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Conformal Yamabe Solitons on $(LCS)_n$-Manifolds

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Abstract The object of the present paper is to study some properties of $(LCS)_n$-manifolds whose metric is a conformal Yamabe soliton. We define certain characteristics of $(LCS)_n$-manifolds at the point where the soliton stabilizes. Further, few specific curvature conditions of $(LCS)_n$-manifolds that accept conformal Yamabe solitons were examined.

1 Introduction

Shaikh [1] in 2003 presented the concept of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds). Shaikh and Baishya[2, 3] examined the use of $(LCS)_n$-manifolds in cosmology and general theory of relativity in 2005 and 2006. Further, M. Ateceken[4], S. K. Hui[5], D. Narain[6], S.K. Yadav[7, 8, 9], A.A. Shaikh[10, 11], S. Roy[12, 13] worked on $(LCS)_n$-manifolds.
In differential geometry and mathematical physics particularly in the study of Riemannian manifolds the concept of the Yamabe flow was first presented by Hamilton[14] which is defined as follows:

$$\frac{\partial}{\partial t}g(t) = -rg(t),$$

where r is the scalar curvature of the metric $g(t)$. A Yamabe soliton is a Riemannian manifold equipped with a conformal vector field that satisfies certain differential equation. Yamabe solitons are solutions to the Yamabe flow equation that move by diffeomorphisms and dilations without changing shape[15]. More formally, a Riemannian manifold (M, g) is called a Yamabe soliton if there exists a vector field V on M and a constant λ such that the following equation holds:

$$\pounds_V g + 2(r - \lambda)g = 0, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative with respect to the vector field V, r is the scalar curvature of metric g and λ is a constant.

The concept of Conformal Yamabe soliton was introduced by Roy et al.[12] and further studied by Jhantu Das[16], Haseeb et al.[17, 18]. Conformal Yamabe solitons are interesting objects in Riemannian geometry because they provide solutions to the Yamabe soliton equation with the additional geometric structure of conformal invariance.

Definition 1.1. A Riemannian or pseudo- Riemannian manifold (M, g) of dimension n is said to admit Conformal Yamabe soliton if

$$\pounds_\xi g + \left[2\lambda - 2r - (p + \frac{2}{n})\right]g = 0, \tag{1.2}$$

for all vector fields X, Y where $\pounds_V$ is the Lie derivative of the metric g along the vector field V, r is the scalar curvature, λ is a constant and p is a scalar non-dynamical field (time dependent scalar field or conformal pressure).
The Conformal Yamabe soliton has been referred to as shrinking if $\lambda < 0$, expanding if $\lambda > 0$, and steady for $\lambda = 0$.

In a Riemannian manifold (M^n, g) the Riemannian-Christoffel curvature tensor R[19, 20], the conharmonic curvature tensor H[21, 22], the projective curvature tensor P[23], the concircular curvature tensor $\tilde{C}$[24, 25] and the W_2-curvature tensor W_2[24, 26] are defined by:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.3}$$

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[g(Y,Z)QX - g(X,)QY$$
$$+S(Y,Z)X - S(X,Z)Y], \tag{1.4}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[g(QY,Z)X - g(QX,Z)Y], \tag{1.5}$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.6}$$

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)QY - g(Y,Z)QX], \tag{1.7}$$

where Q represents the Ricci operator and is defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, and the scalar curvature is represented by $r = tr(S)$, where $tr(S)$ represents the trace of S and $X, Y, Z \in \chi(M)$, $\chi(M)$ is the Lie algebra of vector fields of M.

The outline of the article goes as follows:
In section 2, after a brief introduction we are concerned with the rudiments of $(LCS)_n$-manifolds. In section 3, we have studied Conformal Yamabe soliton on $(LCS)_n$-manifolds. Here, we examined that if when $(LCS)_n$-manifold admits conformal Yamabe soliton, then the manifold becomes K-$(LCS)_n$-manifold and Ricci symmetric. In this section, we have also shown that $(LCS)_n$-manifold admitting Conformal Yamabe soliton is ξ-Projectively flat, ξ-Concircularly flat and ξ-Conharmonically flat if and only if the soliton becomes steady. In the last Section, we studied curvature properties on $(LCS)_n$-manifold admitting conformal Yamabe soliton. Here, we have some findings regarding the conformal Yamabe soliton that satisfy the conditions of the following type:
$S(\xi, X) \cdot R = 0$, $S(\xi, X) \cdot W_2 = 0$. Also, we have found that if the manifold admits Conformal Yamabe soliton then $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$.

