Palestine Journal of Mathematics

Vol 14(1)(2025) , 654–670

Existence and uniqueness of fixed point to a mixed monotone vector operator and application to a system of fractional boundary value problems

Abdelhak El Haddouchi and Abdellatif Sadrati

Communicated by Hichem Ben-El-Mechaiekh

MSC 2010 Classifications: 47H07, 47H10, 34B10, 34B15, 34K37.

Keywords and phrases: Mixed monotone vector operator; Existence and uniqueness solutions; System of fractional differential equations.

The authors express their gratitude to the reviewers and editor for their constructive comments and valuable suggestions, which have enhanced the quality of our paper.

Abstract This paper is devoted to the study of the existence and uniqueness of fixed points for a class of sum-type vector operators in ordered Banach space, and their applications. In particular, we show that some fixed point results for such operators in the scalar case can be extended to the vector case and applied to obtain sufficient conditions for the existence and uniqueness of a solution for a nonlinear fractional boundary value problem.

1 Introduction

Since their introduction by Guo and Lakshmikantham in [8], the study of mixed monotone operators has been an active area of research in nonlinear functional analysis. For further reading, the reader is referred to the following sources: [4, 8, 9, 15, 24, 32, 36].

Later, H. Wang et al. [25] studied the existence and uniqueness of fixed points for nonlinear sum operators of the form $Ax + Bx + C(x, x)$, where A is an increasing α-concave (or sub-homogeneous) operator, B is a decreasing operator, and C is a mixed monotone operator. In [26], the authors studied a different sum-type operator equation of the form $A(x, x) + B(x, x) + Cx = x$, where A and B are mixed monotone operators, and C is an increasing operator. In [21], Y. Sang et al. established the existence and uniqueness of a solution for the operator equation $A(x, x) + B(x, x) + Cx + e = x$. The authors generalized the results obtained in [34] from the cone case to the non-cone case. In all three works mentioned above, the authors applied their fixed point results to the solvability of some nonlinear fractional differential equations.

In 2021, the authors [20] studied the existence and uniqueness of a fixed point for the vector operator equation $\Phi(x, y, x, y) = (A_1(x, x, y), A_2(x, y, y)) = (x, y)$, where $\Phi : P_h \times P_k \times P_h \times P_k \to P_h \times P_k$ has certain mixed monotone properties. They applied these existence results to obtain the existence of a positive solution to a nonlinear Neumann boundary value problem. Building upon [20, 22] and other works, we present in this paper some fixed point theorems for a class of sum-type vector operators with specific mixed monotone properties. We apply these theorems to study the existence and uniqueness of solutions to systems of fractional differential equations.

The paper is organized as follows. In Section 2, we recall definitions and results from the theory of mixed monotone operators and cones in Banach spaces. Section 3 is devoted to the theoretical existence results, where we provide two fixed point theorems for specific vector operators in a partially ordered Banach space. In brief, we first consider the existence and uniqueness of solutions to the following operator system:

$$\begin{aligned} A_1(x, x, y) &= x, \\ A_2(x, y, y) &= y, \end{aligned} \tag{1.1}$$

where A_1 and A_2 are operators with certain mixed monotone properties. Our theoretical theorem generalizes a result in [20] from cone mappings to the non-cone case. This result is then used to establish the existence and uniqueness of a solution to a class of sum-type operator systems.

$$\begin{aligned} A_1(x,x,y)+B_1(x,x)+e=x,\\ A_2(x,y,y)+B_2(y,y)+f=y, \end{aligned} \tag{1.2}$$

where A_1, A_2, B_1, and B_2 are mixed monotone operators, and $e,f\in P$, where P is a cone in a Banach space E. In the last section (Section 4), we demonstrate that such more general vector operators can be applied to establish the existence and uniqueness of solutions to a system of nonlinear fractional differential equations of the type:

$$\begin{cases} -D^{\alpha}_{0^+}x(t)=F_1(t,x(t),y(t))+G_1(t,x(t))-a,\ 0<t<1,\\ -D^{\beta}_{0^+}y(t)=F_2(t,x(t),y(t))+G_2(t,y(t))-b,\ 0<t<1,\\ x^{(i)}(0)=0=y^{(i)}(0),\ 0\le i\le n-2,\\ [D^{\gamma}_{0^+}x(t)]_{t=1}=0=[D^{\delta}_{0^+}y(t)]_{t=1}, \end{cases} \tag{1.3}$$

where $D^{\alpha}_{0^+}, D^{\beta}_{0^+}, D^{\gamma}_{0^+}, D^{\delta}_{0^+}$ are the Riemann-Liouville fractional derivatives of orders $\alpha,\beta,\gamma,\delta$, respectively. Here, $n-1\le\alpha,\beta\le n$, and $1\le\gamma,\delta\le n-2$, with $n>3$ $(n\in\mathbb{N})$. Additionally, $a,b>0$ are constants, and F_i,G_i $(i=1,2)$ represent appropriate functions specified later.

2 Preliminaries

For the reader's convenience, we begin with definitions and lemmas that will be used in the proof of our main results. For more details, we refer to [1, 5, 8, 10, 20, 28, 36] and the references therein. Throughout this paper, $(E,\|.\|)$ is a real Banach space ordered by a cone $P\subset E$, i.e., $x\preceq y$ if and only if $y-x\in P$. Recall that a nonempty closed and convex set $P\subset E$ is a cone if it satisfies $(i)x\in P,\lambda\ge 0\Rightarrow\lambda x\in P$, $(ii)x\in P,-x\in P\Rightarrow x=\theta$, here θ is the zero element in E. A cone P is called normal if there exists a constant $N>0$ such that $\theta\preceq x\preceq y$ implies $\|x\|\le N\|y\|$; in this case N is called the normality constant of P. Given $h\succ\theta$ (i.e. $h\succeq\theta$ and $h\neq\theta$), P_h is the set $P_h:=\{x\in E:\ \text{there exist}\ \lambda>0,\mu>0\ \text{such that}\ \lambda h\preceq x\preceq\mu h\}$. It is easy to see that $P_h\subset P$ is convex and $\lambda P_h=P_h$ for all $\lambda>0$. If $\overset{\circ}{P}\neq\varnothing$ and $h\in\overset{\circ}{P}$, then $P_h=\overset{\circ}{P}$.

For an element $h\in P$ with $h\neq\theta$ and $e\in P$ with $\theta\preceq e\preceq h$, we denote

$$P_{h,e}=\{x\in E\ :x+e\in P_h\}.$$

Remark 2.1. (i) It is clear that $P_{h,\theta}=P_h$ for each $h\succ\theta$.

(ii) P_h and $P_{h,e}$ are of different nature. In fact, one can observe that $P_h\subset P\setminus\{\theta\}$ for any $h\succ\theta$, while $P_{h,e}$ need not be a subset of the cone P for some $h\succ\theta$, $e\succeq\theta$ with $e\preceq h$.

Lemma 2.2 ([33]). *If $x\in P_{h,e}$, then $\lambda x+(\lambda-1)e\in P_{h,e}$ for $\lambda>0$.*

Lemma 2.3 ([33]). *If $x,u\in P_{h,e}$, then there exist reals μ and γ, with $0<\mu<1$ and $\gamma>1$, such that*

$$\mu u+(\mu-1)e\preceq x\preceq\gamma u+(\gamma-1)e.$$

Furthermore, we can choose a small real number $r\in(0,1)$, such that

$$ru+(r-1)e\preceq x\preceq r^{-1}u+(r^{-1}-1)e.$$

Definition 2.4 ([3]). Let $(X,\preceq)$ be a partially ordered set and $B:X\times X\longrightarrow X$ be an operator. We say that B has the mixed monotone property if $B(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y\in X$,

$$\begin{aligned} x_1,x_2\in X,\ \ x_1\preceq x_2\ \Rightarrow\ B(x_1,y)\preceq B(x_2,y),\\ y_1,y_2\in X,\ \ y_1\preceq y_2\ \Rightarrow\ B(x,y_2)\preceq B(x,y_1). \end{aligned}$$

An element $(x,y)\in X\times X$ is called a coupled fixed point of B if $B(x,y)=x$ and $B(y,x)=y$. In this case, if $x=y$ then x is called a fixed point of B, that is, $B(x,x)=x$.

Definition 2.5 ([2]). Let $(X,\preceq)$ be a partially ordered set and $A: X\times X\times X\longrightarrow X$. Then the trivariate operator A is said to have the mixed monotone property if $A(.,u,y)$ and $A(x,u,.)$ are monotone non-decreasing, and $A(x,.,y)$ is monotone non-increasing, i.e., for any $x,u,y\in X$

$$\begin{aligned}
&x_1,x_2\in X,\ x_1\preceq x_2 \Longrightarrow A(x_1,u,y)\preceq A(x_2,u,y),\\
&u_1,u_2\in X,\ u_1\preceq u_2 \Longrightarrow A(x,u_1,y)\succeq A(x,u_2,y),\\
&y_1,y_2\in X,\ y_1\preceq y_2 \Longrightarrow A(x,u,y_1)\preceq A(x,u,y_2).
\end{aligned}$$

In the sequel, if $(X,\preceq)$ is a partially ordered set and, if $A_1,A_2: X\times X\times X\longrightarrow X$ are two operators, then we define the vector operator $\Phi: X\times X\times X\times X\longrightarrow X\times X$, noted $\Phi=(A_1,A_2)$, by

$$\Phi(x,y,u,v)=(A_1(x,u,y),A_2(x,v,y)),\ \forall x,y,u,v\in X. \tag{2.1}$$

Definition 2.6 ([20]). Let $(X,\preceq)$ be a partially ordered set. Let $A_1,A_2: X\times X\times X\longrightarrow X$ be two operators and $\Phi=(A_1,A_2)$ be given as in (2.1).

