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An obstacle problem for nonlocal elliptic hemivariational inequalities

M.Idrissi, M.Khouakhi, M.Masmodi and C.Yazough

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Corresponding Author: Mustapha Idrissi

Abstract *In this paper, we investigated the existence of a weak solution to a unilateral obstacle problem for a nonlocal hemivariational inequalities governed by a variable-order fractional Laplace operator. The basic tools used in our paper are the surjectivity result for pseudomonotone mappings, and the Moreau-Yosida approximation.*

1 Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. We consider the following unilateral obstacle problem

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \partial K(u) + \partial_c\Phi(u) \ni f & \text{in } \Omega, \\ u(x) \leq \psi(x) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^{\complement} := \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $s(.) : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ is a continuous function with $N > 2s(x,y)$ for all $(x,y) \in \Omega \times \Omega$. The operator $(-\Delta)^{s(\cdot)}$ is the variable-order fractional Laplace operator defined by

$$(-\Delta)^{s(\cdot)}u(x) = 2P.V \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s(x,y)}} dy \quad \text{for all } x \in \mathbb{R}^N,$$

along any $u \in C_0^\infty(\Omega)$, where P.V denotes the Cauchy principle value. Note that, if $s(.) \equiv$ constant, then $(-\Delta)^{s(\cdot)}$ reduces to the usual fractional Laplace operator. $\partial K(\cdot)$ stands for the generalized Clarke subdifferential operator of a locally Lipschitz functional J, $\partial_c\Phi(.)$ denotes the convex subdifferential operator of a convex functional Φ.

In recent years, great deal of attention has been devoted to the study of nonlocal hemivariational inequalities governed by a variable-order fractional Laplace operator which is a generalization of variational inequalities based on the notion of the Clarke subgradient defined for locally Lipschitz functions.

Nonlocal operators, such as the fractional laplacian $(-\Delta)^s$, appear in dynamics population, game theory and continuum mechanics (for more details see [1], [2], [3], [4], [5], [6], [7]).

In [8], the author considered the fractional Laplace operator $(-\Delta)^s$ with $s \in (0,1)$ and using the surjectivity for pseudomonotone and coercive operators to show the existence of at least one solution for the nonlocal elliptic hemivariational inequalities with $\Phi \equiv 0$. Unfortunately, the main results of Liu and Tan [8] cannot be applied directly to problems which are controlled by a convex subdifferential operator (Φ is not null- function). To overcome this difficulty, we used the Moreau-Yosida approximation method. In this paper, we consider a unilateral obstacle problem

with a nonlocal hemivariational inequalities governed by a variable-order fractional Laplace operator $(-\Delta)^{s(\cdot)}$, *and we establish the existence of at least one weak solution for the problem* (1.1).
This paper is organized as follows. In Section 2 we give some notations and preliminaries. In Section 3 we present the main results of this paper.

2 Notations and Preliminaries

Let Ω *be a nonempty open subset of* $\mathbb{R}^N$ *and let* $s(\cdot):\mathbb{R}^N\times\mathbb{R}^N\to(0,1)$ *be a measurable function satisfying*
(H_1) $\quad 0<s^-:=\min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y)\leq s^+:=\max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y)<1.$
(H_2) $\quad s(\cdot)$ *is symmetric, that is,* $s(x,y)=s(y,x)$ *for all* $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$.

Remark 2.1. In general, if $s(.):\mathbb{R}^N\times\mathbb{R}^N\to(0,1)$ is a measurable function, then the variable-order fractional Laplacian can be defined as follows

$$\int_{\mathbb{R}^N} v(x)(-\Delta)^{s(\cdot)}u(x)dx=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s(x,y)}}dydx,$$

for any $u,v\in C_0^\infty(\Omega)$.

Let us introduce the space X_0 *defined by*

$$X_0=\left\{u\in L^2(\mathbb{R}^N):\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}dxdy\right)^{1/2}<\infty\quad and\quad u=0\quad for\ a.e\quad x\in\Omega^c\right\}$$

In the following, we collect some important properties of the function space X_0.

