**Palestine Journal of Mathematics**

**Vol 14(1)(2025)** , 596–605 

# On Balancing and Lucas-balancing Sedenions

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B39; Secondary 11R52, 20G20.

Keywords and phrases: Balancing numbers, Lucas-balancing numbers, Sedenions.

*The authors would like to thank the reviewers and editor of "Palestine Journal of Mathematics" for their constructive comments and valuable suggestions that improved the quality of our paper.*

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**Abstract.** Here, we introduce a new recurrence sequence using balancing and Lucas-balancing numbers known as balancing and Lucas-balancing sedenions involving some interesting results. We find various types of generating functions and Binet formulas with some well-known identities for both balancing and Lucas-balancing sedenions. Additionally, we give some combinatorial properties for balancing and Lucas-balancing sedenions.

## 1 Introduction

The quaternions were first introduced in 1843 by William Rowan Hamilton. Quaternions form a 4-dimensional real vector space with a multiplicative operation. The quaternions have many applications in applied sciences such as physics, computer science, and Clifford algebras in mathematics. A quaternion with real coefficients is of the form $q = a + be_1 + ce_2 + de_3$, where $\{1, e_1, e_2, e_3\}$ is the quaternion basis satisfying

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.$$

In abstract algebra, the sedenions form a 16-dimensional non-commutative, non-associative, and non-alternative but power-associative algebra over the real numbers, obtained by the Cayley-Dickson construction. The well-known sedenion algebra plays a great role in mathematics, coding theory, physics, robotics, computer science, etc. In recent years, several authors have studied the quaternions, octonions, sedenions and their generalizations [1, 2, 3, 4, 5, 6, 8, 15, 20].
A sedenion is defined as follows

$$\mathcal{S} = \sum_{i=0}^{15} a_i e_i, \tag{1.1}$$

where $a_0,\ a_1,\ a_2,\ \ldots,\ a_{15} \in \mathbb{R}$ and $e_0 = 1,\ e_1,\ e_2,\ \ldots,\ e_{15}$, is the sedenion basis satisfying the multiplication table [2, 3].
Panda and Ray [14] introduced balancing numbers $n, r \in \mathbb{Z}^+$, as a solution of the equation

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r), \tag{1.2}$$

where $n$ is a balancing number with balancer $r$. For example 6, 35, 204, ... are balancing numbers with balancer 2, 14, 84, ..., respectively.
The nth balancing number $B_n$ is given by

$$B_n = 6B_{n-1} - B_{n-2},\ for\ \ n \geqslant 2, \tag{1.3}$$

with initial values $B_0 = 0$ and $B_1 = 1$. The recurrence relation for Lucas-balancing number is

$$C_n = 6C_{n-1} - C_{n-2},\ for\ n \geqslant 2, \tag{1.4}$$

with initial values $C_0 = 1$ and $C_1 = 3$. The characteristic equation for balancing number is

$$x^2 - 6x + 1 = 0, \tag{1.5}$$

with roots $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$. Behera et.al[14] established the generating function for balancing number is

$$G(x) = \frac{x}{1 - 6x + x^2} \tag{1.6}$$

and the Binet formula for balancing and Lucas-balancing numbers are given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \tag{1.7}$$

and

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}. \tag{1.8}$$

In [7]Horadam quaternions are important steps in the development of contemporary the Caley-Dickson algebra theory. Later, in [6] Halici gives Binet's formulas, generating functions, and some properties of Fibonacci and Lucas numbers. Patel and Ray [15] introduced two new classes of quaternions known as balaning and Lucas-balancing quaternions in 2021, and in [1] Asci and Aydinyuz present new kinds of sequences of quaternions called as Gaussian balancing and Gaussian cobalancing quaternions that are based on balancing and Lucas-balancing numbers. In addition, "Bi-periodic balancing quaternions" were studied by Sevgi and Tasci in [19], and for some related studies, see [11, 12, 16].
In this article, we introduced the balancing sedenions and Lucas-balancing sedenions with some interesting properties, generating functions, Binet formula, various identities, etc.

## 2 Balancing and Lucas-balancing Sedenions

In this section we define balancing and Lucas-balancing sedenions and calculate some properties of these sedenions.