(i) We say that the operator $\Phi=(A_1,A_2)$ is a cooperative mixed monotone vector operator if A_1, A_2 are mixed monotone as in Definition 2.5.

(ii) We say that $\Phi=(A_1,A_2)$ is a competitive mixed monotone vector operator if $A_1(.,u,y)$, $A_2(x,u,.)$ are monotone non-decreasing, and $A_1(x,.,y)$, $A_1(x,u,.)$, $A_2(x,.,y)$, $A_2(.,u,y)$ are monotone non-increasing.

3 Fixed point theorems

First, we present a fixed point theorem which generalizes Theorem 2.3 in [20] on the cone mappings to non-cone case. let $h,k\in P$ be such that $h\neq\theta$, $k\neq\theta$ and choose $e,f\in P$ with $\theta\preceq e\preceq h$, $\theta\preceq f\preceq k$.

Theorem 3.1. *Let P be a normal cone in a real Banach space E. Let $A_1: P_{h,e}\times P_{h,e}\times P_{k,f}\to E$ and $A_2: P_{h,e}\times P_{k,f}\times P_{k,f}\to E$ be two operators such that the vector operator $\Phi=(A_1,A_2): P_{h,e}\times P_{k,f}\times P_{h,e}\times P_{k,f}\to E\times E$ satisfies the following assumptions.*

(H_1) *$\Phi=(A_1,A_2)$ is cooperative mixed monotone. Moreover,*

$$A_1(h,h,k)\in P_{h,e} \text{ and } A_2(h,k,k)\in P_{k,f}; \tag{3.1}$$

(H_2) *There exist positive-valued functions τ_1,τ_2 defined on the interval (a,b), and φ defined on the square $(a,b)\times(a,b)$ such that*

(i) $\tau_1,\tau_2:(a,b)\to(0,1)$ are surjections.

(ii) $1>\varphi(t,s)>\min\{\tau_1(t),\tau_2(s)\}$, for all $t,s\in(a,b)$.

(iii) For any $x,u\in P_{h,e}$, for any $y,v\in P_{k,f}$ and any $t,s\in(a,b)$

$$\begin{aligned}
&A_1\Big(\tau_1(t)x+(\tau_1(t)-1)e,\frac{1}{\tau_1(t)}u+(\frac{1}{\tau_1(t)}-1)e,\tau_2(s)y+(\tau_2(s)-1)f\Big)\\
&\succeq \varphi(t,s)A_1(x,u,y)+(\varphi(t,s)-1)e,\\
&A_2\Big(\tau_1(t)x+(\tau_1(t)-1)e,\frac{1}{\tau_2(t)}v+(\frac{1}{\tau_2(s)}-1)f,\tau_2(s)y+(\tau_2(s)-1)f\Big)\\
&\succeq \varphi(t,s)A_2(x,v,y)+(\varphi(t,s)-1)f.
\end{aligned}$$

Then,

(i) $A_1: P_{h,e}\times P_{h,e}\times P_{k,f}\longrightarrow P_{h,e}$, $A_2: P_{h,e}\times P_{k,f}\times P_{k,f}\longrightarrow P_{k,f}$.

(ii) There exist $x_0, u_0 \in P_{h,e}$, $y_0, v_0 \in P_{k,f}$ *and* $r \in (0,1)$ *such that*

$$\begin{cases} ru_0 \preceq x_0 \prec u_0, \\ rv_0 \preceq y_0 \prec v_0 \end{cases} \quad \text{and} \quad \begin{cases} x_0 \preceq A_1(x_0,u_0,y_0) \preceq A_1(u_0,x_0,v_0) \preceq u_0, \\ y_0 \preceq A_2(x_0,v_0,y_0) \preceq A_2(u_0,y_0,v_0) \preceq v_0. \end{cases} \tag{3.2}$$

(iii) Φ *has a unique fixed point* $(x^*, y^*) \in P_{h,e} \times P_{k,f}$.

(iv) For any initial values $x_0', u_0' \in P_{h,e}$ *and* $y_0', v_0' \in P_{k,f}$, *constructing successively the sequences*

$$\begin{aligned} x_n' &= A_1(x_{n-1}', u_{n-1}', y_{n-1}'), \quad y_n' = A_2(x_{n-1}', v_{n-1}', y_{n-1}'), \\ u_n' &= A_1(u_{n-1}', x_{n-1}', v_{n-1}'), \quad v_n' = A_2(u_{n-1}', y_{n-1}', v_{n-1}'), \end{aligned} \quad n = 1,2,..., \tag{3.3}$$

we have $\|x_n' - x^*\| \to 0$, $\|u_n' - x^*\| \to 0$ *and* $\|y_n' - y^*\| \to 0$, $\|v_n' - y^*\| \to 0$ *(as* $n \longrightarrow \infty$*).*

Proof. 1) It is easy to see that for any $x, u \in P_{h,e}$ and any $y, v \in P_{k,f}$, there exists $\sigma_* \in (0,1)$ such that

$$\sigma_* h + (\sigma_* - 1)e \preceq x, u \preceq \frac{1}{\sigma_*}h + (\frac{1}{\sigma_*} - 1)e,$$
$$\sigma_* k + (\sigma_* - 1)f \preceq y, v \preceq \frac{1}{\sigma_*}k + (\frac{1}{\sigma_*} - 1)f.$$

It follows from $(H_2)(i)$ that there exist $t_*, s_* \in (a,b)$ such that $\tau_1(t_*) = \sigma_* = \tau_2(s_*)$, hence

$$\begin{aligned} \tau_1(t_*)h + (\tau_1(t_*) - 1)e &\preceq x, u \preceq \frac{1}{\tau_1(t_*)}h + (\frac{1}{\tau_1(t_*)} - 1)e, \\ \tau_2(s_*)k + (\tau_2(s_*) - 1)f &\preceq y, v \preceq \frac{1}{\tau_2(s_*)}k + (\frac{1}{\tau_2(s_*)} - 1)f. \end{aligned} \tag{3.4}$$

Furthermore, by $(H_2)(ii)$ we have

$$\begin{aligned} &A_1\Big(\frac{1}{\tau_1(t)}x + (\frac{1}{\tau_1(t)} - 1)e, \tau_1(t)u + (\tau_1(t) - 1)e, \frac{1}{\tau_2(s)}y + (\frac{1}{\tau_2(s)} - 1)f\Big) \\ &\preceq \frac{1}{\varphi(t,s)}A_1(x,u,y) + (\frac{1}{\varphi(t,s)} - 1)e, \\ &A_2\Big(\frac{1}{\tau_1(t)}x + (\frac{1}{\tau_1(t)} - 1)e, \tau_2(s)v + (\tau_2(s) - 1)f, \frac{1}{\tau_2(s)}y + (\frac{1}{\tau_2(s)} - 1)f\Big) \\ &\preceq \frac{1}{\varphi(t,s)}A_2(x,u,y) + (\frac{1}{\varphi(t,s)} - 1)f. \end{aligned}$$

Then, by the mixed monotone properties of A_1, A_2 and (3.4), we have

$$\begin{aligned} &A_1(x,u,y) \\ &\succeq A_1\Big(\tau_1(t_*)h + (\tau_1(t_*) - 1)e, \frac{1}{\tau_1(t_*)}h + (\frac{1}{\tau_1(t_*)} - 1)e, \tau_2(s_*)k + (\tau_2(s_*) - 1)f\Big) \\ &\succeq \varphi(t_*, s_*)A_1(h,h,k) + (\varphi(t_*, s_*) - 1)e \end{aligned}$$

and

$$\begin{aligned} &A_1(x,u,y) \\ &\preceq A_1\Big(\frac{1}{\tau_1(t_*)}h + (\frac{1}{\tau_1(t_*)} - 1)e, \tau_1(t_*)h + (\tau_1(t_*) - 1)e, \frac{1}{\tau_2(s_*)}k + (\frac{1}{\tau_2(s_*)} - 1)f\Big) \\ &\preceq \frac{1}{\varphi(t_*, s_*)}A_1(h,h,k) + (\frac{1}{\varphi(t_*, s_*)} - 1)e. \end{aligned}$$

Since $A_1(h,h,k) \in P_{h,e}$, it follows that $A_1(x,u,y) \in P_{h,e}$, and hence $A_1 : P_{h,e} \times P_{h,e} \times P_{k,f} \longrightarrow P_{h,e}$. Similarly, we get $A_2 : P_{h,e} \times P_{k,f} \times P_{k,f} \longrightarrow P_{k,f}$.