Lemma 2.2. *([9]) Let* Ω *be a nonempty, bounded, open subset of* $\mathbb{R}^N$ *with Lipschitz boundary and let* $s(\cdot):\mathbb{R}^N\times\mathbb{R}^N\to(0,1)$ *be a continuous function satisfying* (H_1). *There exist two constants* $0<s_0<s_1<1$ *such that* $s_0\leq s(x,y)\leq s_1$ *for all* $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $N>2s_0$ *and* $2^*_{s_0}=\frac{2N}{N-2s_0}$. *Then, we have*

(i) X_0 *is a Hilbert space with the inner product*

$$<u,v>_{X_0}:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s(x,y)}}dydx$$

for all $u,v\in X_0$ *and the associated norm denoted by* $\|.\|_{X_0}$.

(ii) If $p\in\left[1,2^*_{s_0}\right]$, *then there exists a positive constant* $c_p=C\left(N,p,s^+,s^-\right)>0$ *such that*

$$\|u\|_{L^p(\Omega)}\leq c_p\|u\|_{X_0}\quad \text{for all } u\in X_0$$

.

(iii) The embedding from X_0 *to* $L^p\left(\mathbb{R}^N\right)$ *is compact for any* $p\in\left[1,2^*_{s_0}\right)$.

Note that $X_0\subset L^2(\Omega)\subset X_0^*$ *and* $2<2^*_{s_0}=\frac{2N}{N-2s_0}$, *where* X_0^* *is the dual space of* X_0 *and using Lemma* (2.2) *, we can see that the embedding from* X_0 *to* $L^2(\Omega)$ *is compact.*

Proposition 2.3. *(See [10], Proposition 3.8) Let* X *and* Y *be topological spaces and* $A:X\to 2^Y$ *be a set-valued mapping. Then* A *is upper semicontinuous, if and only if, for each closed set* $D\subset Y$, *the set* $A^-(D)=\{x\in X|F(x)\cap D\neq\emptyset\}$ *is closed in* X.

Definition 2.4. Let $K:X\to\mathbb{R}$ be a locally Lipschitz function and let $u,v\in X$. The generalized directional derivative $K^0(u;v)$ of K at the point u in the direction v is defined by

$$K^0(u;v):=\lim_{w\to u,t\downarrow 0}\frac{K(w+tv)-K(w)}{t}.$$

The generalized gradient $\partial K:X\to 2^{X^*}$ of $K:X\to\mathbb{R}$ is defined by

$$\partial K(u):=\left\{\mu\in X^*|K^0(u;v)\geq\langle\mu,v\rangle_{X^*\times X}\text{ for all } v\in X\right\}\text{ for all } u\in X.$$

Proposition 2.5. *(See [10], Proposition 3.23) Let $K : X \to \mathbb{R}$ be a locally Lipschitz function of rank $L_u > 0$ at $u \in X$. Then, we have*

(a) The function $v \mapsto K^0(u;v)$ is positively homogeneous, subadditive, and satisfies

$$\left|K^0(u;v)\right| \leq L_u\|v\|_X \text{ for all } v \in X.$$

(b) $(u,v) \mapsto K^0(u;v)$ is upper semicontinuous.

(c) For each $u \in X$, $\partial K(u)$ is a nonempty, convex and weak compact subset of X^* with $\|\mu\|_{X^*} \leq L_u$ for all $\mu \in \partial K(u)$.*

(d) $K^0(u;v) = \max\{\langle \mu, v\rangle_{X^* \times X} | \mu \in \partial K(u)\}$ for all $v \in X$.

(e) The multivalued function $X \ni u \mapsto \partial K(u) \subset X^*$ is upper semicontinuous from X into $w^* - X^*$.

Definition 2.6. Let X be a real reflexive Banach space. The operator $A : X \to 2^{X^*}$ is called pseudomonotone if the following conditions hold

(i) The set $A(u)$ is nonempty, bounded, closed and convex for all $u \in X$.

(ii) A is upper semicontinuous from each finite-dimensional subspace of X to the weak topology on X^*.

