).$$

Some famous results regarding HHI utilizing RLFI are given below.

Theorem 1.2. *[12] Let $\Omega : [\Upsilon, \Phi] \to \mathbb{R}$ be a +ve mapping having $0 \leq \Upsilon < \Phi$ and $\Omega \in L_1[\Upsilon, \Phi]$. If Ω is a convex function on $[\Upsilon, \Phi]$, then*

$$\Omega\left(\frac{\Upsilon+\Phi}{2}\right) \leq \frac{\Gamma(\tilde{u}+1)}{2(\Phi-\Upsilon)^{\tilde{u}}}\left[J^{\tilde{u}}_{\Upsilon+}\Omega(\Phi) + J^{\tilde{u}}_{\Phi-}\Omega(\Upsilon)\right] \leq \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}$$

holds, involving RLFI with $\tilde{u} > 0$.

Theorem 1.3. *[14] Let $\Omega : [\Upsilon, \Phi] \to \mathbb{R}$ be two times differentiable mapping such that, there exist m and M (real constants) so that $m \leq \Omega'' \leq M$, then*

$$m\frac{(\Phi-\Upsilon)^2}{24} \leq \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2} - \frac{1}{\Phi-\Upsilon}\int_{\Upsilon}^{\Phi}\Omega(\vartheta)d\vartheta \leq M\frac{(\Phi-\Upsilon)^2}{24}$$

holds.

Theorem 1.4. *[6], [10] By considering the conditions of theorem 1.3, then*

$$m\frac{(\Phi-\Upsilon)^2}{24} \leq \frac{1}{\Phi-\Upsilon}\int_{\Upsilon}^{\Phi}\Omega(\vartheta)d\vartheta - \Omega\left(\frac{\Upsilon+\Phi}{2}\right) \leq M\frac{(\Phi-\Upsilon)^2}{24}$$

holds.

Theorem 1.5. *[4] Let $\Omega : [\Upsilon, \Phi] \longrightarrow \mathbb{R}$ be twice differentiable mapping with $\Upsilon < \Phi$ and $\Omega \in L_1[\Upsilon, \Phi]$ if Ω'' is bounded $[\Upsilon, \Phi]$ then we get*

$$\begin{aligned} &\frac{m\tilde{u}}{2(\Phi-\Upsilon)^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}\left(\frac{\Upsilon+\Phi}{2}-\vartheta\right)^2\left[(\Phi-\vartheta)^{\tilde{u}-1}+(\Upsilon-\vartheta)^{\tilde{u}-1}\right]d\vartheta \qquad (1.1)\\ \leq\ &\frac{\Gamma(\tilde{u}+1)}{2(\Phi-\Upsilon)^{\tilde{u}}}\left[J^{\varphi}_{\Upsilon+}\Omega(\Phi)+J^{\varphi}_{\Phi-}\Omega(\Upsilon)\right]-\Omega\left(\frac{\Upsilon+\Phi}{2}\right)\\ \leq\ &\frac{M\tilde{u}}{2(\Phi-\Upsilon)^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}\left(\frac{\Upsilon+\Phi}{2}-\vartheta\right)^2\left[(\Phi-\vartheta)^{\tilde{u}-1}+(\Upsilon-\vartheta)^{\tilde{u}-1}\right]d\vartheta \end{aligned}$$

with $\tilde{u} > 0$, where

$$m = \inf_{\delta\in[\Upsilon,\Phi]}\Omega''(\delta), \quad M = \sup_{\delta\in[\Upsilon,\Phi]}\Omega''(\delta).$$