**Definition 2.1.** We define the balancing and Lucas-balancing sedenions over the sedenion algebra $\mathbb{S}$. The nth balancing and Lucas- balancing sedenions are defined respectively as

$$SB_n = B_n e_0 + B_{n+1} e_1 + \ldots + B_{n+15} e_{15} = \sum_{s=0}^{15} B_{n+s} e_s, \tag{2.1}$$

and

$$SC_n = C_n e_0 + C_{n+1} e_1 + \ldots + C_{n+15} e_{15} = \sum_{s=0}^{15} C_{n+s} e_s, \tag{2.2}$$

where $e_0, e_1, e_2, \ldots, e_{15}$ are the standard basis vectors in $\mathbb{R}^{16}$.

**Proposition 2.2.** *The recurrence relations for balancing and Lucas-balancing sedenions are respectively*

$$SB_n = 6SB_{n-1} - SB_{n-2} \quad \textit{and} \quad SC_n = 6SC_{n-1} - SC_{n-2}, \; for\ n \geq 2,$$

*where* $SB_0 = \sum_{s=0}^{15} B_s e_s$, $SB_1 = \sum_{s=0}^{15} B_{1+s} e_s$ *and* $SC_0 = \sum_{s=0}^{15} C_s e_s$,
$SC_1 = \sum_{s=0}^{15} C_{1+s} e_s$.

*Proof.* Using the recurrence relation of $\{B_n\}_{n\geq 2}$, we have

$$\begin{aligned} SB_n &= \sum_{s=0}^{15} B_{n+s}e_s \\ &= \sum_{s=0}^{15}(6B_{n-1+s} - B_{n-2+s})e_s \\ &= 6\sum_{s=0}^{15} B_{n+s-1}e_s - \sum_{s=0}^{15} B_{n+s-2}e_s \\ &= 6SB_{n-1} - SB_{n-2}, \end{aligned}$$

which completes the proof. The proof is similar for Lucas-balancing sedenions. □

We can observe from the equations (2.1) and (2.2) that addition, subtraction, and multiplication of these sedenions can be obtained as follows:

$$SB_n \pm SC_n = \sum_{s=0}^{15}(B_s \pm C_s)e_s,$$

and

$$SB_n \times SC_n = S_{SB_n}S_{SC_n} + S_{SB_n}V_{SC_n} + V_{SB_n}S_{SC_n} - V_{SB_n}V_{SC_n} + V_{SB_n}V_{SC_n},$$

where $S_{SB_n}$, $S_{SC_n}$ are scalar part and $V_{SB_n}$, $V_{SC_n}$ are vector part of balancing and Lucas-balancing sedenions respectively.

**Definition 2.3.** The conjugates of $SB_n$ and $SC_n$ are respectively defined as

$$\overline{SB_n} = B_ne_0 - B_{n+1}e_1 - B_{n+2}e_2 - \ldots - B_{n+15}e_{15} = B_n - \sum_{s=1}^{15} B_{n+s}e_s$$

$$\text{and} \quad \overline{SC_n} = C_ne_0 - C_{n+1}e_1 - C_{n+2}e_2 - \ldots - C_{n+15}e_{15} = C_n - \sum_{s=1}^{15} C_{n+s}e_s,$$

and the norms of $SB_n$ and $SC_n$ are respectively defined as

$$N_{SB_n} = \overline{SB_n}SB_n = B_n^2 + B_{n+1}^2 + B_{n+2}^2 + \ldots + B_{n+15}^2 = \sum_{s=0}^{15} B_{n+s}^2$$

$$\text{and} \quad N_{SC_n} = \overline{SC_n}SC_n = C_n^2 + C_{n+1}^2 + C_{n+2}^2 + \ldots + C_{n+15}^2 = \sum_{s=0}^{15} C_{n+s}^2.$$

**Proposition 2.4.** *For all $n \geq 0$, we have*

*(i)* $SB_n + \overline{SB_n} = 2B_n$.

*(ii)* $SC_n + \overline{SC_n} = 2C_n$.

*(iii)* $SB_n^2 + SB_n\overline{SB_n} = 2B_nSB_n$.

*(iv)* $SC_n^2 + SC_n\overline{SC_n} = 2C_nSC_n$.