2 Preliminaries

Let (M, g) be an n-dimensional Lorentzian manifold that admits a unit timelike concircular vector field ξ called the structure vector field of the manifold. Then we have $g(\xi, \xi) = -1$.
Since ξ is a unit concircular vector field, $g(X, \xi) = \eta(X)$ implies the existence of a nonzero 1-form η. Also ξ satisfies $\nabla\xi = \alpha(I + \eta \otimes \xi)$ with a nowhere zero smooth function α on M, verifying the equation $\nabla_X\alpha = (X\alpha) = d\alpha(X) = \rho\eta(X)$ for $\rho \in C^\infty(M)$ where ∇ is the Levi-Civita connection of g and X is a vector field. Moreover, in this case, ϕ is the $(1,1)$ tensor field denoted by $\phi := \frac{1}{\alpha}\nabla\xi$.
The Lorentzian para-Sasakian manifold notion was first proposed by K. Matsumoto [27]. According to A. A. Shaikh, the Lorentzian manifold M, the unit timelike concircular vector field ξ, a 1-form η, and a $(1,1)$ tensor field ϕ are collectively referred to as a Lorentzian concircular structure manifold $(M, g, \xi, \eta, \phi, \alpha)$[28, 29]. In an n-dimensional $(LCS)_n$-manifold the follow-

ing relations had denoted by $(LCS)_n$,

$$\phi^2 = I + \eta \otimes \xi,\ \eta(\xi) = -1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \quad and \quad g(\phi X, Y) = g(X, \phi Y), \tag{2.2}$$

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]. \tag{2.3}$$

for any $X, Y \in \chi(M)$.

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.4}$$

$$\eta(\nabla_X \xi) = 0, \nabla_\xi \xi = 0, \tag{2.5}$$

$$R(X, Y)Z = (\alpha^2 - \rho)[g(Y, Z)X - g(X, Z)Y], \tag{2.6}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.7}$$

$$R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \tag{2.8}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]. \tag{2.9}$$

for any $X, Y, Z \in \chi(M)$.

$$\eta(R(X, Y)\xi) = 0, \tag{2.10}$$

$$S(X, Y) = (\alpha^2 - \rho)(n - 1)g(X, Y), \tag{2.11}$$

$$r = n(n - 1)(\alpha^2 - \rho), \tag{2.12}$$

$$\nabla\eta = \alpha(g + \eta \otimes \eta), \nabla_\xi \eta = 0, \tag{2.13}$$

$$\pounds_\xi \phi = 0, \pounds_\xi \eta = 0, \pounds_\xi g = 2\nabla\eta = 2\alpha(g + \eta \otimes \eta), \tag{2.14}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor, r is the scalar curvature, ∇ is the Levi-Civita connection associated with g and $\pounds_\xi$ denotes the Lie derivative along the vector field ξ.

3 Conformal Yamabe soliton on $(LCS)_n$-manifold

Let $(M, g, \xi, \eta, \phi, \alpha)$ be an n-dimensional $(LCS)_n$-manifold. Consider the conformal Yamabe soliton on M as:

$$\frac{1}{2}\pounds_\xi g = \left[r - \lambda + (\frac{p}{2} + \frac{1}{n})\right]g. \tag{3.1}$$

Then from (2.14), we get

$$\alpha\left[g(X, Y) + \eta(X)\eta(Y)\right] = \left[r - \lambda + (\frac{p}{2} + \frac{1}{n})\right]g(X, Y), \tag{3.2}$$

which implies

$$\left[r - \lambda + (\frac{p}{2} + \frac{1}{n}) - \alpha\right]g(X, Y) - \alpha\eta(X)\eta(Y) = 0. \tag{3.3}$$