Theorem 1.6. *[4] Assume the conditions of Theorem 3, then*

$$\begin{aligned} &\frac{-M\tilde{u}}{2(\Phi-\Upsilon)^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}(\vartheta-\Upsilon)(\Phi-\vartheta)\left[(\Phi-\vartheta)^{\tilde{u}-1}+(\Upsilon-\vartheta)^{\tilde{u}-1}\right]d\vartheta \qquad (1.2)\\ \leq\ &\frac{\Gamma(\tilde{u}+1)}{2(\Phi-\Upsilon)^{\tilde{u}}}\left[J^{\varphi}_{\Upsilon+}\Omega(\Phi)+J^{\varphi}_{\Phi-}\Omega(\Upsilon)\right]-\frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}\\ \leq\ &\frac{-m\tilde{u}}{2(\Phi-\Upsilon)^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}(\vartheta-\Upsilon)(\Phi-\vartheta)\left[(\Phi-\vartheta)^{\tilde{u}-1}+(\Upsilon-\vartheta)^{\tilde{u}-1}\right]d\vartheta \end{aligned}$$

holds, with $\tilde{u} > 0$, where

$$m = \inf_{\delta\in[\Upsilon,\Phi]}\Omega''(\delta), \quad M = \sup_{\delta\in[\Upsilon,\Phi]}\Omega''(\delta).$$

Below definitions of ψ-Hilfer fractional integrals are presented by [8, 11].

Definition 1.7. Let $\psi : [\Upsilon, \Phi] \to \mathbb{R}$ be an increasing and positive monotone function on $(\Upsilon, \Phi]$, having a continuous derivative $\psi'(\vartheta)$ on (Υ, Φ). The left-sided fractional integral of Ω w.r.t the function ψ on $[\Upsilon, \Phi]$ of order $\tilde{u} > 0$ is defined by

$$I^{\tilde{u}}_{\Upsilon+;\psi}\Omega(\vartheta) = \frac{1}{\Gamma(\tilde{u})}\int_{\Upsilon}^{\vartheta}(\psi(\vartheta)-\psi(\delta))^{\tilde{u}-1}\psi'(\delta)\Omega(\delta)d\delta, \vartheta > \Upsilon$$

provided that the integral exists.
The right-sided fractional integral of Ω w.r.t the function ψ on $[\Upsilon, \Phi]$ of order $\tilde{u} > 0$ is defined by

$$I^{\tilde{u}}_{\Phi-;\psi}\Omega(\vartheta) = \frac{1}{\Gamma(\tilde{u})}\int_{\vartheta}^{\Phi}(\psi(\delta)-\psi(\vartheta))^{\tilde{u}-1}\psi'(\delta)\Omega(\delta)d\delta, \vartheta < \Phi,$$

respectively.

Theorem 1.8. *[7] Let Ω is a convex function on $[\Upsilon, \Phi]$ and $\tilde{u} > 0$, then*

$$\begin{aligned} & \Omega\left(\frac{\Upsilon+\Phi}{2}\right) \\ \leq\ & \frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\check{F}(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\check{F}(\Upsilon)\right] \\ \leq\ & \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2} \end{aligned} \tag{1.3}$$

holds, where

$$\check{F}(\delta)=\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta).$$

Further related results can be read in [28]- [32].
The main aim of this article is to give extensions, refinements, and generalizations of Hermite-Hadamard inequalities (H-H-I) for ψ-Hilfer along with generalizations of Trapezoid and Midpoint inequalities using fractional integrals.

2 Main Results

Let $\psi : [\Upsilon, \Phi] \to \mathbb{R}$ be an increasing and positive monotone function on $(\Upsilon, \Phi]$ and a continuous derivative $\psi'(\vartheta)$ on (Υ, Φ).