**Lemma 2.5.**

$$SC_n + \sqrt{8}SB_n = u\lambda_1^n \quad \textit{and} \quad SC_n - \sqrt{8}SB_n = v\lambda_2^n,$$

*where* $u = \sum_{s=0}^{15} \lambda_1^s e_s,\ v = \sum_{s=0}^{15} \lambda_2^s e_s$.

*Proof.* Using Binet formula for $C_n$, $B_n$ we have $C_n + \sqrt{8}B_n = \lambda_1^n$. Thus we have

$$\begin{aligned} SC_n + \sqrt{8}SB_n &= \sum_{s=0}^{15} C_{n+s}e_s + \sqrt{8}\sum_{s=0}^{15} B_{n+s}e_s \\ &= \sum_{s=0}^{15}(C_{n+s} + \sqrt{8}B_{n+s})e_s \\ &= \sum_{s=0}^{15} \lambda_1^{n+s}e_s \\ &= \lambda_1^n \sum_{s=0}^{15} \lambda_1^s e_s \\ &= u\lambda_1^n. \end{aligned}$$

Using the identity $C_n - \sqrt{8}B_n = \lambda_2^n$, we can easily obtain $SC_n - \sqrt{8}SB_n = v\lambda_2^n$. □

**Proposition 2.6.** *For integers $m,\ n \geqslant 0$, then we have*

$$\begin{aligned} SB_{m+n} &= B_m SC_n + C_m SB_n \\ \textit{and} \quad SC_{m+n} &= C_m SC_n + 8B_m SB_n. \end{aligned}$$

*Proof.* Using the result $B_{m+n} = B_m C_n + C_m B_n$, we have

$$\begin{aligned} SB_{m+n} &= \sum_{s=0}^{15} B_{m+n+s}e_s \\ &= \sum_{s=0}^{15}(B_m C_{n+s} + C_m B_{n+s})e_s \\ &= B_m \sum_{s=0}^{15} C_{n+s}e_s + C_m \sum_{s=0}^{15} B_{n+s}e_s \\ &= B_m SC_n + C_m SB_n. \end{aligned}$$

Similarly, using $C_{m+n} = C_m C_n + 8B_m B_n$, we can easily obtain the second result. This proves the result. □

**Proposition 2.7.** *For $n \geqslant 2$, we have*

1. $SB_n = 3SB_{n-1} + SC_{n-1}$.
2. $SC_n = 8SB_{n-1} + 3SC_{n-1}$.
3. $2SC_n = SB_{n+1} - SB_{n-1}$.

*Proof.* Using the identity $B_n = 3B_{n-1} + C_{n-1}$, we have

$$\begin{aligned} SB_n &= \sum_{s=0}^{15} B_{n+s}e_s \\ &= \sum_{s=0}^{15}(3B_{n-1+s} + C_{n-1+s})e_s \\ &= 3\sum_{s=0}^{15} B_{n-1+s}e_s + \sum_{s=0}^{15} C_{n-1+s}e_s \\ &= 3SB_{n-1} + SC_{n-1}. \end{aligned}$$

Using the result $C_n = 8B_{n-1} + 3C_{n-1}$, we have

$$\begin{aligned} SC_n &= \sum_{s=0}^{15} C_{n+s} e_s \\ &= \sum_{s=0}^{15} (8B_{n-1+s} + 3C_{n-1+s}) e_s \\ &= 8\sum_{s=0}^{15} B_{n-1+s} e_s + 3\sum_{s=0}^{15} C_{n-1+s} e_s \\ &= 8SB_{n-1} + 3SC_{n-1}. \end{aligned}$$

Again by employing $2C_n = B_{n+1} - B_{n-1}$, we have

$$\begin{aligned} 2SC_n &= 2\sum_{s=0}^{15} C_{n+s} e_s \\ &= \sum_{s=0}^{15} (B_{n+1+s} - B_{n-1+s}) e_s \\ &= SB_{n+1} - SB_{n-1}. \end{aligned}$$

Thus the required results. □

**Theorem 2.8.** *(Binet formula) For $n \geqslant 0$, we have*

$$SB_n = \frac{u\lambda_1^n - v\lambda_2^n}{\lambda_1 - \lambda_2}$$

*and*

$$SC_n = \frac{u\lambda_1^n + v\lambda_2^n}{2},$$

*where $u = \sum_{s=0}^{15} \lambda_1^s e_s$ and $v = \sum_{s=0}^{15} \lambda_2^s e_s$.*