(iii) If $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ in X and if $u_n^* \in A(u_n)$ is such that

$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X^* \times X} \leq 0,$$

then, to each element $v \in X$, exists $u^*(v) \in A(u)$ with

$$\langle u^*(v), u - v\rangle_{X^* \times X} \leq \liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X^* \times X}.$$

Theorem 2.7. *([10]) Let X be a reflexive Banach space and $A : X \to 2^{X^*}$ be pseudomonotone and coercive. Then A is surjective, i.e., for every $u^* \in X^*$, there exists $u \in X$ such that $u^* \in A(u)$.*

Lemma 2.8. *([11]) Let X be a Banach space and $\varphi : X \to \overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function. Hence, for $\varepsilon > 0$, the Moreau-Yosida approximation $\varphi_\varepsilon : X \to \mathbb{R}$ of φ defined by*

$$\varphi_\varepsilon(u) = \inf_{v\in X}\left(\frac{\|u-v\|_X^2}{2\varepsilon} + \varphi(v)\right)$$

for all $u \in X$, satisfies

(i) φ_ε is convex, lower semicontinuous, and Gâteaux differentiable.

(ii) The differential operator $\varphi_\varepsilon' : X \to X^*$ is bounded, monotone, and demicontinuous.

(iii) If $u_\varepsilon \to u$ weakly in X, then,

$$\limsup_{\varepsilon\to 0} \varphi_\varepsilon(v) \leq \varphi(v) \quad \text{for all } v \in X,$$
$$\varphi(u) \leq \liminf_{\varepsilon\to 0} \varphi_\varepsilon(u_\varepsilon)$$

as $\varepsilon \to 0$.

3 Existence result

We impose the following assumptions for the data of problem (1.1).
(A_1) : $k : \Omega \times \mathbb{R} \to \mathbb{R}$ *is such that*

(i) $x \mapsto k(x,r)$ *is measurable on* Ω *for all* $r \in \mathbb{R}$ *where* $x \mapsto k(x,0)$ *belongs to* $L^1(\Omega)$.

(ii) $r \mapsto k(x,r)$ *is locally Lipschitzienne function for a.e.* $x \in \Omega$.

(iii) There exist $c > 0$, $p \geq 1$, *and* $b \in L^{p/(p-1)}(\Omega)$ *such that*

$$|\mu| \leq b(x) + c|r|^{p-1} \quad \text{for all } \mu \in \partial k(x,r) \text{ and a.e. } x \in \Omega.$$

(A_2) : $\Phi : X_0 \to \overline{\mathbb{R}}$ *is a proper, convex, and lower semicontinuous.*
(A_3) : $f \in L^{p'}(\Omega)$.
Let us define the function $K : L^p(\Omega) \to \mathbb{R}$ *where*

$$K(u) = \int_\Omega k(x,u(x))\mathrm{d}x \text{ for all } u \in L^p(\Omega).$$

Next lemma is a consequence of theorem 3.47 *of Migórski et al. [10].*

Lemma 3.1. *If we suppose* (A_1), *then we have*

(i) $K : L^p(\Omega) \to \mathbb{R}$ *is locally Lipschitz continuous.*

(ii) We have the inequality

$$K^0(u;v) \leq \int_\Omega k^0(x,u(x);v(x))\mathrm{d}x$$

for all $u, v \in L^p(\Omega)$.

(iii) There exists c_1 *a positive constant such that*

$$\|\mu\|_{L^{p'}(\Omega)} \leq c_1 \left(1 + \|u\|^{p-1}_{L^p(\Omega)}\right) \quad \text{for all } \mu \in \partial \left(K|_{L^p(\Omega)}\right)(u) \text{ and } u \in L^p(\Omega),$$

where $1/p + 1/p' = 1$.

We define the following subset C *of* X_0

$$C = \{u \in X_0 \quad | \quad u(x) \leq \psi(x) \quad \text{for a.e} \quad x \in \Omega\},$$

where

$$\psi : \Omega \to [0, +\infty] \quad \text{is a function.}$$

Remark 3.2. It is obvious that the set C is a nonempty, closed and convex subset of X_0 and $0 \in C$.

Definition 3.3. We say that $u \in C$ is a weak solution of problem (1.1), if there exist $\mu \in \partial K(u)$ and $\eta \in \partial_c \Phi(u)$ as follows

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s(x,y)}} dydx + \int_{\mathbb{R}^N} (\mu(x) + \eta(x))\, v(x)dx \quad = \int_{\mathbb{R}^N} f(x)v(x)dx$$

for all $v \in C$.