Theorem 2.1. *Let $\Omega : [\Upsilon, \Phi] \to \mathbb{R}$ be a two times differentiable mapping s. that there exist (m and M) (real constants) so if $m \leq \Omega'' \leq M$, then*

$$\begin{aligned} & \frac{m\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}} H_{\psi}(\delta)\left(\frac{\Upsilon+\Phi}{2}-\delta\right)^{2}d\delta \\ \leq\ & \frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\check{F}(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\check{F}(\Upsilon)\right]-\Omega\left(\frac{\Upsilon+\Phi}{2}\right) \\ \leq\ & \frac{M\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}} H_{\psi}(\delta)\left(\frac{\Upsilon+\Phi}{2}-\delta\right)^{2}d\delta \end{aligned} \tag{2.1}$$

holds, where

$$\begin{aligned} & H_{\psi}(\delta) \\ =\ & \left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\psi'(\delta) \\ & +\left[(\psi(\Phi)-\psi(\Upsilon+\Phi-\delta))^{\tilde{u}-1}+(\psi(\Upsilon+\Phi-\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right] \\ & \times\psi'(\Upsilon+\Phi-\delta). \end{aligned} \tag{2.2}$$

Proof. With the help of Definition 1.7, we get

$$\begin{aligned} & \frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\check{F}(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\check{F}(\Upsilon)\right] \\ =\ & \frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \times\left[\frac{1}{\Gamma(\tilde{u})}\int_{\Upsilon}^{\Phi}(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}\psi'(\delta)\Omega(\delta)d\delta\right. \\ & \left.+\frac{1}{\Gamma(\tilde{u})}\int_{\Upsilon}^{\Phi}(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\psi'(\delta)\Omega(\delta)d\delta\right] \\ =\ & \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \times\int_{\Upsilon}^{\Phi}\left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\psi'(\delta)\Omega(\delta)d\delta \\ =\ & \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \int_{\Upsilon}^{\Phi}\left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right] \\ & \times\psi'(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta \\ =\ & \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \times\left[\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}\left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\right. \end{aligned}$$

$$\times\psi'(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta$$
$$+\int_{\frac{\Upsilon+\Phi}{2}}^{\Phi}\left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]$$
$$\times\psi'(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta\Big].$$

By utilizing change of variables, we can write,

$$\begin{aligned}
&\frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I_{\Upsilon+;\psi}^{\tilde{u}}\breve{F}(\Phi)+I_{\Phi-;\psi}^{\tilde{u}}\breve{F}(\Upsilon)\right] \qquad (2.3)\\
=\;&\frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\\
&\Bigg[\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}\left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\\
&\times\psi'(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta\\
&+\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}\left[(\psi(\Phi)-\psi(\Upsilon+\Phi-\delta))^{\tilde{u}-1}+(\psi(\Upsilon+\Phi-\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\\
&\times\psi'(\Upsilon+\Phi-\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta\Bigg]\\
=\;&\frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}\Big[\left[(\psi(\Phi)-\psi(\delta))^{\tilde{u}-1}+(\psi(\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\psi'(\delta)\\
&+\left[(\psi(\Phi)-\psi(\Upsilon+\Phi-\delta))^{\tilde{u}-1}+(\psi(\Upsilon+\Phi-\delta)-\psi(\Upsilon))^{\tilde{u}-1}\right]\psi'(\Upsilon+\Phi-\delta)\Big]\\
&\times\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta\\
=\;&\frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_{\psi}(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta.
\end{aligned}$$

By using equality (2.3), we have

$$\begin{aligned}
&\frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I_{\Upsilon+;\psi}^{\tilde{u}}\breve{F}(\Phi)+I_{\Phi-;\psi}^{\tilde{u}}\breve{F}(\Upsilon)\right]-\Omega\left(\frac{\Upsilon+\Phi}{2}\right) \qquad (2.4)\\
=\;&\frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\\
&\times\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_{\psi}(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)-2\Omega\left(\frac{\Upsilon+\Phi}{2}\right)\right]d\delta.
\end{aligned}$$