Taking $Y = \xi$ in the above equation and using (2.1), we get,

$$\left[r - \lambda + (\frac{p}{2} + \frac{1}{n}) - \alpha\right]\eta(X) + \alpha\eta(X) = 0, \tag{3.4}$$

From (3.4), we have

$$\left[r - \lambda + (\frac{p}{2} + \frac{1}{n})\right]\eta(X) = 0. \tag{3.5}$$

Since $\eta(X) \neq 0$, so we get,

$$r = \lambda - (\frac{p}{2} + \frac{1}{n}). \tag{3.6}$$

Using the above equation in (3.1), we have,

$$\pounds_\xi g = 0. \tag{3.7}$$

Thus, ξ is a killing vector field and consequently, M is a K-$(LCS)_n$-manifold. Since λ is constant, the scalar curvature r is also constant. This brings us to the following theorem:

Theorem 3.1. *: If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a conformal Yamabe soliton $(g, \xi), \xi$ being the Reeb vector field of the Lorentzian concircular structure, then the scalar curvature is constant and the manifold is a K-$(LCS)_n$-manifold.*

Now from (2.12) and (3.6), we get,

$$\lambda = (\frac{p}{2} + \frac{1}{n}) + n(n-1)(\alpha^2 - \rho). \tag{3.8}$$

Then using (2.11) and (3.8), we obtain,

$$S(X,Y) = \frac{1}{n}\Big[\lambda - (\frac{p}{2} + \frac{1}{n})\Big] g(X,Y), \tag{3.9}$$

for all vector fields X, Y on M.
This brings us to the following

Proposition 3.2. *If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a conformal Yamabe soliton $(g, \xi), \xi$, then the manifold becomes η-Einstein manifold.*

Now replacing the expression of S from (3.9) in

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$$

we get,

$$(\nabla_X S)(Y,Z) = \frac{1}{n}\Big[\lambda - (\frac{p}{2} + \frac{1}{n})\Big](\nabla_X g)(Y,Z), \tag{3.10}$$

which implies that,

$$\nabla S = 0. \tag{3.11}$$

This brings us to the following:

Proposition 3.3. *If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a conformal Yamabe soliton $(g, \xi), \xi$, then the manifold becomes Ricci symmetric.*

Again, let the Ricci tensor S of the $(LCS)_n$-manifold be η-recurrent i.e.,

$$\nabla S = \eta \otimes S,$$

which implies that,

$$(\nabla_X)(Y,Z) = \eta(X)S(Y,Z), \tag{3.12}$$

for all vector fields X, Y and Z on M. Then using (3.11) and (3.9), we get

$$\frac{1}{n}\Big[\lambda - (\frac{p}{2} + \frac{1}{n})\Big]\eta(X)g(Y,Z) = 0, \tag{3.13}$$

Taking $Y = \xi$, $Z = \xi$ in the above equation we obtain,

$$\Big[\lambda - (\frac{p}{2} + \frac{1}{n})\Big]\eta(X) = 0. \tag{3.14}$$

As $\eta(X) \neq 0$, we have $\lambda - (\frac{p}{2} + \frac{1}{n}) = 0$. Also from (3.6), we get $r = 0$.
This brings us to the following:

Proposition 3.4. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$-manifold, admitting a conformal Yamabe soliton $(g, \xi), \xi$. If the Ricci tensor S of the manifold is η-recurrent, then the soliton is steady and the manifold becomes flat.*

Let us assume that a symmetric $(0, 2)$ tensor field $h = \pounds_\xi g - 2rg$ is parallel with respect to the Levi-Civita connection associated with g.
Then

$$h(\xi, \xi) = \pounds_\xi g(\xi, \xi) - 2rg(\xi, \xi) = 2\lambda, \tag{3.15}$$

implies that,

$$\lambda = \frac{1}{2}h(\xi, \xi). \tag{3.16}$$

Now h is parallel with respect to g, then from[30], we get,

$$h(X, Y) = -h(\xi, \xi)g(X, Y), \tag{3.17}$$

for all vector fields X, Y on M. which leads to,

$$\pounds_\xi g(X, Y) = 2\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big]g(X, Y). \tag{3.18}$$