Theorem 3.4. *We suppose that* (A_1), (A_2) *and* (A_3) *hold with* $1 \leq p < 2$ *or* $p = 2$ *such that* $2c_1(c_p)^p < 1$. *Then, the problem* (1.1) *has at least one weak solution.*

Proof. First, we introduce the auxiliary problem

$$(-\Delta)^{s(.)}u_\varepsilon + \partial K(u_\varepsilon) + \Phi'_\varepsilon(u_\varepsilon) \ni f, \tag{3.1}$$

with $u_\varepsilon \in C$ and $\varepsilon > 0$.
Where $\Phi_\varepsilon : X_0 \to \mathbb{R}$ is the Moreau-Yosida approximation of Φ defined by

$$\Phi_\varepsilon(u) := \inf_{v \in X_0} \left(\frac{\|u - v\|^2_{X_0}}{2\varepsilon} + \Phi(v) \right)$$

for all $u \in X_0$.
The proof of the existence of a solution to (3.1) is divided into three steps.
Step1. $(-\Delta)^{s(.)} : X_0 \mapsto X_0^*$ is a continuous, bounded and strongly monotone operator.
We have

$$\begin{aligned} < (-\Delta)^{s(.)}u, v >_{X_0} &= \int_{\mathbb{R}^N} v(x)(-\Delta)^{s(.)}u(x)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s(x,y)}}dydx \\ &= < u, v >_{X_0} \end{aligned}$$

for all $u, v \in X_0$.
Therefore, $(-\Delta)^{s(.)}$ is linear, bounded and

$$\|(-\Delta)^{s(.)}u\|_{X_0^*} \leq \|u\|_{X_0} \quad \text{for all} \quad u \in X_0,$$

then, $(-\Delta)^{s(.)}$ is linear and continuous. On the other hand, we have the following equality

$$< (-\Delta)^{s(.)}u - (-\Delta)^{s(.)}v, u - v >_{X_0} = \|u - v\|^2_{X_0} \quad \text{for all} \quad u, v \in X_0,$$

which indicates that $(-\Delta)^{s(.)}$ is strongly monotone with constant $m = 1$.

Step2. $X_0 \ni u \mapsto (-\Delta)^{s(.)}u + \partial K(u) \subset X_0^*$ is bounded and pseudomonotone.
Using proposition (2.5), we have that $\partial K(u)$ is nonempty, convex, weak-compact subset of X_0^*.
Then, for each $u \in X_0$, $(-\Delta)^{s(.)}u + \partial K(u)$ is nonempty, bounded, closed and convex subset of X_0^*.
By Proposition(2.3), it is sufficient to verify that the set $\left((-\Delta)^{s(.)} + \partial K\right)^- (D)$ is closed in X_0, for any weakly closed subset D in X_0^*.
Let $\{u_n\} \subset \left((-\Delta)^{s(.)} + \partial K\right)^- (D)$ be a sequence such that

$$u_n \to u \quad in \quad X_0 \quad as \quad n \to \infty, \quad \text{for some} \quad u \in X_0. \tag{3.2}$$

Therefore, for each $n \in \mathbb{N}$ there exists $\mu_n \in \partial K(u_n)$ satisfying

$$u_n^* = (-\Delta)^{s(.)}u_n + \mu_n \in \Big((-\Delta)^{s(.)}(u_n) + \partial K(u_n)\Big) \cap D.$$

The continuity of $(-\Delta)^{s(.)}$ proves that $(-\Delta)^{s(.)}(u_n) \to (-\Delta)^{s(.)}(u)$ in X_0^*, as $n \to \infty$. Furthermore, by Lemma (3.1) (iii) and the convergence (3.2), we get that the sequence $\{\mu_n\}$ is bounded in X_0^*, then $\mu_n \to \mu$ in X_0^* for a subsequence, as $n \to \infty$, with some $\mu \in X_0^*$. By Proposition (2.5), we have that ∂K is upper semicontinuous from X_0 to $w - X_0^*$ and has bounded, convex, closed values, hence, it has a closed graph in $X_0 \times w - X_0^*$ (see cf. Kamenskii et al.[12], Theorem 1.1.4). But, due to the weak closedness of D, we obtain that $(-\Delta)^{s(.)}(u) + \mu \in D$ and $\mu \in \partial K(u)$, which implies that $u \in ((-\Delta)^{s(.)} + \partial K)^-(D)$. Therefore, $(-\Delta)^{s(.)} + \partial K$ is upper semicontinuous from X_0 to X_0^*.
Now, we will prove that $(-\Delta)^{s(.)} + \partial K$ is pseudomonotone. Let $\{u_n\}$ and $\{u_n^*\}$ be sequences such that