**Theorem 2.9.** *(Generating function) The generating function of the balancing sedenions is given by*

$$G_{SB_n}(t) = \frac{SB_0 + t(SB_1 - 6SB_0)}{1 - 6t + t^2}.$$

*Proof.* Let

$$G_{SB_n}(t) = \sum_{n=0}^{\infty} SB_n t^n$$

be the generating function for $SB_n$.

$$\begin{aligned} G_{SB_n}(t) &= \sum_{n=0}^{\infty} SB_n t^n \\ &= SB_0 + SB_1 t + \sum_{n=2}^{\infty} SB_n t^n \\ &= SB_0 + SB_1 t + \sum_{n=2}^{\infty} [6SB_{n-1} - SB_{n-2}] t^n \\ &= SB_0 + SB_1 t + 6\sum_{n=2}^{\infty} SB_{n-1} t^n - \sum_{n=2}^{\infty} SB_{n-2} t^n \\ &= SB_0 + SB_1 t + 6t\sum_{n=1}^{\infty} SB_n t^n - t^2 \sum_{n=0}^{\infty} SB_n t^n \\ &= SB_0 + SB_1 t + 6t[G_{SB_n}(t) - SB_0] - t^2 G_{SB_n}(t), \end{aligned}$$

by making necessary arrangement, the generating function of balancing sedenions is found as follows:

$$G_{SB_n}(t) = \frac{SB_0 + t(SB_1 - 6SB_0)}{1 - 6t + t^2}.$$

□

**Theorem 2.10.** *(Ordinary even and odd indexed generating functions) The ordinary generating function of even and odd indexed for $SB_n$ are given by respectively,*

$$G_{SB_{2n}}(t) = \frac{SB_0 + t(6SB_1 - 35SB_0)}{1 + t^2 - 34t}$$

$$\text{and} \quad G_{SB_{2n+1}}(t) = \frac{SB_1 + t(SB_1 - 6SB_0)}{1 + t^2 - 34t}.$$

*Proof.* We have

$$G_{SB_{2n}}(t) = \sum_{n=0}^{\infty} SB_{2n} t^n = \frac{G_{SB_n}(\sqrt{t}) + G_{SB_n}(-\sqrt{t})}{2}$$

$$\text{and} \quad G_{SB_{2n+1}}(t) = \sum_{n=0}^{\infty} SB_{2n+1} t^n = \frac{G_{SB_n}(\sqrt{t}) - G_{SB_n}(-\sqrt{t})}{2\sqrt{t}}.$$

Now using Theorem 2.9 and after some mathematical calculation, we have $G_{SB_n}(t)+G_{SB_n}(-t) = \frac{2[SB_0+t^2(6SB_1-35SB_0)]}{1+t^4-34t^2}$ and $G_{SB_n}(t) - G_{SB_n}(-t) = \frac{2t[SB_1+t^2(SB_1-6SB_0)]}{1+t^4-34t^2}$. Thus, we have

$$G_{SB_{2n}}(t) = \frac{SB_0 + t(6SB_1 - 35SB_0)}{1 + t^2 - 34t}$$

$$\text{and} \quad G_{SB_{2n+1}}(t) = \frac{SB_1 + t(SB_1 - 6SB_0)}{1 + t^2 - 34t}$$

are the required results. □

**Theorem 2.11.** *( Exponential Generating function) The exponential generating function of the balancing sedenions is*

$$E_{SB_n}(t) = \frac{ue^{\lambda_1 t} - ve^{\lambda_2 t}}{\lambda_1 - \lambda_2}.$$

*Proof.* Let

$$E_{SB_n}(t) = \sum_{n=0}^{\infty} SB_n \frac{t^n}{n!}$$

be the exponential generating function for $SB_n$.