By using the facts that

$$\Omega(\delta)-\Omega\left(\frac{\Upsilon+\Phi}{2}\right)=\int_{\frac{\Upsilon+\Phi}{2}}^{\delta}\Omega'(s)ds$$

and

$$\Omega(\Upsilon+\Phi-\delta)-\Omega\left(\frac{\Upsilon+\Phi}{2}\right)=\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\Omega'(s)ds,$$

we have

$$\begin{aligned}
&\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)-2\Omega\left(\frac{\Upsilon+\Phi}{2}\right) \qquad (2.5)\\
=\;&\int_{\frac{\Upsilon+\Phi}{2}}^{\delta}\Omega'(s)ds+\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\Omega'(s)ds\\
=\;&\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\Omega'(s)ds-\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\Omega'(\Upsilon+\Phi-s)ds\\
=\;&\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\left[\Omega'(s)-\Omega'(\Upsilon+\Phi-s)\right]ds.
\end{aligned}$$

We also get

$$\Omega'(s)-\Omega'(\Upsilon+\Phi-s)=\int_{\Upsilon+\Phi-s}^{s}\Omega''(y)dy. \qquad (2.6)$$

By using $m\leq\Omega''(y)\leq M\ for\ all\ y\in[\Upsilon,\Phi]$, with help of the equality (2.6), we have

$$\int_{\Upsilon+\Phi-s}^{s}mdy\leq\int_{\Upsilon+\Phi-s}^{s}\Omega''(y)dy\leq\int_{\Upsilon+\Phi-s}^{s}Mdy,$$

which gives

$$m(2s-\Upsilon-\Phi)\leq\Omega'(s)-\Omega'(\Upsilon+\Phi-s)\leq M(2s-\Upsilon-\Phi).$$

By equality (2.5), we can write

$$\begin{aligned} & m\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}(2s-\Upsilon-\Phi) \\ \leq & \int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\left[\Omega'(s)-\Omega'(\Upsilon+\Phi-s)\right]ds \\ \leq & M\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}(2s-\Upsilon-\Phi). \end{aligned}$$

That is,

$$\begin{aligned} & m\left(\frac{\Upsilon+\Phi}{2}-\delta\right)^2 \\ \leq & \Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)-2\Omega\left(\frac{\Upsilon+\Phi}{2}\right)\leq M\left(\frac{\Upsilon+\Phi}{2}-\delta\right)^2. \end{aligned} \tag{2.7}$$

Multiplying the inequality (2.7) by

$$\frac{\tilde{u}H_\psi(\delta)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}$$

and then integrating w.r.t δ on the interval $\left[\Upsilon,\frac{\Upsilon+\Phi}{2}\right]$, we get

$$\begin{aligned} & \frac{m\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)\left(\frac{\Upsilon+\Phi}{2}-\delta\right)^2d\delta \\ \leq & \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \times\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)-2\Omega(\frac{\Upsilon+\Phi}{2})\right]d\delta \\ \leq & \frac{M\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)\left(\frac{\Upsilon+\Phi}{2}-\delta\right)^2d\delta. \end{aligned}$$

With help of the equality (2.4), we obtain (2.2). □

Remark 2.2. By choosing $\psi(\delta)=\delta$ in Theorem 2.1, inequality (2.1) will become inequality (1.1).

Theorem 2.3. *Let $\Omega:[\Upsilon,\Phi]\to\mathbb{R}$ be a twice differentiable function such that $\exists$ real constants $(m$ and $M)$, so if $m\leq\Omega''\leq M$, then*

$$\begin{aligned} & \frac{m\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)(\Phi-\delta)(\delta-\Upsilon)d\delta \\ \leq & \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}-\frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I_{\Upsilon+;\psi}^{\tilde{u}}\check{F}(\Phi)+I_{\Phi-;\psi}^{\tilde{u}}\check{F}(\Upsilon)\right] \\ \leq & \frac{M\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)(\Phi-\delta)(\delta-\Upsilon)d\delta \end{aligned} \tag{2.8}$$

holds, where H is defined by as in (2.2).