2) Since $A_1(h,h,k) \in P_{h,e}$ and $A_2(h,k,k) \in P_{k,f}$, we can choose a small enough number $\sigma_0 \in (0,1)$ such that

$$\sigma_0 h + (\sigma_0 - 1)e \preceq A_1(h,h,k) \preceq \frac{1}{\sigma_0}h + (\frac{1}{\sigma_0} - 1)e,$$
$$\sigma_0 k + (\sigma_0 - 1)f \preceq A_2(h,k,k) \preceq \frac{1}{\sigma_0}k + (\frac{1}{\sigma_0} - 1)f.$$

Again, by $(H_2)(i)$, there exist $t_0, s_0 \in (a,b)$ such that $\tau_1(t_0) = \sigma_0 = \tau_2(s_0)$, and by $(H_2)(ii)$, we can choose a positive integer m such that

$$\Big(\frac{\varphi(t_0,s_0)}{\sigma_0}\Big)^m \geq \frac{1}{\sigma_0}.$$

Put

$$\overline{x}_n = \sigma_0^n h + (\sigma_0^n - 1)e,\ \overline{u}_n = \frac{1}{\sigma_0^n}h + (\frac{1}{\sigma_0^n} - 1)e,$$
$$\overline{y}_n = \sigma_0^n k + (\sigma_0^n - 1)f,\ \overline{v}_n = \frac{1}{\sigma_0}k + (\frac{1}{\sigma_0^n} - 1)f,\ \ n = 1,2,...$$

Then, we have

$$\overline{x}_n = \sigma_0 \overline{x}_{n-1} + (\sigma_0 - 1)e,\ \overline{u}_n = \frac{1}{\sigma_0}\overline{u}_{n-1} + (\frac{1}{\sigma_0} - 1)e,$$
$$\overline{y}_n = \sigma_0 \overline{y}_{n-1} + (\sigma_0 - 1)f,\ \overline{v}_n = \frac{1}{\sigma_0}\overline{v}_{n-1} + (\frac{1}{\sigma_0} - 1)f,\ \ n = 1,2,...$$

Take $x_0 = \overline{x}_m$, $u_0 = \overline{u}_m$, $y_0 = \overline{y}_m$ and $v_0 = \overline{v}_m$. Then, it is clear that $x_0, u_0 \in P_{h,e}$ and $y_0, v_0 \in P_{k,f}$ with $x_0 = \sigma_0^{2m} u_0 + (\sigma_0^{2m} - 1)e \prec u_0$, $y_0 = \sigma_0^{2m} v_0 + (\sigma_0^{2m} - 1)f \prec v_0$ and for any $r \in (0, \sigma_0^{2m}) \subset (0,1)$ we have $ru_0 \preceq x_0$ and $rv_0 \preceq y_0$. Also, by the mixed monotonicity of A_1 and A_2, $A_1(x_0,u_0,y_0) \preceq A_1(u_0,x_0,v_0)$ and $A_2(x_0,v_0,y_0) \preceq A_2(u_0,y_0,v_0)$. In addition,

$$\begin{aligned}
A_1(x_0,u_0,y_0) &= A_1(\overline{x}_m, \overline{u}_m, \overline{y}_m) \\
&= A_1\Big(\tau_1(t_0)\overline{x}_{m-1} + (\tau_1(t_0) - 1)e, \frac{1}{\tau_1(t_0)}\overline{u}_{m-1} + (\frac{1}{\tau_1(t_0)} - 1)e, \\
&\qquad \tau_2(s_0)\overline{y}_{m-1} + (\tau_2(s_0) - 1)f\Big) \\
&\succeq \varphi(t_0,s_0)A_1(\overline{x}_{m-1}, \overline{u}_{m-1}, \overline{y}_{m-1}) + (\varphi(t_0,s_0) - 1)e \\
&\succeq \varphi(t_0,s_0)^2 A_1(\overline{x}_{m-2}, \overline{u}_{m-2}, \overline{y}_{m-2}) + (\varphi(t_0,s_0)^2 - 1)e \\
&\succeq ... \succeq \varphi(t_0,s_0)^m A_1(h,h,k) + (\varphi(t_0,s_0)^m - 1)e \\
&\succeq \sigma_0^{m-1}(\sigma_0 h + (\sigma_0 - 1)e) + (\sigma_0^{m-1} - 1)e \\
&= \sigma_0^m h + (\sigma_0^m - 1)e = \overline{x}_m = x_0
\end{aligned}$$

and

$$\begin{aligned}
A_1(u_0, x_0, v_0) &= A_1(\overline{u}_m, \overline{x}_m, \overline{v}_m)\\
&= A_1\Big(\frac{1}{\tau_1(t_0)}\overline{u}_{m-1} + (\frac{1}{\tau_1(t_0)} - 1)e, \tau_1(t_0)\overline{x}_{m-1} + (\tau_1(t_0) - 1)e,\\
&\quad \frac{1}{\tau_2(s_0)}\overline{v}_{m-1} + (\frac{1}{\tau_2(s_0)} - 1)f\Big)\\
&\preceq \frac{1}{\varphi(t_0, s_0)}A_1(\overline{u}_{m-1}, \overline{x}_{m-1}, \overline{v}_{m-1}) + (\frac{1}{\varphi(t_0, s_0)} - 1)e\\
&\preceq \frac{1}{\varphi(t_0, s_0)^2}A_1(\overline{u}_{m-2}, \overline{x}_{m-2}, \overline{v}_{m-2}) + (\frac{1}{\varphi(t_0, s_0)^2} - 1)e\\
&\preceq ... \preceq \frac{1}{\varphi(t_0, s_0)^m}A_1(h, h, k) + (\frac{1}{\varphi(t_0, s_0)^m} - 1)e\\
&\preceq \frac{1}{\sigma_0^{m-1}}\Big(\frac{1}{\sigma_0}h + (\frac{1}{\sigma_0} - 1)e\Big) + (\frac{1}{\sigma_0^{m-1}} - 1)e\\
&= \frac{1}{\sigma_0^m}h + (\frac{1}{\sigma_0^m} - 1)e = \overline{u}_m = u_0.
\end{aligned}$$

Similarly, we get

$$y_0 \preceq A_2(x_0, v_0, y_0) \text{ and } A_2(u_0, y_0, v_0) \preceq v_0.$$

3) Constructing successively the sequences

$$\begin{aligned}
x_n &= A_1(x_{n-1}, u_{n-1}, y_{n-1}), \quad y_n = A_2(x_{n-1}, v_{n-1}, y_{n-1}),\\
u_n &= A_1(u_{n-1}, x_{n-1}, v_{n-1}), \quad v_n = A_2(u_{n-1}, y_{n-1}, v_{n-1}),
\end{aligned} \quad n = 1, 2, ...,$$

Then we have, $x_1 \preceq u_1$ and $y_1 \preceq v_1$. Combining with the mixed monotone properties of A_1 and A_2, we obtain

$$\begin{aligned}
x_0 \preceq x_1 \preceq ... \preceq x_n \preceq ... \preceq u_n \preceq ... \preceq u_1 \preceq u_0,\\
y_0 \preceq y_1 \preceq ... \preceq y_n \preceq ... \preceq v_n \preceq ... \preceq v_1 \preceq v_0.
\end{aligned} \tag{3.5}$$

Furthermore, for all $n \geq 1$

$$\begin{aligned}
x_n \succeq x_0 \succeq ru_0 + (r-1)e \succeq ru_n + (r-1)e,\\
y_n \succeq y_0 \succeq rv_0 + (r-1)f \succeq rv_n + (r-1)f.
\end{aligned}$$

Set

$$r_n = \sup\{r > 0 : x_n \succeq ru_n + (r-1)e \text{ and } y_n \succeq rv_n + (r-1)f\}.$$

It follows that,

$$\begin{aligned}
x_{n+1} \succeq x_n \succeq r_n u_n + (r_n - 1)e \succeq r_n u_{n+1} + (r_n - 1)e,\\
y_{n+1} \succeq y_n \succeq r_n v_n + (r_n - 1)f \succeq r_n v_{n+1} + (r_n - 1)f,
\end{aligned} \quad n = 1, 2, ...,$$

Therefore, $r_{n+1} \geq r_n$, that is $\{r_n\}$ is an increasing sequence with $\{r_n\} \subset (0, 1]$, which implies that $\{r_n\}$ is convergent. Assume that $r_n \to r^*$ as $n \to \infty$, then necessarily $r^* = 1$. In fact, if we suppose to the contrary, that is, $0 < r_n \leq r^* < 1$, then by $(H_2)(i)$ there exist $t^*, s^* \in (a, b)$ such that $\tau_1(t^*) = r^* = \tau_2(s^*)$. We need to distinguish two cases.

Case 1 : There exist an integer N such that $r_N = r^*$. In this case, for all $n \geq N$ we have $r_n = r^*$ and

$$\begin{aligned}
x_{n+1} &= A_1(x_n, u_n, y_n)\\
&\succeq A_1\Big(r_n u_n + (r_n - 1)e, \frac{1}{r_n}x_n + (\frac{1}{r_n} - 1)e, r_n v_n + (r_n - 1)f\Big)\\
&\succeq \varphi(t^*, s^*)A_1(u_n, x_n, v_n) + (\varphi(t^*, s^*) - 1)e\\
&= \varphi(t^*, s^*)u_{n+1} + (\varphi(t^*, s^*) - 1)e
\end{aligned}$$

and

$$\begin{aligned} y_{n+1} &= A_2(x_n, v_n, y_n) \\ &\succeq A_2\Big(r_n u_n + (r_n - 1)e, \frac{1}{r_n} y_n + (\frac{1}{r_n} - 1)f, r_n v_n + (r_n - 1)f\Big) \\ &\succeq \varphi(t^*, s^*) A_2(u_n, y_n, v_n) + (\varphi(t^*, s^*) - 1)f \\ &= \varphi(t^*, s^*) v_{n+1} + (\varphi(t^*, s^*) - 1)f \end{aligned}$$

Thus, $r_{n+1} = r^* \geq \varphi(t^*, s^*) > r^*$, which is a contradiction.