$$\begin{aligned}
E_{SB_n}(t) &= \sum_{n=0}^{\infty} SB_n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{u\lambda_1^n - v\lambda_2^n}{\lambda_1 - \lambda_2}\right) \frac{t^n}{n!} \\
&= \frac{u}{\lambda_1 - \lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!} - \frac{v}{\lambda_1 - \lambda_2} \sum_{n=0}^{\infty} \frac{(\lambda_2 t)^n}{n!} \\
&= \frac{1}{\lambda_1 - \lambda_2}(ue^{\lambda_1 t} - ve^{\lambda_2 t}),
\end{aligned}$$

which is the required result. □

**Theorem 2.12.** *[Exponential odd and even indexed Generating function] The exponential generating function of even and odd indexed for $SB_n$ are given by respectively*

$$E_{SB_{2n}}(t) = \frac{ucosh(\lambda_1\sqrt{t}) - vcosh(\lambda_2\sqrt{t})}{\lambda_1 - \lambda_2}$$

$$\text{and} \quad E_{SB_{2n+1}}(t) = \frac{usinh(\lambda_1\sqrt{t}) - vsinh(\lambda_2\sqrt{t})}{\sqrt{t}(\lambda_1 - \lambda_2)}.$$

*Proof.* Using the values of $E_{SB_n}(t) \pm E_{SB_n}(-t)$ obtain from Theorem 2.11 and the result $2sinh(kt) = e^{kt} - e^{-kt}$ and $2cosh(kt) = e^{kt} + e^{-kt}$, we have the required results. □

**Theorem 2.13.** *The generating function of the Lucas-balancing sedenions is*

$$G_{SC_n}(t) = \frac{SC_0 + t(SC_1 - 6SC_0)}{1 - 6t + t^2}.$$

*Proof.* The proof is analogous to Theorem 2.9. □

**Theorem 2.14.** *(Ordinary even and odd indexed Generating functions) The ordinary generating function of even and odd indexed for $SB_n$ are given by respectively*

$$G_{SC_{2n}}(t) = \frac{SC_0 + t(6SC_1 - 35SC_0)}{1 + t^2 - 34t}$$

$$\text{and} \quad G_{SC_{2n+1}}(t) = \frac{SC_1 + t(SC_1 - 6SC_0)}{1 + t^2 - 34t}.$$

*Proof.* The result is similar to Theorem 2.10. □

**Theorem 2.15.** *( Exponential Generating function) The exponential generating function of the Luas-balancing sedenions is*

$$E_{SC}(t) = \frac{ue^{\lambda_1 t} + ve^{\lambda_2 t}}{2}.$$

*Proof.* The proof is similar to Theorem 2.11. □

**Theorem 2.16.** *[Exponential odd and even indexed Generating function] The exponential generating function of even and odd indexed for $SC_n$ are given by respectively*

$$E_{SC_{2n}}(t) = \frac{ucosh(\lambda_1\sqrt{t}) + vcosh(\lambda_2\sqrt{t})}{2}$$

$$\text{and} \quad E_{SC_{2n+1}}(t) = \frac{usinh(\lambda_1\sqrt{t}) + vsinh(\lambda_2\sqrt{t})}{2\sqrt{t}}.$$

*Proof.* It can be proved by similar argument to Theorem 2.12. □

**Theorem 2.17.** *(Catalan's identity) For $n \geq 0$, let $s \in \mathbb{N}$ be such that $n \geq s$, then we have*

$$SB_n^2 - SB_{n+s}SB_{n-s} = \frac{1}{(\lambda_1 - \lambda_2)^2}[uv(\lambda_1^{2s} - 1) + vu(\lambda_2^{2s} - 1)]$$

*and*

$$SC_n^2 - SC_{n+s}SC_{n-s} = \frac{1}{4}[uv(1 - \lambda_1^{2s}) + vu(1 - \lambda_2^{2s})].$$

*Proof.* We have

$$\begin{aligned} SB_n^2 - SB_{n+s}SB_{n-s} &= \Big(\frac{u\lambda_1^n - v\lambda_2^n}{\lambda_1 - \lambda_2}\Big)^2 - \Big(\frac{u\lambda_1^{n+s} - v\lambda_2^{n+s}}{\lambda_1 - \lambda_2}\Big)\Big(\frac{u\lambda_1^{n-s} - v\lambda_2^{n-s}}{\lambda_1 - \lambda_2}\Big) \\ &= \frac{1}{(\lambda_1 - \lambda_2)^2}[-uv\lambda_1^n\lambda_2^n - vu\lambda_2^n\lambda_1^n + uv\lambda_1^{n+s}\lambda_2^{n-s} + vu\lambda_2^{n+s}\lambda_1^{n-s}], \end{aligned}$$

after some mathematical calculations, we have

$$SB_n^2 - SB_{n+s}SB_{n-s} = \frac{1}{(\lambda_1 - \lambda_2)^2}[uv(\lambda_1^{2s} - 1) + vu(\lambda_2^{2s} - 1)].$$