With this, we may assert the following theorem:

Theorem 3.5. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$-manifold. Assume that a symmetric $(0, 2)$ tensor field $h = \pounds_\xi g - 2rg$ is parallel with respect to the Levi-Civita connection of g. Then (g, ξ) yields a conformal Yamabe soliton on M.*

We know,

$$(\nabla_\xi Q)X = \nabla_\xi QX - Q(\nabla_\xi X), \tag{3.19}$$

and

$$(\nabla_\xi S)(X, Y) = \xi S(X, Y) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y), \tag{3.20}$$

for any vector fields X, Y on M.
Now using (3.9) we obtain,

$$QX = \frac{1}{n}\Big[\lambda - (\frac{p}{2} + \frac{1}{n})\Big]X, \tag{3.21}$$

for any vector fields X on M.
Then in view of (3.9) and (3.21), the equations (3.19) and (3.20) become

$$(\nabla_\xi Q)X = 0 \ and \tag{3.22}$$
$$(\nabla_\xi S)(X, Y) = 0, \tag{3.23}$$

respectively, for any vector fields X, Y on M.
This leads us to the following conclusion:

Theorem 3.6. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$-manifold, admitting a conformal Yamabe soliton $(g, \xi), \xi$ being the Reeb vector field. Then Q and S are parallel along ξ, where Q is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$ and S is the Ricci tensor of M.*

Also in view of (3.21), we obtain

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y) = 0, \tag{3.24}$$

for any vector fields X, Y on M.
And we have

Corollary 3.7. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$-manifold, admitting a conformal Yamabe soliton $(g, \xi), \xi$ then Q is parallel to any arbitrary vector field on M.*

Let a conformal Yamabe soliton is defined on an n-dimensional $(LCS)_n$-manifold M as,

$$\frac{1}{2}\pounds_V g = \Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big] g, \tag{3.25}$$

where $\pounds_V g$ denotes the Lorentzian derivative of the metric g along a vector field V and r, λ is as defined in (1.1).
Let V be pointwise co-linear with ξ i.e., $V = b\xi$ where b is a function on M. Then the equation (3.25) becomes,

$$\pounds_{b\xi} g(X, Y) = 2\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big] g(X, Y), \tag{3.26}$$

for any vector fields X, Y on M.
Applying the property of Lie derivative and Levi-Civita connection we have,

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) = 2\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big] g(X, Y). \tag{3.27}$$

Using (2.4), the above equation reduces to,

$$b\alpha g(\phi X, Y) + (Xb)\eta(Y) + b\alpha g(\phi Y, X) + (Yb)\eta(X) = 2\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big] g(X, Y). \tag{3.28}$$

Taking $Y = \xi$ in the above equation, we obtain,

$$-Xb + (\xi b)\eta(X) = 2\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big]\eta(X). \tag{3.29}$$

Again putting $X = \xi$ in the above equation, we obtain,

$$\xi b = r - \lambda + (\frac{p}{2} + \frac{1}{n}), \tag{3.30}$$

Then using (3.30), (3.29) becomes,

$$Xb = -\Big[\Big(r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big)\eta(X)\Big]. \tag{3.31}$$

Applying exterior differentiation in (3.31), we have,

$$\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big] d\eta = 0. \tag{3.32}$$

Now in an n-dimensional $(LCS)_n$-manifold we have,

$$(d\eta)(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]),$$

which implies

$$\begin{aligned}(d\eta)(X, Y) &= g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)\\ &= \alpha g(Y, X) + \eta(X)\eta(Y) - \alpha g(Y, X) + \eta(X)\eta(Y)\\ &= 0.\end{aligned} \tag{3.33}$$

Hence the 1-form η is closed.
Then using the above equation, (3.32) implies that, either $r \neq \lambda$ or $r = \lambda$. Now if $r \neq \lambda$ then from (3.25), we have,

$$\pounds_V g = 2\Big[r - \lambda + (\frac{p}{2} + \frac{1}{n})\Big] g \tag{3.34}$$

which implies V is a conformal killing vector field. Again if $r = \lambda$ then from (3.31), we get,

$$Xb = 0, \tag{3.35}$$

implies that b is constant. This brings us to the following theorem:

Theorem 3.8. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$-manifold, admitting a conformal Yamabe soliton $(g, V), V$ being a vector field on M. If V is pointwise co-linear with ξ then either V is a conformal killing vector field, provided $r \neq \lambda$, or V is a constant multiple of ξ, where ξ being the Reeb vector field of the Lorentzian concircular structure, r is the scalar curvature and λ is a constant.*

Also if $r = \lambda$ then from (3.25), we obtain,

$$\pounds_V g = 0, \tag{3.36}$$

implies that V is a killing vector field. Then we have,

Corollary 3.9. *Let $(M, g, \xi, \eta, \phi, \alpha)$ be an $(LCS)_n$-manifold, admitting a conformal Yamabe soliton $(g, V), V$ being a vector field on M. If V is pointwise co-linear with ξ and $r = \lambda$ then V becomes killing vector field, where ξ being the Reeb vector field of the Lorentzian concircular structure, r is the scalar curvature and λ is a constant.*

From the definition of Projective curvature tensor (1.5), defined on an n-dimensional $(LCS)_n$-manifold, we have,

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y], \tag{3.37}$$

for any vector fields X, Y and Z on M.
Putting $Z = \xi$, we get

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-1)}[S(Y,\xi)X - S(X,\xi)Y]. \tag{3.38}$$

Using (2.7) and (3.9), we obtain,

$$P(X,Y)\xi = \left[(\alpha^2 - \rho) - \frac{1}{n(n-1)}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right][\eta(Y)X - \eta(X)Y]. \tag{3.39}$$

Again using (3.8), we get,

$$P(X,Y)\xi = 0. \tag{3.40}$$

This brings us to the following:

Proposition 3.10. *An $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admitting a conformal Yamabe soliton $(g, \xi), \xi$ is ξ-Projectively flat.*

From the definition of concircular curvature tensor (1.6), defined on an n-dimensional $(LCS)_n$-manifold, we have,

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.41}$$

for any vector fields X, Y and Z on M.
Putting $Z = \xi$ we get,

$$\tilde{C}(X,Y)\xi = R(X,Y)\xi - \frac{r}{n(n-1)}[g(Y,\xi)X - g(X,\xi)Y], \tag{3.42}$$

Using (2.7) and (3.9), we obtain,

$$\tilde{C}(X,Y)\xi = \left[(\alpha^2 - \rho) - \frac{1}{n(n-1)}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right][\eta(Y)X - \eta(X)Y]. \tag{3.43}$$

Again using (3.8), we get,

$$\tilde{C}(X,Y)\xi = 0. \tag{3.44}$$

This brings us to the following:

Proposition 3.11. *An $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admitting a conformal Yamabe soliton $(g, \xi), \xi$ being the Reeb vector field of the Lorentzian Concircular structure, is ξ-concircularly flat.*

From the definition of conharmonic curvature tensor (1), defined on an n-dimensional $(LCS)_n$-manifold, we have,

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y], \tag{3.45}$$

for any vector fields X, Y and Z on M.
Putting $Z = \xi$ we get,

$$H(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)}[g(Y,\xi)QX - g(X,\xi)QY + S(Y,\xi)X - S(X,\xi)Y]. \tag{3.46}$$

Using (2.7), (3.9) and (3.21), we obtain,

$$H(X,Y)\xi = \left[(\alpha^2 - \rho) - \frac{2}{n(n-2)}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right][\eta(Y)X - \eta(X)Y], \tag{3.47}$$

Again using (3.8), we get,

$$H(X,Y)\xi = -\left[\frac{1}{(n-1)(n-2)}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right][\eta(Y)X - \eta(X)Y]. \tag{3.48}$$

This implies that $H(X,Y)\xi = 0$ if and only if $\lambda = 0$.
This brings us to the following:

Proposition 3.12. *An $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admitting a conformal Yamabe soliton $(g, \xi), \xi$ is ξ-conharmonically flat if and only if the soliton is steady.*