Case 2 : For all integer n, $r_n < r^* < 1$. By $(H_2)(i)$, there exist $\alpha_n, \beta_n \in (a, b)$ such that $\tau_1(\alpha_n) = \frac{r_n}{r^*} = \tau_2(\beta_n)$. Then we have

$$\begin{aligned} x_{n+1} &= A_1(x_n, u_n, y_n) \\ &\succeq A_1\Big(r_n u_n + (r_n - 1)e, \frac{1}{r_n} x_n + (\frac{1}{r_n} - 1)e, r_n v_n + (r_n - 1)f\Big) \\ &= A_1\Big(\frac{r_n}{r^*}(r^* u_n + (r^* - 1)e) + (\frac{r_n}{r^*} - 1)e, \\ &\quad \frac{r^*}{r_n}\Big(\frac{1}{r^*} x_n + (\frac{1}{r^*} - 1)e\Big) + \Big(\frac{r^*}{r_n} - 1\Big)e, \\ &\quad \frac{r_n}{r^*}(r^* v_n + (r^* - 1)f) + (\frac{r_n}{r^*} - 1)f\Big) \\ &= A_1\Big(\tau_1(\alpha_n)(r^* u_n + (r^* - 1)e) + (\tau_1(\alpha_n) - 1)e, \\ &\quad \frac{1}{\tau_1(\alpha_n)}\Big(\frac{1}{r^*} x_n + (\frac{1}{r^*} - 1)e\Big) + \Big(\frac{1}{\tau_1(\alpha_n)} - 1\Big)e, \\ &\quad \tau_2(\beta_n)(r^* v_n + (r^* - 1)f) + (\tau_2(\beta_n) - 1)f\Big) \\ &\succeq \varphi(\alpha_n, \beta_n) A_1\Big(r^* u_n + (r^* - 1)e, \frac{1}{r^*} x_n + (\frac{1}{r^*} - 1)e, r^* v_n + (r^* - 1)f\Big) \\ &+ (\varphi(\alpha_n, \beta_n) - 1)e \\ &\succeq \varphi(\alpha_n, \beta_n)\varphi(t^*, s^*) A_1(u_n, x_n, v_n) + (\varphi(\alpha_n, \beta_n)\varphi(t^*, s^*) - 1)e. \end{aligned}$$

Analogously, we get

$$y_{n+1} \succeq \varphi(\alpha_n, \beta_n)\varphi(t^*, s^*) A_2(u_n, y_n, v_n) + (\varphi(\alpha_n, \beta_n)\varphi(t^*, s^*) - 1)f.$$

It follows from the definition of r_n that

$$r_{n+1} \geq \varphi(\alpha_n, \beta_n)\varphi(t^*, s^*) > \frac{r_n}{r^*}\varphi(t^*, s^*).$$

If we take $n \to +\infty$, we get $r^* \succeq \varphi(t^*, s^*) > r^*$. This is also a contradiction. Consequently, $r^* = 1$.

Now, since P is a normal cone, we have

$$\begin{aligned} &\|x_{n+p} - x_n\| \leq N(1 - r_n)\|u_0 + e\|, \ \|u_n - u_{n+p}\| \leq N(1 - r_n)\|u_0 + e\|, \\ &\|y_{n+p} - y_n\| \leq N(1 - r_n)\|v_0 + f\|, \ \|v_n - v_{n+p}\| \leq N(1 - r_n)\|v_0 + f\|, \end{aligned}$$

where N is the normality constant. Taking $n \to +\infty$, we obtain that $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{v_n\}$ are Cauchy sequences in the Banach space E. Hence, it converge to $x^*, u^*, y^*, v^* \in E$, respectively. It follows by (3.5) that

$$x_0 \preceq x_n \preceq x^* \preceq u^* \preceq u_n \preceq u_0 \text{ and } y_0 \preceq y_n \preceq y^* \preceq v^* \preceq v_n \preceq v_0,$$

which implies that $x^*, u^* \in P_{h,e}$ and $y^*, v^* \in P_{k,f}$. Moreover,

$$\begin{aligned} \theta \prec u^* - x^* \preceq u_n - x_n \preceq (1 - r_n)(u_0 + e), \\ \theta \prec v^* - y^* \preceq v_n - y_n \preceq (1 - r_n)(v_0 + f). \end{aligned}$$

again, by the normality of P we get

$$\|u^* - x^*\| \le N(1-r_n)\|u_0 + e\| \text{ and } \|v^* - y^*\| \le N(1-r_n)\|v_0 + f\|.$$

Taking $n \to +\infty$, we obtain $u^* = x^*$ and $v^* = y^*$. Furthermore,

$$\begin{aligned} x_{n+1} &= A_1(x_n, u_n, y_n) \preceq A_1(x^*, x^*, y^*) \preceq A_1(u_n, x_n, v_n) = u_{n+1}, \\ y_{n+1} &= A_2(x_n, v_n, y_n) \preceq A_2(x^*, y^*, y^*) \preceq A_2(u_n, y_n, v_n) = v_{n+1}. \end{aligned}$$

Let $n \to +\infty$, we get $x^* = A_1(x^*, x^*, y^*)$ and $y^* = A_2(x^*, y^*, y^*)$. Thus (x^*, y^*) is a fixed point of Φ in $P_{h,e} \times P_{k,f}$.

Let us prove that (x^*, y^*) is the unique fixed point of the operator Φ in $P_{h,e} \times P_{k,f}$. Suppose that (u^*, v^*) is any fixed point of Φ in $P_{h,e} \times P_{k,f}$. Then by Lemma 2.3, there exists $r > 0$ such that

$$\begin{aligned} ru^* + (r-1)e \preceq x^* \preceq \frac{1}{r}u^* + (\frac{1}{r} - 1)e, \\ rv^* + (r-1)f \preceq y^* \preceq \frac{1}{r}v^* + (\frac{1}{r} - 1)f. \end{aligned} \tag{3.6}$$

Set

$$\overline{r} = \sup\{r > 0 \text{ such that (3.6) is satisfied}\}.$$

We claim that $\overline{r} \ge 1$. In deed, if $0 < \overline{r} < 1$, then by $(H_2(i))$ there exist $\alpha, \beta \in (a, b)$ such that $\tau_1(\alpha) = \overline{r} = \tau_2(\beta)$. Combining (3.6) with the mixed monotonicity of A_1 and A_2, we get

$$\begin{aligned} x^* &= A_1(x^*, x^*, y^*) \succeq \varphi(\alpha, \beta)A_1(u^*, u^*, v^*) + (\varphi(\alpha, \beta) - 1)e, \\ y^* &= A_2(x^*, y^*, y^*) \succeq \varphi(\alpha, \beta)A_2(u^*, v^*, v^*) + (\varphi(\alpha, \beta) - 1)f. \end{aligned}$$

From the definition of $\overline{r}$, we obtain $\overline{r} \ge \varphi(\alpha, \beta) > \overline{r}$, which is a contradiction. Thus $\overline{r} \ge 1$. It follows that $x^* \succeq \overline{r}u^* + (\overline{r} - 1)e \succeq \overline{r}u^* \succeq u^*$ and $y^* \succeq \overline{r}v^* + (\overline{r} - 1)f \succeq \overline{r}v^* \succeq v^*$. Also, in a similar way, we get $u^* \succeq x^*$ and $v^* \succeq y^*$. Consequently, $x^* = u^*$ and $y^* = v^*$.

4) For any initial points $x_0', u_0' \in P_{h,e}$ and $y_0', v_0' \in P_{k,f}$, construct successively sequences as in (3.3) and choose a small number $\eta_0 \in (0, 1)$ verifying

$$\begin{aligned} \eta_0 h + (\eta_0 - 1)e \preceq x_0', u_0' \preceq \frac{1}{\eta_0}h + (\frac{1}{\eta_0} - 1)e, \\ \eta_0 k + (\eta_0 - 1)f \preceq y_0', v_0' \preceq \frac{1}{\eta_0}k + (\frac{1}{\eta_0} - 1)f. \end{aligned}$$

Again, from $(H_2)(i)$, there exist $\mu_0, \nu_0 \in (a, b)$ such that $\tau_1(\mu_0) = \eta_0 = \tau_2(\nu_0)$. Take a sufficiently large positive integer ℓ satisfying

$$\Big(\frac{\varphi(\mu_0, \nu_0)}{\eta_0}\Big)^\ell \ge \frac{1}{\eta_0},$$

and Put

$$\begin{aligned} x_0'' = \eta_0^\ell h + (\eta_0^\ell - 1)e,\ u_0'' = \frac{1}{\eta_0^\ell}h + (\frac{1}{\eta_0^\ell} - 1)e, \\ y_0'' = \eta_0^\ell k + (\eta_0^\ell - 1)f,\ v_0'' = \frac{1}{\eta_0^\ell}k + (\frac{1}{\eta_0^\ell} - 1)f. \end{aligned}$$

It is clear that $x_0'', u_0'' \in P_{h,e}$ and $y_0'', v_0'' \in P_{k,f}$ with $x_0'' < x_0', u_0' < u_0''$ and $y_0'' < y_0', v_0' < v_0''$. Define the sequences

$$\begin{aligned} x_n'' &= A_1(x_{n-1}'', u_{n-1}'', y_{n-1}''), \quad y_n'' = A_2(x_{n-1}'', v_{n-1}'', y_{n-1}''), \\ u_n'' &= A_1(u_{n-1}'', x_{n-1}'', v_{n-1}''), \quad v_n'' = A_2(u_{n-1}'', y_{n-1}'', v_{n-1}''), \end{aligned} \quad n = 1, 2, ...,$$

repeating the same reasoning as in 3), we obtain the existence of $(x^{**}, y^{**}) \in P_{h,e} \times P_{k,f}$ such that $\Phi(x^{**}, x^{**}, y^{**}, y^{**}) = (x^{**}, y^{**})$, $\lim_{n\to\infty} x_n'' = \lim_{n\to\infty} u_n'' = x^{**}$ and $\lim_{n\to\infty} y_n'' = \lim_{n\to\infty} v_n'' = y^{**}$.