$$u_n \rightharpoonup u \text{ in } X_0, \tag{3.3}$$

$$u_n^* \in (-\Delta)^{s(.)}(u_n) + \partial K(u_n) \qquad \text{with} \qquad \limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle_{X_0} \leq 0. \tag{3.4}$$

It is sufficient to prove that for each $v \in X_0$, we can find $u^*(v) \in (-\Delta)^{s(.)}(u) + \partial K(u)$ satisfying

$$\liminf_{n\to\infty} \langle u_n^*, u_n - v\rangle_{X_0} \geq \langle u^*(v), u - v\rangle_{X_0}. \tag{3.5}$$

Using (3.4), there exists a sequence $\{\mu_n\} \subset X_0^*$ such that for each $n \in \mathbb{N}$, $\mu_n \in \partial K(u_n)$ and

$$u_n^* = (-\Delta)^{s(.)}(u_n) + \mu_n.$$

Combining with the inequality in (3.4) we have

$$\limsup_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}u_n,u_n-u\right\rangle_{X_0}+\liminf_{n\to\infty}\langle\mu_n,u_n-u\rangle_{X_0}\leq 0. \tag{3.6}$$

By (3.3) and the compactness of the embedding of X_0 into $L^p(\Omega)$ yields that

$$u_n\to u \text{ in } L^p(\Omega) \text{ as } n\to\infty.$$

And using Theorem 2.2 of Chang [13], we have

$$\partial\left(K|_{X_0}\right)(u)\subset\partial\left(K|_{L^p(\Omega)}\right)(u) \text{ for all } u\in X_0,$$

consequently,

$$\langle\mu_n,u_n-u\rangle_{X_0}=\langle\mu_n,u_n-u\rangle_{L^p(\Omega)}. \tag{3.7}$$

Therefore, by Lemma (3.1)(iii) and the boundedness of the sequence $\{u_n\}$ in X_0 we get that the sequence $\{\mu_n\}$ is contained in $L^{p'}(\Omega)$. Hence, passing to the limit in (3.7) as $n\to\infty$ we have

$$\lim_{n\to\infty}\langle\mu_n,u_n-u\rangle_{X_0}=\lim_{n\to\infty}\langle\mu_n,u_n-u\rangle_{L^p(\Omega)}=0.$$

Then, by (3.6) we have

$$\limsup_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}u_n,u_n-u\right\rangle_{X_0}=\limsup_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}u_n,u_n-u\right\rangle_{X_0}+\liminf_{n\to\infty}\langle\mu_n,u_n-u\rangle_{X_0}\leq 0.$$

The monotonicity of $(-\Delta)^{s(\cdot)}$ yields that

$$\begin{aligned}0&\geq\limsup_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}u_n-(-\Delta)^{s(\cdot)}u+(-\Delta)^{s(\cdot)}u,u_n-u\right\rangle_{X_0}\\&\geq\liminf_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}u,u_n-u\right\rangle_{X_0}+\limsup_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}u_n-(-\Delta)^{s(\cdot)}u,u_n-u\right\rangle_{X_0}\\&\geq\limsup_{n\to\infty}\|u_n-u\|^2_{X_0}.\end{aligned}$$

Then, $u_n\to u$ in X_0, as $n\to\infty$. The reflexivity of X_0^* and boundedness of $\{\mu_n\}\subset X_0^*$ allows us to summarize that

$$\mu_n\to\mu \text{ in } X_0^* \text{ for some } \mu\in X_0^*.$$

As before, it is easy to see that $\mu\in\partial K(u)$ (see, e.g., Kamenskii et al.[12], Theorem 1.1.4). Therefore,

$$\liminf_{n\to\infty}\langle u_n^*,u_n-v\rangle_{X_0}=\liminf_{n\to\infty}\left\langle(-\Delta)^{s(\cdot)}(u_n)+\mu_n,u_n-v\right\rangle_{X_0}=\langle(-\Delta)^{s(\cdot)}(u)+\mu,u-v\rangle_{X_0},$$

and it is clear that (3.5) holds with $u^*=(-\Delta)^{s(\cdot)}(u)+\mu\in(-\Delta)^{s(\cdot)}(u)+\partial K(u)$. Then, we conclude that $(-\Delta)^{s(\cdot)}+\partial K$ is pseudomonotone.