4 Curvature properties on $(LCS)_n$-manifold admitting Conformal Yamabe soliton

We know,

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z), \tag{4.1}$$

for any vector fields X, Y and Z on M.
Using (2.8), we obtain,

$$R(\xi, X) \cdot S = S((\alpha^2 - \rho)(g(X,Y)\xi - \eta(Y)X, Z) + S(Y, (\alpha^2 - \rho)g(X,Z)\xi - \eta(Z)X)). \tag{4.2}$$

Then using (3.9), we get,

$$R(\xi, X) \cdot S = \frac{(\alpha^2 - \rho)}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})][g(X,Y)\eta(Z) - g(X,Z)\eta(Y) + g(X,Z)\eta(Y) - g(X,Y)\eta(Z)], \tag{4.3}$$

which implies that

$$R(\xi, X) \cdot S = 0.$$

With this, we may assert the following theorem:

Theorem 4.1. *If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a conformal Yamabe soliton (g, ξ), ξ being the Reeb vector field of the Lorentzian concircular structure, then $R(\xi, X) \cdot S = 0$, i.e., the manifold is ξ-Semi Symmetric.*

Again the condition $S(\xi, X) \cdot R = 0$ implies that,

$$\begin{aligned} &S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W \\ &+ S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X \\ &= 0. \end{aligned} \tag{4.4}$$

for any vector fields X, Y, Z and W on M.
Taking the inner product with ξ, the above equation becomes,

$$\begin{aligned} &- S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) \\ &- S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) - S(\xi, Z)\eta(R(Y, X)W) \\ &+ S(X, W)\eta(R(Y, Z)\xi) - S(\xi W)\eta(R(Y, Z)X) = 0. \end{aligned} \tag{4.5}$$

Replacing the expression of S from (3.9) and taking $Z = \xi$, $W = \xi$, we get,

$$\begin{aligned} &\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})][-g(X, R(Y, \xi)\xi) - \eta(R(Y, \xi)\xi)\eta(X) + g(X, Y)\eta(R(\xi, \xi)\xi\xi) \\ &- \eta(Y)\eta(R(X, \xi)\xi) + \eta(X)\eta(R(Y, \xi)\xi) - \eta(\xi)\eta(R(Y, X)\xi) \\ &+ \eta(X)\eta(R(Y, \xi)\xi) - \eta(\xi)\eta(R(Y, \xi)X)] = 0, \end{aligned} \tag{4.6}$$

Now using (2.7), (2.9), (2.10), we get on simplification,

$$\frac{(\alpha^2 - \rho)}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})][g(X, Y) + \eta(X)\eta(Y)] = 0, \tag{4.7}$$

Using (2.2), the above equation becomes,

$$\frac{(\alpha^2 - \rho)}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})][g(\phi X, \phi Y)] = 0, \tag{4.8}$$

for any vector fields X, Y on M.
This implies that,

$$\frac{(\alpha^2 - \rho)}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})] = 0, \tag{4.9}$$

Then using (3.8), we get,

$$\frac{[\lambda - (\frac{p}{2} + \frac{1}{n})]^2}{n^2(n-1)} = 0.$$

implying that $\lambda = 0$.
Hence using (3.6), we get $r = 0$.
With this, we have the following theorem:

Theorem 4.2. *If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a conformal Yamabe soliton (g, ξ), ξ satisfies $S(\xi, X) \cdot R = 0$ then the manifold becomes flat and the soliton is steady, where R is the Riemannian curvature tensor and S is the Ricci tensor.*

We know,

$$W_2(\xi, X) \cdot S = S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z), \tag{4.10}$$

for any vector fields X, Y and Z on M.
Replacing the expression of S from (3.9) and using the definition of W_2-curvature tensor from (1.7), we get,

$$W_2(\xi, X) \cdot S = \left[\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right] g(R(\xi, X)Y + \frac{1}{n-1}[g(\xi, Y)QX - g(X, Z)Q\xi], Z) + \left[\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right] g(Y, R(\xi, X)Z + [g(\xi, Z)QX - g(X, Z)Q\xi]). \tag{4.11}$$