By the uniqueness of fixed point of the operator Φ in $P_{h,e} \times P_{k,f}$, we have $x^* = x^{**}$ and $y^* = y^{**}$. Moreover, by induction we get $x''_n \preceq x'_n, u'_n \preceq u''_n$ and $y''_n \preceq y'_n, v'_n \preceq v''_n$, for $n = 1, 2, ...$ Finally, by the normality of the cone P we deduce that $\lim_{n\to\infty} x'_n = \lim_{n\to\infty} u'_n = x^*$ and $\lim_{n\to\infty} y'_n = \lim_{n\to\infty} v'_n = y^*$. □

In the following, we use the above theorem to prove the existence and uniqueness of solution to the operators system (1.2). Choose $h, k \in P$ such that $h \neq \theta$, $k \neq \theta$ and take $e, f \in P$ with $\theta \preceq e \preceq h$, $\theta \preceq f \preceq k$.

Theorem 3.2. *Let P be a normal cone in a real Banach space E. Let $A_1 : P_{h,e} \times P_{h,e} \times P_{k,f} \to E$ and $A_2 : P_{h,e} \times P_{k,f} \times P_{k,f} \to E$ be two operators such that the vector operator $\Phi = (A_1, A_2) : P_{h,e} \times P_{k,f} \times P_{h,e} \times P_{k,f} \to E \times E$ is cooperative mixed monotone. Let $B_1 : P_{h,e} \times P_{h,e} \to E$ and $B_2 : P_{k,f} \times P_{k,f} \to E$ be two mixed monotone operators. Suppose that*

(S_1) *$A_1(h,h,k), B_1(h,h) \in P_{h,e}$ and $A_2(h,k,k), B_2(k,k) \in P_{k,f}$;*

(S_2) *There exist positive-value τ_1, τ_2 on interval (a,b), ψ on $(a,b) \times (a,b)$ such that*

- *(i) $\tau_1, \tau_2 : (a,b) \to (0,1)$ are surjections.*
- *(ii) $1 > \psi(\mu,\nu) > \min\{\tau_1(\mu), \tau_2(\nu)\}$, for all $\mu, \nu \in (a,b)$.*
- *(iii) For any $x, u \in P_{h,e}$, for any $y, v \in P_{k,f}$ and any $\mu, \nu \in (a,b)$*

$$A_1\Big(\tau_1(\mu)x + (\tau_1(\mu) - 1)e, \frac{1}{\tau_1(\mu)}u + (\frac{1}{\tau_1(\mu)} - 1)e, \tau_2(\nu)y + (\tau_2(\nu) - 1)f\Big)$$
$$\succeq \psi(\mu,\nu)A_1(x,u,y) + (\psi(\mu,\nu) - 1)e,$$
$$A_2\Big(\tau_1(\mu)x + (\tau_1(\mu) - 1)e, \frac{1}{\tau_2(\nu)}v + (\frac{1}{\tau_2(\nu)} - 1)f, \tau_2(\nu)y + (\tau_2(\nu) - 1)f\Big)$$
$$\succeq \psi(\mu,\nu)A_2(x,v,y) + (\psi(\mu,\nu) - 1)f.$$

(S_3) *For any $x, u \in P_{h,e}$, for any $y, v \in P_{k,f}$ and any $\mu \in (0,1)$*

$$B_1\Big(\mu x + (\mu - 1)e, \frac{1}{\mu}u + (\frac{1}{\mu} - 1)e\Big) \succeq \mu B_1(x,u) + (\mu - 1)e,$$
$$B_2\Big(\mu y + (\mu - 1)f, \frac{1}{\mu}v + (\frac{1}{\mu} - 1)f\Big) \succeq \mu B_2(y,v) + (\mu - 1)f.$$

(S_4) *For all $x, u \in P_{h,e}$ and all $y, v \in P_{k,f}$, there exist constants $R_1, R_2 > 0$ such that*

$$A_1(x,u,y) \succeq R_1 B_1(x,u) + (R_1 - 1)e,$$
$$A_2(x,v,y) \succeq R_2 B_2(y,v) + (R_2 - 1)f.$$

Then the operators system (1.2) has a unique solution $(x^*, y^*) \in P_{h,e} \times P_{k,f}$. Moreover, for any initial values $x_0, u_0 \in P_{h,e}$ and $y_0, v_0 \in P_{k,f}$, by constructing successively the sequences

$$\begin{aligned} x_n &= A_1(x_{n-1}, u_{n-1}, y_{n-1}), \quad y_n = A_2(x_{n-1}, v_{n-1}, y_{n-1}), \\ u_n &= A_1(u_{n-1}, x_{n-1}, v_{n-1}), \quad v_n = A_2(u_{n-1}, y_{n-1}, v_{n-1}), \end{aligned} \quad n = 1, 2, ..., \tag{3.7}$$

we have $\|x_n - x^*\| \longrightarrow 0$, $\|u_n - x^*\| \longrightarrow 0$ and $\|y_n - y^*\| \longrightarrow 0$, $\|v_n - y^*\| \longrightarrow 0$ (as $n \longrightarrow \infty$).

Proof. Let $\Phi_1 : P_{h,e} \times P_{h,e} \times P_{k,f} \to E$ and $\Phi_2 : P_{h,e} \times P_{k,f} \times P_{k,f} \to E$ be two operators defined by

$$\Phi_1(x,u,y) = A_1(x,u,y) + B_1(x,u) + e,$$
$$\Phi_2(x,v,y) = A_2(x,v,y) + B_2(y,v) + f,$$

for any $x, u \in P_{h,e}$ and any $y, v \in P_{k,f}$. Consider the vector operator $\Phi : P_{h,e} \times P_{k,f} \times P_{h,e} \times P_{k,f} \to E \times E$ defined by $\Phi(x,y,u,v) = (\Phi_1(x,u,y), \Phi_2(x,v,y))$. Then, using the mixed monotone properties of the operators A_1, A_2, B_1 and B_2, it is easy to see that Φ is a cooperative mixed monotone vector operator. Moreover, since $A_1(h,h,k), B_1(h,h) \in P_{h,e}$ we have $\Phi_1(h,h,k) \in P_{h,e}$, and since $A_2(h,k,k), B_2(k,k) \in P_{k,f}$ we have $\Phi_2(h,k,k) \in P_{k,f}$.

Next, from (S_4) we have, for any $x, u \in P_{h,e}$ and any $y, v \in P_{k,f}$

$$\begin{aligned} A_1(x,u,y) + R_1A_1(x,u,y) + R_1e &\succeq R_1B_1(x,u) + (R_1-1)e \\ &+ R_1A_1(x,u,y) + R_1e, \\ A_2(x,v,y) + R_2A_2(x,v,y) + R_2f &\succeq R_2B_2(y,v) + (R_2-1)f \\ &+ R_2A_2(x,v,y) + R_2f. \end{aligned}$$

Therefore,

$$\begin{aligned} A_1(x,u,y) &\succeq \frac{R_1}{1+R_1}\Phi_1(x,u,y) - \frac{e}{1+R_1}, \\ A_2(x,v,y) &\succeq \frac{R_2}{1+R_2}\Phi_2(x,v,y) - \frac{f}{1+R_2}. \end{aligned} \tag{3.8}$$

Using (S_2), (S_3) and (3.8), we have for any $x, u \in P_{h,e}$, for any $y, v \in P_{k,f}$ and for any $\mu, \nu \in (a,b)$

$$\begin{aligned} &\Phi_1\Big(\tau_1(\mu)x + (\tau_1(\mu)-1)e, \frac{1}{\tau_1(\mu)}u + (\frac{1}{\tau_1(\mu)} - 1)e, \tau_2(\nu)y + (\tau_2(\nu)-1)f\Big) \\ &- \tau_1(\mu)\Phi_1(x,u,y) \\ &= A_1\Big(\tau_1(\mu)x + (\tau_1(\mu)-1)e, \frac{1}{\tau_1(\mu)}u + (\frac{1}{\tau_1(\mu)} - 1)e, \tau_2(\nu)y + (\tau_2(\nu)-1)f\Big) \\ &+ B_1\Big(\tau_1(\mu)x + (\tau_1(\mu)-1)e, \frac{1}{\tau_1(\mu)}u + (\frac{1}{\tau_1(\mu)} - 1)e\Big) + e \\ &- \tau_1(\mu)(A_1(x,u,y) + B_1(x,u) + e) \\ &\succeq \psi(\mu,\nu)A_1(x,u,y) + (\psi(\mu,\nu)-1)e + \tau_1(\mu)B_1(x,u), \\ &+ (\tau_1(\mu)-1)e + e - \tau_1(\mu)A_1(x,u,y) - \tau_1(\mu)B_1(x,u) - \tau_1(\mu)e \\ &= (\psi(\mu,\nu) - \tau_1(\mu))A_1(x,u,y) + (\psi(\mu,\nu)-1)e \\ &\succeq (\psi(\mu,\nu) - \tau_1(\mu))\Big(\frac{R_1}{1+R_1}\Phi_1(x,u,y) - \frac{e}{1+R_1}\Big) + (\psi(\mu,\nu)-1)e \\ &= \frac{R_1(\psi(\mu,\nu) - \tau_1(\mu))}{1+R_1}\Phi_1(x,u,y) + \Big(\psi(\mu,\nu) - 1 - \frac{\psi(\mu,\nu) - \tau_1(\mu)}{1+R_1}\Big)e. \end{aligned}$$