Analogously, the second identity follows, which completes the proof. □

Cassini's identity is a special case of Catalan's identity, where $s = 1$. So Cassini's identities for balancing sedenion and Lucas-balancing sedenion are following

**Corollary 2.18.** *(Cassini's identity) For $n \geq 1$, we have*

$$SB_n^2 - SB_{n+1}SB_{n-1} = \frac{1}{(\lambda_1 - \lambda_2)^2}[uv(\lambda_1^2 - 1) + vu(\lambda_2^2 - 1)]$$

$$\textit{and} \quad SC_n^2 - SC_{n+1}SC_{n-1} = \frac{1}{4}[uv(1 - \lambda_1^2) + vu(1 - \lambda_2^2)].$$

**Theorem 2.19.** *(d'Ocagne's identity) If $m,\ n \in \mathbb{N}$ with $n \geqslant m$, then we have*

$$SB_{m+1}SB_n - SB_mSB_{n+1} = \frac{vu\lambda_2^m\lambda_1^n - uv\lambda_1^m\lambda_2^n}{2\sqrt{8}}$$

*and*

$$SC_{m+1}SC_n - SC_mSC_{n+1} = \sqrt{2}(uv\lambda_1^m\lambda_2^n - vu\lambda_2^m\lambda_1^n).$$

*Proof.* Using Binet's formula, We have

$$\begin{aligned} SB_{m+1}SB_n - SB_mSB_{n+1} &= \Big(\frac{u\lambda_1^{m+1} - v\lambda_2^{m+1}}{\lambda_1 - \lambda_2}\Big)\Big(\frac{u\lambda_1^n - v\lambda_2^n}{\lambda_1 - \lambda_2}\Big) - \Big(\frac{u\lambda_1^m - v\lambda_2^m}{\lambda_1 - \lambda_2}\Big) \\ &\quad \Big(\frac{u\lambda_1^{n+1} - v\lambda_2^{n+1}}{\lambda_1 - \lambda_2}\Big) \\ &= \frac{1}{(\lambda_1 - \lambda_2)^2}[u^2\lambda_1^{m+n+1} - uv\lambda_1^{m+1}\lambda_2^n - vu\lambda_2^{m+1}\lambda_1^n + v^2\lambda_2^{m+1+n} \\ &\quad -u^2\lambda_1^{m+n+1} + uv\lambda_1^m\lambda_2^{n+1} + vu\lambda_2^m\lambda_1^{n+1} - v^2\lambda_2^{m+1+n}], \end{aligned}$$

after simplification, we find the result

$$SB_{m+1}SB_n - SB_mSB_{n+1} = \frac{vu\lambda_2^m\lambda_1^n - uv\lambda_1^m\lambda_2^n}{2\sqrt{8}}. \tag{2.3}$$

Again, we have

$$\begin{aligned} SC_{m+1}SC_n - SC_mSC_{n+1} &= \Big(\frac{u\lambda_1^{m+1} + v\lambda_2^{m+1}}{2}\Big)\Big(\frac{u\lambda_1^n + v\lambda_2^n}{2}\Big) - \Big(\frac{u\lambda_1^m + v\lambda_2^m}{2}\Big) \\ &\quad \Big(\frac{u\lambda_1^{n+1} + v\lambda_2^{n+1}}{2}\Big) \\ &= \frac{1}{4}[u^2\lambda_1^{m+n+1} + uv\lambda_1^{m+1}\lambda_2^n + vu\lambda_2^{m+1}\lambda_1^n + v^2\lambda_2^{m+1+n} \\ &\quad -u^2\lambda_1^{m+n+1} - uv\lambda_1^m\lambda_2^{n+1} - vu\lambda_2^m\lambda_1^{n+1} - v^2\lambda_2^{m+1+n}], \end{aligned}$$

after simplifying, we find the result

$$\begin{aligned} SC_{m+1}SC_n - SC_mSC_{n+1} &= \frac{1}{4}[(\lambda_1 - \lambda_2)(uv\lambda_1^m\lambda_2^n - vu\lambda_2^m\lambda_1^n)] \\ &= \sqrt{2}(uv\lambda_1^m\lambda_2^n - vu\lambda_2^m\lambda_1^n). \end{aligned} \tag{2.4}$$