Step3. $X_0\ni u\mapsto(-\Delta)^{s(\cdot)}u+\partial K(u)+\Phi'_\varepsilon(u)\subset X_0^*$ is pseudomonotone and coercive. The operator $\Phi'_\varepsilon:X_0\to X_0^*$ is bounded, demicontinuous, monotone. And using theorem 3.69 (see [10]), we get that Φ'_ε is pseudomonotone. Then, we conclude that $X_0\ni u\mapsto(-\Delta)^{s(\cdot)}u+\partial K(u)+\Phi'_\varepsilon(u)\subset X_0^*$ is pseudomonotone.
For any $\mu\in\partial K(u)$, we have

$$\begin{aligned}\left\langle(-\Delta)^{s(\cdot)}u+\mu+\Phi'_\varepsilon(u),u\right\rangle_{X_0}&\geq\|u\|^2_{X_0}-\|\mu\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)}+\langle\Phi'_\varepsilon(0),u\rangle_{X_0}+\langle\Phi'_\varepsilon(u)-\Phi'_\varepsilon(0),u\rangle_{X_0}\\&\geq\|u\|^2_{X_0}-c_1\|u\|_{L^p(\Omega)}-c_1\|u\|^p_{L^{p'}(\Omega)}-\|\Phi'_\varepsilon(0)\|_{X_0^*}\|u\|_{X_0}\\&\geq\|u\|^2_{X_0}-c_1c_p\|u\|_{X_0}-c_1(c_p)^p\|u\|^p_{X_0}-\|\Phi'_\varepsilon(0)\|_{X_0^*}\|u\|_{X_0}\end{aligned}$$

for all $u\in X_0$. This shows that $X_0\ni u\mapsto(-\Delta)^{s(\cdot)}u+\partial K(u)+\Phi'_\varepsilon(u)\subset X_0^*$ is coercive where $1\leq p<2$ or $p=2$ with $2c_1(c_p)^p<1$. Therefore, we apply Theorem(2.7), there exists $u_\varepsilon\in X_0$

such that (3.1) holds.
Now, let us prove that the sequence $\{u_\varepsilon\}_\varepsilon$ is bounded in X_0, we suppose here that $1 \leq p < 2$ (the proof of the case when $p = 2$ is similar).
Let $v \in C$, multiplying (3.1) by $v - u_\varepsilon$, then we have

$$\left\langle (-\Delta)^{s(.)} u_\varepsilon + \mu_\varepsilon + \Phi'_\varepsilon (u_\varepsilon), v - u_\varepsilon \right\rangle_{X_0} = \langle f, v - u_\varepsilon \rangle_{X_0},$$

where $\mu_\varepsilon \in \partial K(u_\varepsilon)$. By convexity of Φ_ε we have

$$\left\langle (-\Delta)^{s(.)} u_\varepsilon + \mu_\varepsilon, v - u_\varepsilon \right\rangle_{X_0} + \Phi_\varepsilon(v) - \Phi_\varepsilon(u_\varepsilon) \geq \langle f, v - u_\varepsilon \rangle_{X_0}. \tag{3.8}$$

Then,

$$\left\langle (-\Delta)^{s(.)} u_\varepsilon + \mu_\varepsilon, u_\varepsilon \right\rangle_{X_0} + \Phi_\varepsilon(u_\varepsilon) \leq \langle f, u_\varepsilon - v \rangle_{X_0} + \Phi_\varepsilon(v) + \left\langle (-\Delta)^{s(.)} u_\varepsilon + \mu_\varepsilon, v \right\rangle_{X_0}.$$