It follows that

$$\begin{aligned} &\Phi_1\Big(\tau_1(\mu)x + (\tau_1(\mu)-1)e, \frac{1}{\tau_1(\mu)}u + (\frac{1}{\tau_1(\mu)} - 1)e, \tau_2(\nu)y + (\tau_2(\nu)-1)f\Big) \\ &\succeq \Big(\frac{R_1(\psi(\mu,\nu) - \tau_1(\mu))}{1+R_1} + \tau_1(\mu)\Big)\Phi_1(x,u,y) + \Big(\psi(\mu,\nu) - 1 - \frac{\psi(\mu,\nu) - \tau_1(\mu)}{1+R_1}\Big)e \\ &= \frac{R_1\psi(\mu,\nu) + \tau_1(\mu)}{1+R_1}\Phi_1(x,u,y) + \Big(\frac{R_1\psi(\mu,\nu) + \tau_1(\mu)}{1+R_1} - 1\Big)e. \end{aligned}$$

In the same way we obtain

$$\begin{aligned} &\Phi_2\Big(\tau_1(\mu)x + (\tau_1(\mu)-1)e, \frac{1}{\tau_2(\nu)}v + (\frac{1}{\tau_2(\nu)} - 1)f, \tau_2(\nu)y + (\tau_2(\nu)-1)f\Big) \\ &\succeq \frac{R_2\psi(\mu,\nu) + \tau_2(\nu)}{1+R_2}\Phi_2(x,v,y) + \Big(\frac{R_2\psi(\mu,\nu) + \tau_2(\nu)}{1+R_2} - 1\Big)f. \end{aligned}$$

Define the function $\varphi(\mu,\nu) = \min\{\frac{R_1\psi(\mu,\nu)+\tau_1(\mu)}{1+R_1}, \frac{R_2\psi(\mu,\nu)+\tau_2(\nu)}{1+R_2}\}$. Then φ satisfies the hypothesis (H_2) in Theorem 3.1. Consequently, all hypotheses of Theorem 3.1 hold. So we obtain the conclusion of Theorem 3.2. □

4 Application to systems of fractional differential equations

Recently, the theory of fractional calculus, notably fractional differential equations, have been of great interest to many researchers because of its wide range of applications in various fields, such as physics, mechanics, engineering, biology, etc. We refer to, e.g., [6, 11, 12, 13, 18, 19, 23] and references therein for studies on fractional differential equations. A main motivation for this last section is the lower number of papers concerned with systems of fractional differential equations. We shall apply Theorem 3.2 in section 3 to prove the existence and uniqueness of a solution for the system (1.3). For the sake of convenience, we begin by giving some definitions and well known results concerning our problem. For more details, we refer to [14, 16, 17, 27, 29, 30, 31, 35].

Let $E = C[0,1]$ be the Banach space of continuous, real-valued functions on the unit interval $[0,1]$ with the standard norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Consider the set

$$P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\}.$$

Then, it is easy to show that P is a normal cone in the space $C[0,1]$ of which the normality constant is 1.

Definition 4.1 ([13]). Let $\alpha > 0$ be a real number and $x : (0,+\infty) \to \mathbb{R}$ be a continuous function. The Riemann-Liouville derivative of order α is defined as

$$D^{\alpha}_{0^+}x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}} ds$$

and the Riemann-Liouville integral of order α is defined as

$$I^{\alpha}_{0^+}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α. $\Gamma(\alpha)$ is the Euler Gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Lemma 4.2 ([7]). *Let $f \in C([0,1])$ be given. Then the fractional boundary value problem*

$$\begin{cases} -D^{\alpha}_{0^+}x(t) = f(t), \ 0 < t < 1, \ n-1 < \alpha < n, \\ x^{(i)}(0) = 0, \ 0 \leq i \leq n-2, \\ [D^{\beta}_{0^+}x(t)]_{t=1} = 0, \ 1 \leq \beta \leq n-2 \end{cases} \tag{4.1}$$

has a unique solution

$$x(t) = \int_0^1 G_{\alpha,\beta}(t,s) f(s) ds,$$

where

$$G_{\alpha,\beta}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, \ 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \ 0 \leq t \leq s \leq 1 \end{cases} \tag{4.2}$$

is the Green's function for this problem.

Lemma 4.3 ([7, 31]). *The Green's function $G_{\alpha,\beta}(t,s)$ in Lemma 4.2 has the following properties.*

(i) $G_{\alpha,\beta}(t,s)$ *is a continuous fonction on* $[0,1] \times [0,1]$*;*

(ii) $G_{\alpha,\beta}(t,s) \geq 0$ *for each* $(t,s) \in [0,1] \times [0,1]$*;*

(iii) For all $t, s \in [0,1]$*,*

$$[1-(1-s)^{\beta}](1-s)^{\alpha-\beta-1}t^{\alpha-1} \leq \Gamma(\alpha)G_{\alpha,\beta}(t,s) \leq (1-s)^{\alpha-\beta-1}t^{\alpha-1};$$

(iv) $\max_{t\in[0,1]} G_{\alpha,\beta}(t,s) = G_{\alpha,\beta}(1,s)$*, for each* $s \in [0,1]$.

Next, regarding problem (1.3) we define the functions

$$e(t) = a\int_0^1 G_{\alpha,\gamma}(t,s)ds = \frac{a}{(\alpha-\gamma)\Gamma(\alpha)}\Big(t^{\alpha-1} - \frac{\alpha-\gamma}{\alpha}t^{\alpha}\Big), t \in [0,1],$$

$$f(t) = b\int_0^1 G_{\beta,\delta}(t,s)ds = \frac{b}{(\beta-\delta)\Gamma(\beta)}\Big(t^{\beta-1} - \frac{\beta-\delta}{\beta}t^{\beta}\Big), t \in [0,1]$$

and put $e^* = \max\{e(t) : t \in [0,1]\}$, $f^* = \max\{f(t) : t \in [0,1]\}$. It is obvious that $e(t) \geq 0$ and $f(t) \geq 0$ for all $t \in [0,1]$. Furthermore, we denote $h(t) = L_1 t^{\alpha-1}$ and $k(t) = L_2 t^{\beta-1}$ for all $t \in [0,1]$ with $L_1 \geq \frac{a}{(\alpha-\gamma)\Gamma(\alpha)}$ and $L_2 \geq \frac{b}{(\beta-\delta)\Gamma(\beta)}$. Then we have

$$e(t) \leq \frac{a}{(\alpha-\gamma)\Gamma(\alpha)}t^{\alpha-1} \leq L_1 t^{\alpha-1} = h(t),$$

$$f(t) \leq \frac{b}{(\beta-\delta)\Gamma(\beta)}t^{\beta-1} \leq L_2 t^{\beta-1} = k(t).$$

So we get $P_{h,e} = \{x \in E : x + e \in P_h\}$ and $P_{k,f} = \{x \in E : x + f \in P_k\}$.

Now, we are ready to present and prove the main result in this section.

Theorem 4.4. *Assume that $F_1(t,x,y) = f_1(t,x,x,y)$, $F_2(t,x,y) = f_2(t,x,y,y)$, $G_1(t,x) = g_1(t,x,x)$, $G_2(t,y) = g_2(t,y,y)$, such that f_1, f_2, g_1, g_2 are functions satisfying the following hypotheses.*

(C_1) *i)* $f_1 : [0,1] \times [-e^*,+\infty) \times [-e^*,+\infty) \times [-f^*,+\infty) \longrightarrow (-\infty,+\infty)$;
ii) $f_2 : [0,1] \times [-e^*,+\infty) \times [-f^*,+\infty) \times [-f^*,+\infty) \longrightarrow (-\infty,+\infty)$;
iii) $g_1 : [0,1] \times [-e^*,+\infty) \times [-e^*,+\infty) \longrightarrow (-\infty,+\infty)$;
iv) $g_2 : [0,1] \times [-f^*,+\infty) \times [-f^*,+\infty) \longrightarrow (-\infty,+\infty)$

are continuous functions. In addition, for all $t \in [0,1]$, $g_i(t,0,L_i) \geq 0$ and $\mathring{A}_i = \varnothing$, where $A_i = \{t \in [0,1] : g_i(t,0,L_i) = 0\}$ and $i = 1,2$.