Proof. By the equality (2.3), we have

$$\begin{aligned} & \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}-\frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I_{\Upsilon+;\psi}^{\tilde{u}}\check{F}(\Phi)+I_{\Phi-;\psi}^{\tilde{u}}\check{F}(\Upsilon)\right] \\ = & \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2} \\ & -\frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)\right]d\delta \\ = & \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \times\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_\psi(\delta)\left[\Omega(\Upsilon)+\Omega(\Phi)-(\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta))\right]d\delta. \end{aligned} \tag{2.9}$$

With help of the equalities

$$\Omega(\Upsilon)-\Omega(\delta)=-\int_{\Upsilon}^{\delta}\Omega'(s)ds$$

and

$$\Omega(\Phi)-\Omega(\Upsilon+\Phi-\delta)=\int_{\Upsilon+\Phi-\delta}^{\Phi}\Omega'(s)ds,$$

we get

$$\begin{aligned} &\Omega(\Upsilon)+\Omega(\Phi)-(\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)) \\ =\ & \int_{\Upsilon+\Phi-\delta}^{\Phi}\Omega'(s)ds-\int_{\Upsilon}^{\delta}\Omega'(s)ds \\ =\ & \int_{\Upsilon}^{\delta}\Omega'(\Upsilon+\Phi-s)ds-\int_{\Upsilon}^{\delta}\Omega'(s)ds \\ =\ & \int_{\Upsilon}^{\delta}\left[\Omega'(\Upsilon+\Phi-s)-\Omega'(s)\right]ds. \end{aligned} \tag{2.10}$$

We also have

$$\Omega'(\Upsilon+\Phi-s)-\Omega'(s)=\int_{s}^{\Upsilon+\Phi-s}\Omega''(y)dy. \tag{2.11}$$

By (2.11) and condition $(m\leq\Omega''\leq M)$, we get

$$\begin{aligned} & m\left(\Upsilon+\Phi-2s\right) \\ \leq\ & \Omega'(s)-\Omega'(\Upsilon+\Phi-s)\leq M\left(\Upsilon+\Phi-2s\right). \end{aligned} \tag{2.12}$$

By using (2.10) and (2.12), we have

$$\begin{aligned} & \int_{\Upsilon}^{\delta}m(\Upsilon+\Phi-2s)ds \\ \leq\ & \int_{\Upsilon}^{\delta}\left[\Omega'(s)-\Omega'(\Upsilon+\Phi-s)\right]ds\leq\int_{\Upsilon}^{\delta}M(\Upsilon+\Phi-2s)ds \end{aligned}$$

i.e.

$$\begin{aligned} & m(\Phi-\delta)(\delta-\Upsilon) \\ \leq\ & \Omega(\Upsilon)+\Omega(\Phi)-(\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)) \\ \leq\ & M(\Phi-\delta)(\delta-\Upsilon). \end{aligned} \tag{2.13}$$

Multiplying the inequality (2.13) by $\frac{\tilde{u}H_{\psi}(\delta)}{4[\psi(\Phi)-\psi(\Upsilon)]^{\tilde{u}}}$ and then integrating w.r.t δ on the interval $\left[\Upsilon,\frac{\Upsilon+\Phi}{2}\right]$, we can write

$$\begin{aligned} & \frac{m\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_{\psi}(\delta)(\Phi-\delta)(\delta-\Upsilon)d\delta \\ \leq\ & \frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}-\frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\breve{F}(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\breve{F}(\Upsilon)\right] \\ \leq\ & \frac{M\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_{\psi}(\delta)(\Phi-\delta)(\delta-\Upsilon)d\delta. \end{aligned}$$

This completes the proof. □

Remark 2.4. Putting $\psi(\delta)=\delta$ in theorem 2.3, shows that (2.8) is the generalized version of (1.2).

Now we will investigate the proof of Theorem 1.8 by using different conditions.