Since Φ_ε is a proper, convex, and lower semicontinuous functional, it is bounded from below by an affine function. Therefore, there exist two constants $\alpha, \beta \in \mathbb{R}$ such that

$$\Phi_\varepsilon(u) \geq \alpha \|u\|_{X_0} + \beta$$

for all $u \in X_0$. As a consequence, we obtain

$$\begin{aligned}
&\|u_\varepsilon\|^2_{X_0} - c_1 \|u_\varepsilon\|_{L^p(\Omega)} - c_1 \|u_\varepsilon\|^p_{L^p(\Omega)} + \alpha \|u_\varepsilon\|_{X_0} + \beta \\
&\leq \|f\|_{X_0^*} \left(\|u_\varepsilon\|_{X_0} + \|v\|_{X_0} \right) + \left\| (-\Delta)^{s(.)} u_\varepsilon \right\|_{X_0^*} \|v\|_{X_0} + \|\mu_\varepsilon\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)} + \Phi_\varepsilon(v).
\end{aligned}$$

Using Young's inequality, with $\delta > 0$ we have

$$\begin{aligned}
c_1 (c_p)^p \|u\|^p_{X_0} &\leq \delta \|u\|^2_{X_0} + \frac{2-p}{2} \left[\left(\sqrt{\frac{p}{2\delta}} \right)^p c_1 (c_p)^p \right]^{2/(2-p)} \\
(c_1 c_p + |\alpha|) \|u\|_{X_0} &\leq \delta \|u\|^2_{X_0} + \frac{1}{4\delta} (c_1 c_p + |\alpha|)^2 \\
\|f\|_{X_0^*} \|u\|_{X_0} &\leq \delta \|u\|^2_{X_0} + \frac{1}{4\delta} \|f\|^2_{X_0^*} \\
c_1 (c_p)^p \|u\|^{p-1}_{X_0} \|v\|_{X_0} &\leq \frac{3-p}{2} \left[\left(\sqrt{\frac{p-1}{2\delta}} \right)^{p-1} c_1 (c_p)^p \|v\|_{X_0} \right]^{2/(3-p)} + \delta \|u\|^p_{X_0},
\end{aligned}$$

and by Lemma (2.2) we conclude

$$\frac{1}{2}(1 - 10\delta) \|u_\varepsilon\|^2_{X_0} \leq m_0 \left(1 + \|v\|^2_{X_0} + \|v\|^{2/(3-p)}_{X_0} \right) + \Phi_\varepsilon(v),$$

where $m_0 > 0$ is independent of ε. Choosing $\delta < \frac{1}{10}$ and $v \in dom\Phi$ (the effective domain), we have

$$\|u_\varepsilon\|_{X_0} \leq m_1,$$

where $m_1 > 0$ is independent of ε. Hence, $\{u_\varepsilon\}$ is bounded in X_0.
For a subsequence, if necessary, we may assume that $u_\varepsilon \to u$ weakly in X_0 with $u \in C$. Taking $v = u$ in (3.8), and passing to the limit, by (iii) in Lemma (2.8) we have

$$\limsup_{\varepsilon \to 0} \left\langle (-\Delta)^{s(.)} u_\varepsilon + \mu_\varepsilon, u_\varepsilon - u \right\rangle_{X_0} \leq \limsup_{\varepsilon \to 0} \Phi_\varepsilon(u) - \liminf_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon) \leq 0.$$

Using [13], Theorem 2.2, we have $\partial \left(K|_{X_0} \right)(u) \subset \partial \left(K|_{L^p(\Omega)} \right)(u)$ for all $u \in X_0$.
Then, $\langle \mu, v \rangle_{X_0} \leq \|\mu\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)}$ for all $\mu \in \partial K(u)$.

Hence,

$$\limsup_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon, u_\varepsilon - u\right\rangle_{X_0} - \liminf_{\substack{\varepsilon\to 0\\ \varepsilon\to 0}}\|\mu_\varepsilon\|_{L^{p'}(\Omega)}\|u_\varepsilon - u\|_{L^p(\Omega)}$$

$$\le \liminf_{\varepsilon\to 0}\left(\mu_\varepsilon, u_\varepsilon - u\right\rangle_{X_0} + \limsup_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon, u_\varepsilon - u\right\rangle_{X_0}$$

$$\le \limsup_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon + \mu_\varepsilon, u_\varepsilon - u\right\rangle_{X_0} \le 0.$$