(C_2) *For all $t \in [0,1]$, all $x,u \in [-e^*,+\infty)$ and all $y,v \in [-f^*,+\infty)$, the functions $f_1(t,.,u,y)$, $f_1(t,x,u,.)$, $f_2(t,.,u,y)$, $f_2(t,x,u,.)$, $g_1(t,.,u)$, $g_2(t,.,v)$ are increasing and the functions $f_1(t,x,.,y)$, $f_2(t,x,.,y)$, $g_1(t,x,.)$, $g_2(t,y,.)$ are decreasing.*

(C_3) *There exist positive-value functions τ_1, τ_2 on $(0,1)$ and ϕ on $(0,1)\times(0,1)$ such that*

(i) $\tau_1, \tau_2 : (0,1) \to (0,1)$ *are surjections.*

(ii) $1 > \phi(\mu,\nu) > \min\{\tau_1(\mu), \tau_2(\nu)\}$, *For all $\mu,\nu \in (0,1)$.*

(iii) For all $x,u,y \in (-\infty,+\infty)$, for all $t \in [0,1]$ and all $\mu,\nu \in (0,1)$,

$$f_1\Big(t, \tau_1(\mu)x + (\tau_1(\mu)-1)\rho_1, \frac{1}{\tau_1(\mu)}u + (\frac{1}{\tau_1(\mu)} - 1)\rho_2, \tau_2(\nu)y$$
$$+ (\tau_2(\nu)-1)\sigma_1\Big) \geq \phi(\mu,\nu)f_1(t,x,u,y),$$

$$f_2\Big(t, \tau_1(\mu)x + (\tau_1(\mu)-1)\rho_3, \frac{1}{\tau_2(\nu)}u + (\frac{1}{\tau_2(\nu)} - 1)\sigma_2, \tau_2(\nu)y$$
$$+ (\tau_2(\nu)-1)\sigma_3\Big) \geq \phi(\mu,\nu)f_2(t,x,u,y),$$

$$g_1\Big(t, \mu x + (\mu-1)\rho_4, \frac{1}{\mu}u + (\frac{1}{\mu} - 1)\rho_5\Big) \geq \mu g_1(t,x,u),$$

$$g_2\Big(t, \mu x + (\mu-1)\sigma_4, \frac{1}{\mu}u + (\frac{1}{\mu} - 1)\sigma_5\Big) \geq \mu g_2(t,x,u),$$

where $\rho_i \in [0,e^*]$, $\sigma_i \in [0,f^*]$ with $i \in \{1,2,3,4,5\}$.

(C_4) *For all $t \in [0,1]$, for all $x,u \in [-e^*,+\infty)$ and all $y,v \in [-f^*,+\infty)$, there exist two constants $R_1, R_2 > 0$ such that*

$$f_1(t,x,u,y) \geq R_1.g_1(t,x,u) \text{ and } f_2(t,x,v,y) \geq R_2.g_2(t,y,v).$$

Then problem (1.3) has a unique solution (x^*, y^*) *in* $P_{h,e} \times P_{k,f}$*. Moreover, for any* $x_0, u_0 \in P_{h,e}$ *and any* $y_0, v_0 \in P_{k,f}$*, if we construct the sequences*

$$\begin{aligned}
x_{n+1}(t) &= \int_0^1 G_{\alpha,\gamma}(t,s)[f_1(s, x_n(s), u_n(s), y_n(s)) \\
&+ g_1(s, x_n(s), u_n(s))]ds - e(t), \\
u_{n+1}(t) &= \int_0^1 G_{\alpha,\gamma}(t,s)[f_1(s, u_n(s), x_n(s), v_n(s)) \\
&+ g_1(s, u_n(s), x_n(s))]ds - e(t), \\
y_{n+1}(t) &= \int_0^1 G_{\beta,\delta}(t,s)[f_2(s, x_n(s), v_n(s), y_n(s)) \\
&+ g_2(s, y_n(s), v_n(s))ds - f(t), \\
v_{n+1}(t) &= \int_0^1 G_{\beta,\delta}(t,s)[f_2(s, u_n(s), y_n(s), v_n(s)) \\
&+ g_2(s, v_n(s), y_n(s))ds - f(t),
\end{aligned}$$

for $n \in \mathbb{N}$*, we get* $\{x_n(t)\}$*,* $\{u_n(t)\}$ *both converge to* $x^*(t)$ *and* $\{y_n(t)\}$*,* $\{v_n(t)\}$ *both converge to* $y^*(t)$*, uniformly for all* $t \in [0, 1]$*.*

Proof. We will prove that all hypotheses of Theorem 3.2 are satisfied for suitable operators.

1) By Lemma 4.2, proplem (1.3) has the integral formulation

$$x(t) = \int_0^1 G_{\alpha,\gamma}(t,s)[f_1(s, x(s), x(s), y(s)) + g_1(s, x(s), x(s)) - a]ds,$$
$$y(t) = \int_0^1 G_{\beta,\delta}(t,s)[f_2(s, x(s), y(s), y(s)) + g_2(s, y(s), y(s)) - b]ds.$$

which gives by a simple calculation

$$\begin{aligned}
x(t) &= \int_0^1 G_{\alpha,\gamma}(t,s) f_1(s, x(s), x(s), y(s))ds - e(t) \\
&+ \int_0^1 G_{\alpha,\gamma}(t,s) g_1(s, x(s), x(s))ds - e(t) + e(t), \\
y(t) &= \int_0^1 G_{\beta,\delta}(t,s) f_2(s, x(s), y(s), y(s))ds - f(t) \\
&+ \int_0^1 G_{\beta,\delta}(t,s) g_2(s, y(s), y(s))ds - f(t) + f(t).
\end{aligned}$$

For every $x, u \in P_{h,e}$, every $y, v \in P_{k,f}$ and every $t \in [0, 1]$, define the operators

$$\begin{aligned}
A_1(x, u, y)(t) &= \int_0^1 G_{\alpha,\gamma}(t,s) f_1(s, x(s), u(s), y(s))ds - e(t), \\
A_2(x, v, y)(t) &= \int_0^1 G_{\beta,\delta}(t,s) f_2(s, x(s), v(s), y(s))ds - f(t), \\
B_1(x, u)(t) &= \int_0^1 G_{\alpha,\gamma}(t,s) g_1(s, x(s), u(s))ds - e(t)
\end{aligned}$$

and

$$B_2(y, v)(t) = \int_0^1 G_{\beta,\delta}(t,s) g_2(s, y(s), v(s))ds - f(t).$$

Then, it is not difficult to see that (x,y) is the solution of system (1.3) if and only if $x = A_1(x,x,y)+B_1(x,x)+e$ and $y = A_2(x,y,y)+B_2(y,y)+f$, that is, (x,y) is a fixed point of the vector operatoe $\Psi = (A_1+B_1+e, A_2+B_2+f)$ in $P_{h,e}\times P_{k,f}$.

2) From (C_2), we get easily that $\Phi = (A_1, A_2)$ is a cooperative mixed monotone operator and B_1, B_2 are mixed monotone.

3) We show that (S_1) in Theorem 3.2 is satiseied. Using Lemma 4.2, Lemma 4.3 and (C_2), we obtain

$$\begin{aligned} A_1(h,h,k)(t)+e(t) &= \int_0^1 G_{\alpha,\gamma}(t,s)f_1(s,h(s),h(s),k(s))ds \\ &\geq \frac{h(t)}{L_1\Gamma(\alpha)}\int_0^1 (1-(1-s)^\alpha)(1-s)^{\alpha-\gamma-1}f_1(s,0,L_1,0)ds, \\ B_1(h,h)(t)+e(t) &= \int_0^1 G_{\alpha,\gamma}(t,s)g_1(s,h(s),h(s))ds \\ &\geq \frac{h(t)}{L_1\Gamma(\alpha)}\int_0^1 (1-(1-s)^\alpha)(1-s)^{\alpha-\gamma-1}g_1(s,0,L_1)ds, \\ A_1(h,h,k)(t)+e(t) &= \int_0^1 G_{\alpha,\gamma}(t,s)f_1(s,h(s),h(s),k(s))ds \\ &\leq \frac{h(t)}{L_1\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}f_1(s,L_1,0,L_2)ds \end{aligned}$$

and

$$\begin{aligned} B_1(h,h)(t)+e(t) &= \int_0^1 G_{\alpha,\gamma}(t,s)g_1(s,h(s),h(s))ds \\ &\leq \frac{h(t)}{L_1\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}g_1(s,L_1,0)ds, \end{aligned}$$

Furthermore, from (C_2) and (C_4)

$$f_1(s,L_1,0,L_2) \geq f_1(s,0,L_1,0) \geq R_1 g_1(s,0,L_1),\ \ s\in[0,1],$$

and by (C_1), we get

$$\int_0^1 f_1(s,L_1,0,L_2)ds \geq \int_0^1 f_1(s,0,L_1,0)ds \geq \int_0^1 R_1 g_1(s,0,L_1)ds > 0.$$

It follows, since $\alpha>\gamma$, $\beta>\delta$, $\Gamma(\alpha)>0$ and $\Gamma(\beta)>0$, that

$$\begin{aligned} M_1 &= \frac{1}{L_1\Gamma(\alpha)}\int_0^1 (1-(1-s)^\alpha)(1-s)^{\alpha-\gamma-1}f_1(s,0,L_1,0)ds > 0, \\ N_1 &= \frac{1}{L_1\Gamma(\alpha)}\int_0^1 (1-(1-s)^\alpha)(1-s)^{\alpha-\gamma-1}g_1(s,0,L_1)ds > 0, \\ M_2 &= \frac{1}{L_1\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}f_1(s,L_1,0,L_2)ds > 0, \\ N_2 &= \frac{1}{L_1\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\gamma-1}g_1(s,L_1,0)ds > 0. \end{aligned}$$

Thus, $M_1h(t) \leq A_1(h,h,k)(t)+e(t) \leq M_2h(t)$ and $N_1h(t) \leq B_1(h,h)(t)+e(t) \leq N_2h(t)$, for all $t\in[0,1]$, which means that $A_1(h,h,k), B_1(h,h) \in P_{h,e}$.