Theorem 2.5. *Let $\Omega:[\Upsilon,\Phi]\to\mathbb{R}$ be a positive and differentiable mapping and $\Omega\in L_1[\Upsilon,\Phi]$. If $\Omega'(\Upsilon+\Phi-\vartheta)\geq\Omega'(\vartheta)$ for all $\vartheta\in\left[\Upsilon,\frac{\Upsilon+\Phi}{2}\right]$, then*

$$\begin{aligned} & \Omega\left(\frac{\Upsilon+\Phi}{2}\right) \\ \leq\ & \frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\Omega(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\Omega(\Upsilon)\right] \\ \leq\ & \frac{\Omega\left(\Upsilon\right)+\Omega\left(\Phi\right)}{2} \end{aligned} \tag{2.14}$$

holds for fractional integrals.

Proof. With help of the equalities (2.4) and (2.5), we obtain

$$\begin{aligned} & \frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\breve{F}(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\breve{F}(\Upsilon)\right]-\Omega\left(\frac{\Upsilon+\Phi}{2}\right) \\ =\ & \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\ & \times\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}}H_{\psi}(\delta)\left[\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta)-2\Omega\left(\frac{\Upsilon+\Phi}{2}\right)\right]d\delta \end{aligned}$$

$$\begin{aligned}
&= \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}} H_{\psi}(\delta) \\
&\quad \times \left[\int_{\frac{\Upsilon+\Phi}{2}}^{\Upsilon+\Phi-\delta}\left[\Omega'(s)-\Omega'(\Upsilon+\Phi-s)\right]ds\right]d\delta \\
&= \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}} H_{\psi}(\delta) \\
&\quad \times \left[\int_{\delta}^{\frac{\Upsilon+\Phi}{2}}\left[\Omega'(\Upsilon+\Phi-u)-\Omega'(u)\right]ds\right]d\delta \\
&\geq 0
\end{aligned}$$

which shows the first inequality of (2.14).

Similarly, by using (2.9) and (2.10), we obtain

$$\begin{aligned}
&\frac{\Omega(\Upsilon)+\Omega(\Phi)}{2}-\frac{\Gamma(\tilde{u}+1)}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\left[I^{\tilde{u}}_{\Upsilon+;\psi}\check{F}(\Phi)+I^{\tilde{u}}_{\Phi-;\psi}\check{F}(\Upsilon)\right] \\
&= \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}} \\
&\quad \times \int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}} H_{\psi}(\delta)\left[\Omega(\Upsilon)+\Omega(\Phi)-(\Omega(\delta)+\Omega(\Upsilon+\Phi-\delta))\right]d\delta \\
&= \frac{\tilde{u}}{4\left[\psi(\Phi)-\psi(\Upsilon)\right]^{\tilde{u}}}\int_{\Upsilon}^{\frac{\Upsilon+\Phi}{2}} H_{\psi}(\delta)\left[\int_{\Upsilon}^{\delta}\left[\Omega'(\Upsilon+\Phi-s)-\Omega'(s)\right]ds\right]d\delta \\
&\geq 0.
\end{aligned}$$

□

3 Conclusion

In the present paper, we offered new Hermite-Hadamard inequality (H-H-I) involving ψ–Hilfer fractional integral operator with help of bounded and twice differentiable function. In addition, the ψ–Hilfer fractional Hermite-Hadamard inequality (H-H-I) was investigated under different types of conditions. Researchers who are working in the field of fractional calculus can use given condition:

$$m \leq \Omega''(\delta) \leq M, \ for \ all \ \delta \in [\Upsilon, \Phi]$$

without using convexity, and they can find sharp bounds utilizing

$$\Omega'(\Upsilon+\Phi-\delta)-\Omega'(\delta) \geq 0, \ \delta \in \left[\Upsilon, \frac{\Upsilon+\Phi}{2}\right].$$

In the future work, by using above new idea several researchers can design new inequalities using different type of convexities, and other fractional operators.

Conflicts of Interest: *NO.*

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