Lemma (2.2) (iii), reveals that $u_\varepsilon \to u$ in $L^p(\Omega)$.
Then,

$$\limsup_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon, u_\varepsilon - u\right\rangle_{X_0} \le 0.$$

Combining the above inequality with the strong monotonicity of $(-\Delta)^{s(\cdot)}$, we get

$$0 \le \liminf_{\varepsilon\to 0}\|u - u_\varepsilon\|^2_{X_0} \le \limsup_{\varepsilon\to 0}\|u - u_\varepsilon\|^2_{X_0}$$

$$\le \limsup_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon - (-\Delta)^{s(\cdot)}u, u_\varepsilon - u\right\rangle_{X_0} \le 0.$$

Therefore, $u_\varepsilon \to u$ in X_0 as $\varepsilon \to 0$.
Similarly, we can show that $\{\mu_\varepsilon\}_\varepsilon \subset X_0^*$ is bounded, and we may assume that $\mu_\varepsilon \to \mu$ weakly in X_0^*. Note that the graph of $\partial\left(K|_{X_0}\right)$ is strongly-weakly upper semicontinuous, hence, $\mu \in \partial K(u)$.
Recall that $(-\Delta)^{s(\cdot)} : X_0 \to X_0^*$ is linear, bounded, and strongly monotone. So, it is pseudomonotone, which implies that

$$\liminf_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon, u_\varepsilon - v\right\rangle_{X_0} \ge \left\langle(-\Delta)^{s(\cdot)}u, u - v\right\rangle_{X_0},$$

for all $v \in X_0$. Passing to the limit of (3.8), we get

$$\begin{aligned}\langle\mu, v-u\rangle_{X_0} + \Phi(v) - \Phi(u) - \langle f, v-u\rangle_{X_0} &\ge \limsup_{\varepsilon\to 0}\langle\mu_\varepsilon, v - u_\varepsilon\rangle_{X_0} + \limsup_{\varepsilon\to 0}\Phi_\varepsilon(v) - \liminf_{\varepsilon\to 0}\Phi_\varepsilon(u_\varepsilon) - \liminf_{\varepsilon\to 0}\langle f, v - u_\varepsilon\rangle_{X_0}\\ &\ge \limsup_{\varepsilon\to 0}\left\langle(-\Delta)^{s(\cdot)}u_\varepsilon, u_\varepsilon - v\right\rangle_{X_0} \ge \liminf_{\varepsilon\to 0}\left((-\Delta)^{s(\cdot)}u_\varepsilon, u_\varepsilon - v\right\rangle_{X_0}\\ &\ge \left\langle(-\Delta)^{s(\cdot)}u, u - v\right\rangle_{X_0},\end{aligned}$$

for all $v \in X_0$. Then,

$$\left\langle(-\Delta)^{s(\cdot)}u + \mu, v - u\right\rangle_{X_0} + \Phi(v) - \Phi(u) \ge \langle f, v - u\rangle_{X_0}$$

with $\mu \in \partial K(u)$ for all $v \in C$, which means that there exists $\eta \in \partial_c\Phi(u)$ such that

$$(-\Delta)^{s(\cdot)}u + \mu + \eta \ni f \quad \text{in } X_0^*.$$

Then, (1.1) has at least one weak solution. □

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. *Author information*

M.Idrissi, Department of Mathematics (LAMA), Faculty of Sciences, Sidi Mohamed Ben Abdellah, 1796 Atlas, Fez, Morocco.
E-mail: mustapha.idrissi1@usmba.ac.ma

M.Khouakhi, Department of Mathematics (LAMA), Faculty of Sciences, Sidi Mohamed Ben Abdellah, 1796 Atlas, Fez, Morocco.
E-mail: moussa.khouakhi@gmail.com

M.Masmodi, Department of Mathematics (LAGA), Faculty of Sciences, Ibn Tofail University, B.P. 133, Kenitra, Morocco.
E-mail: mohamed.masmodi@uit.ac.ma

C.Yazough, Department of Mathematics (LAMA), Faculty of Sciences, Sidi Mohamed Ben Abdellah, 1796 Atlas, Fez, Morocco.
E-mail: chihabyazough@gmail.com

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