Now using (2.8) and $g(QX, Y) = S(X, Y)$ after simplifying, we get

$$W_2(\xi, X) \cdot S = \left[\frac{1}{n(n-1)}[\lambda - (\frac{p}{2} + \frac{1}{n})]\right][\eta(Y)S(X, Z) - S(\xi, Z)g(X, Y) + \eta(Z)S(X, Y) - S(\xi, Y)g(X, Z)]. \tag{4.12}$$

Then using (3.9) the above equation becomes,

$$W_2(\xi, X) \cdot S = 0. \tag{4.13}$$

With this, we may assert the following theorem:

Theorem 4.3. *If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admits a conformal Yamabe soliton (g, ξ), ξ then $W_2(\xi, X) \cdot S = 0$.*

Again the condition $S(\xi, X) \cdot W_2 = 0$ implies that,

$$S(X, W_2(Y, Z)V)\xi - S(\xi, W_2(Y, Z)V)X + S(X, Y)W_2(\xi, Z)V - S(\xi, Y)W_2(X, Z)V + S(X, Z)W_2(Y, \xi)V - S(\xi, Z)W_2(Y, X)V + S(X, V)W_2(Y, Z)\xi - S(\xi, V)W_2(Y, Z)X = 0, \tag{4.14}$$

for any vector fields X, Y, Z and V on M. Taking the inner product with ξ, the above equation becomes,

$$- S(X, W_2(Y, Z)V) - S(\xi, W_2(Y, Z)V)\eta(X) + S(X, Y)\eta(W_2(\xi, Z)V) - S(\xi, Y)\eta(W_2(X, Z)V) + S(X, Z)\eta(W_2(Y, \xi)V) - S(\xi, Z)\eta(W_2(Y, X)V) + S(X, V)\eta(W_2(Y, Z)\xi) - S(\xi, V)\eta(W_2(Y, Z)X) = 0. \tag{4.15}$$

Replacing the expression of S from (3.9) and taking $Z = \xi$, $V = \xi$, we get,

$$\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})[-g(X, W_2(Y, \xi)\xi) - \eta(W_2(Y, \xi)\xi)\eta(X) + g(X, Y)\eta(W_2(\xi, \xi)\xi) - \eta(Y)\eta(W_2(X, \xi)\xi) + \eta(X)\eta(W_2(Y, \xi)\xi) - \eta(\xi)\eta(W_2(Y, X)\xi) + \eta(X)\eta(W_2(Y, \xi)\xi) - \eta(\xi)\eta(W_2(Y, \xi)X)] = 0. \tag{4.16}$$

Now using (1.7), (2.7), (2.9) and (2.11), we obtain on simplification,

$$\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})[[g(X, Y) + \eta(X)\eta(Y) - (\alpha^2 - \rho)g(X, Y) - (\alpha^2 - \rho)\eta(X)\eta(Y)] = 0. \tag{4.17}$$

implies that

$$\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})(1 - \alpha^2 + \rho)][g(X, Y) + \eta(X)\eta(Y)] = 0. \tag{4.18}$$

Using (2.2), the above equation becomes,

$$\frac{1}{n}[\lambda - (\frac{p}{2} + \frac{1}{n})(1 - \alpha^2 + \rho)]g(\phi X, \phi Y) = 0, \tag{4.19}$$

for any vector fields X, Y on M.
This implies that,

$$[\lambda - (\frac{p}{2} + \frac{1}{n})(1 - \alpha^2 + \rho)] = 0.$$

Then either $\lambda = 0$, or $\alpha^2 - \rho = 1$.
Now if $\alpha^2 - \rho = 1$, then from (2.12), we have,

$$r = n(n-1).$$

With this, we may assert the following theorem:

Theorem 4.4. *If an $(LCS)_n$-manifold $(M, g, \xi, \eta, \phi, \alpha)$ admitting a conformal Yamabe soliton $(g, \xi), \xi$ satisfies $S(\xi, X) \cdot W_2 = 0$ then either the soliton is steady, or $r = n(n-1)$.*

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