This completes the proof. □

**Theorem 2.20.** *(Honsberger's identity) For integers $m$ and $n$, we have*

$$SB_{m-1}SB_n + SB_mSB_{n+1} = \frac{1}{16}[3(u^2\lambda_1^{m+n} + v^2\lambda_2^{m+n}) - uv\lambda_1^{m-n-1} - vu\lambda_2^{m-n-1}]$$

$$\textit{and} \quad SC_{m-1}SC_n + SC_mSC_{n+1} = \frac{1}{2}[3(u^2\lambda_1^{m+n} + v^2\lambda_2^{m+n}) + uv\lambda_1^{m-n-1} + vu\lambda_2^{m-n-1}].$$

*Proof.* The proof holds using the Binet's formula for balancing sedenion and Lucas- balancing sedenion respectively. □

In context of partial sums, we require the negative balancing sedenion and negative Lucas-balancing sedenions $SB_{-1}$, $SB_{-2}$, $SC_{-1}$ and $SC_{-2}$ which are obtained by the recurrence relations. So considering $SB_{-1} = 6SB_0 - SB_1$ and $SB_{-2} = 6SB_{-1} - SB_0$.

**Theorem 2.21** (Partial sum). *For the balancing sedenion, we have*

1. $$\sum_{j=0}^{n} SB_j = \frac{1}{4}(SB_{n+1} - SB_n + SB_{-1} - SB_0).$$
2. $$\sum_{j=0}^{n} SB_{2j} = \frac{1}{32}(SB_{2(n+1)} - SB_{2n} + SB_{-2} - SB_0).$$
3. $$\sum_{j=0}^{n} SB_{2j-1} = \frac{1}{32}(SB_{(2n-2)} - SB_{2n-4} + SB_{-2} - SB_0).$$

*Proof.* We have

$$\begin{aligned}\sum_{j=0}^{n} SB_j &= \sum_{j=0}^{n} \Big(\frac{u\lambda_1^j - v\lambda_2^j}{\lambda_1 - \lambda_2}\Big)\\ &= \frac{1}{\lambda_1 - \lambda_2}\Big(u\sum_{j=0}^{n} \lambda_1^j - v\sum_{j=0}^{n} \lambda_2^j\Big)\\ &= \frac{1}{\lambda_1 - \lambda_2}\Big[u\Big(\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1}\Big) - v\Big(\frac{\lambda_2^{n+1} - 1}{\lambda_2 - 1}\Big)\Big].\end{aligned}$$

After some mathematical calculation, we have

$$\sum_{j=0}^{n} SB_j = \frac{1}{4}(SB_{n+1} - SB_n + SB_{-1} - SB_0).$$

For the second and third identities, a similar argument holds. □

**Theorem 2.22** (Partial sum). *For the Lucas balancing sedenion, we have*

1. $$\sum_{j=0}^{n} SC_j = \frac{1}{4}(SC_{n+1} - SC_n + SC_{-1} - SC_0).$$
2. $$\sum_{j=0}^{n} SC_{2j} = \frac{1}{32}(SC_{2(n+1)} - SC_{2n} + SC_{-2} - SC_0).$$
3. $$\sum_{j=0}^{n} SC_{2j-1} = \frac{1}{32}(SC_{2n-2} - SC_{2n-4} + SC_{-2} - SC_0).$$

*Proof.* The proof is analogous to Theorem 2.21. □

## 3 Conclusion

In this study, we introduced the balancing and Lucas-balancing sedenions with their various types of generating functions and Binet-type formulas. Also, by means of sedenions, we find different results and some well-known identities like Casini's, Catalan's, d'Ocagane's, and Honsberger's identity with partial sum formulae for both balancing and Lucas-balancing sedenions.

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Received: 2024-03-15
Accepted: 2024-07-21