In a similar way we obtain $A_2(h,k,k), B_2(k,k)\in P_{k,f}$.

4) By using the same reasoning as in the proofs of [21, Theorem 3.1] and [22, Theorem 5.1], we show that (S_2) and (S_3) are satisfied from (C_3), and (S_4) is satisfied from (C_4). The proof is complete. □

Example 4.5. Take $n = 5, \alpha = \frac{9}{2}, \beta = \frac{13}{3}, \gamma = \frac{5}{2}, \delta = \frac{7}{3}, a = 2$ and $b = 4$. Consider the system of boundary value problems

$$\begin{cases} D_{0^+}^{\frac{9}{2}} x(t) + f_1(t, x(t), x(t), y(t)) + g_1(t, x(t), x(t)) - 2 = 0, \\ D_{0^+}^{\frac{13}{3}} y(t) + f_2(t, x(t), y(t), y(t)) + g_2(t, y(t), y(t)) - 4 = 0, \\ x^{(i)}(0) = 0 = y^{(i)}(0), \ i = 0, 1, 2, 3, \\ [D_{0^+}^{\frac{5}{2}} x(t)]_{t=1} = 0 = [D_{0^+}^{\frac{7}{3}} y(t)]_{t=1}, \end{cases} \tag{4.3}$$

where for $t \in [0, 1]$,

$$f_1(t, x, u, y) = \Big(\frac{e(t)}{e^*}x + \frac{f(t)}{f^*}y + e(t) + f(t)\Big)^{\frac{1}{2}} + \Big(\frac{e(t)}{e^*}u + e(t) + 1\Big)^{-\frac{1}{2}},$$

$$f_2(t, x, v, y) = \Big(\frac{e(t)}{e^*}x + \frac{f(t)}{f^*}y + e(t) + f(t)\Big)^{\frac{1}{3}} + \Big(\frac{f(t)}{f^*}v + e(t) + 1\Big)^{-\frac{1}{3}},$$

$$g_1(t, x, u) = (u + e(t) + 2)^{-\frac{1}{2}} \text{ and } g_2(t, y, v) = (v + f(t) + 2)^{-\frac{1}{3}},$$

with $e(t) = \dfrac{t^{\frac{7}{2}} - \frac{4}{9}t^{\frac{9}{2}}}{\Gamma(\frac{9}{2})}$ and $f(t) = \dfrac{2t^{\frac{10}{3}} - \frac{12}{13}t^{\frac{13}{3}}}{\Gamma(\frac{13}{3})}$, for all $t \in [0, 1]$. Hence, we have

$$e^* = \frac{5}{9\Gamma(\frac{9}{2})}, \ f^* = \frac{14}{13\Gamma(\frac{13}{3})}, \ L_1 = \frac{1}{\Gamma(\frac{9}{2})} \text{ and } L_2 = \frac{2}{\Gamma(\frac{13}{3})}.$$

In addition, for any surjective functions $\tau_1, \tau_2 : (0, 1) \to (0, 1)$, $\phi(\mu, \nu) = \Big(\min\{\tau_1(\mu), \tau_2(\nu)\}\Big)^{\frac{1}{2}}$, for all $\mu, \nu \in (0, 1)$.

Therefore, all hypotheses of Theorem 4.4 are satisfied. Thus, system (4.3) has a unique solution (x^*, y^*) in $P_{h,e} \times P_{k,f}$, where $h(t) = L_1 t^{\frac{7}{2}}$ and $k(t) = L_2 t^{\frac{10}{3}}$, for all $t \in [0, 1]$.

References

[1] H. Amann; *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Funct. Anal. 11, (1972), 346-384.

[2] V. Berinde and M. Borcut; *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. 74 (2011), 4889-4897.

[3] T. G. Bhaskar and V. Lakshmikantham; *Fixed point theorems in partially ordered metric spaces and applications]*, Nonlinear Anal. 65 (2006), 1379-1393.

[4] Y. Chen; *Thompson's metric and mixed monotone operators*, J. Math. Anal. Appl. 177 (1993), 31-37.

[5] K. Deimling; *Nonlinear Functional Analysis* Springer, Berlin (1985).

[6] L. Gaul, P. Klein and S. Kempffe; *Damping description involving fractional operators*, Mech. Syst. Signal Process. 5 (1991), 81-88.

[7] C. S. Goodrich; *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett. 23 (2010), 1050-1055.

[8] D. Guo and V. Lakshmikantham; *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. 11 (1987), 623-632.

[9] D. Guo; *Fixed points of mixed monotone operators with application*, Appl. Anal. 34 (1988), 215-224.

[10] D. Guo and V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, New York (1988).

[11] S. Ibrahim and S. Adil Murad; *Solution of fractional differential equations with some existence and stability results*, Palestine Polytechnic University-PPU 2023, Vol. 12(1)(2023) , 482-492.

[12] Y. D. Ji, Y. P. Guo, J. Q. Qiu and L. Y. Yang; *Existence of positive solutions for a boundary value problem of nonlinear fractional differential equations*, Adv. Differ. Equ. 2015, 13 (2015).

[13] V. Lakshmikantham and A.S. Vatsala; *Basic theory of fractional differential equations*, Nonlinear Anal. TMA 69 (2008) 2677-2682.

[14] H. Li, L. Liu, Y. Wu; *Positive solutions for singular nonlinear fractional differential equation with integral boundary conditions]*, Bound. Value Probl., 2015 (2015), 15.

[15] X. Lian and Y. Li; *Fixed point theorems for a class of mixed monotone operators with applications*, Nonlinear Anal. 67 (2007), 2752-2762.

[16] S. Liang and J. Zhang; *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Anal., 71 (2009), 5545-5550.

[17] L. Liu, X. Zhang, J. Jiang and Y. Wu; *The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems*,J. Nonlinear Sci. Appl. 9 (2016), 2943-2958.

[18] K. S. Miller and B. Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).

[19] KB. . Oldham and J. Spanier; *The Fractional Calculus*, Academic Press, New York (1974).

[20] A. Sadratit and M. D. Aouragh; *Fixed Point Theorems for Mixed Monotone Vector Operators with Application to Systems of Nonlinear Boundary Value Problems*, KYUNGPOOK Math. J. 61(2021), 613-629.

[21] Y. B. Sang, L. He, Y. Wang, Y. Ren and N. Shi; *Existence of positive solutions for a class of fractional differential equations with the derivative term via a new fixed point theorem*, Advances in Difference Equations (2021) 2021 (2021),156.

[22] Y. B. Sang and Y. Ren; *Nonlinear sum operator equations and applications to elastic beam equation and fractional differential equation*, Bound. Value Probl. 2019 (2019), 49.

[23] H.G. Schuster; *Reviews of Nonlinear Dynamics and Complexity]*, Wiley-VCH, Weinheim, 2008.

[24] V. Singh Chouhan and R. Sharma; *Coupled fixed point results involving altering distances in partially ordered metric spaces with applications to integral equations*, Palestine Polytechnic University-PPU 2017, Vol. 6(2)(2017) , 419-433.

[25] H. Wang, L. L. Zhang and X. Q. Wang; *Fixed point theorems for a class of nonlinear sum-type operators and application in a fractional differential equation*, Bound. Value Probl. 2018 (2018), 140.

[26] H. Wang and L. Zhang; *The solution for a class of sum operator equation and its application to fractional differential equation boundary value problems*, Bound. Value Probl. 2015 (2015), 203.

[27] Y. Wang, L. Liu and Y. Wu; *Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters*, Adv. Difference Equ., 2014 (2014), 24.

[28] D. Wardowski; *Mixed monotone operators and their application to integral equations*, J. Fixed Point Theory Appl. 19 (2017), 1103-1117.

[29] X. Xu, D. Jiang and C. Yuan; *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation]*, Nonlinear Anal., 71 (2009), 4676-4688.

[30] C. Zhai, W. Yan and C. Yang; *A sum operator method for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems*, Commun. Nonlinear Sci. Numer. Simul., 18 (2013), 858-866.

[31] C. Zhai and M. Hao; *Mixed monotone operator methods for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems*, Bound. Value Probl., 2013 (2013), 13.

[32] C. Zhai and L. Zhang; *New fixed point theorems for mixed monotone operators and local existence–uniqueness of positive solutions for nonlinear boundary value problems*, J. Math. Anal. Appl., 382 (2011), 594-614.

[33] C. B. Zhai and L. Wang; *ϕ-(h, e)-concave operators and applications*, J. Math. Anal. Appl. 454 (2017), 571-584.

[34] L. L. Zhang and H. M. Tian; *Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations]*, Adv. Differ. Equ. 2017, 114 (2017).

[35] X. Zhang, L. Liu and Y. Wu; *The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium*, Appl. Math. Lett., 37 (2014), 26-33.

[36] Z. Zhao; *Existence and uniqueness of fixed points for some mixed monotone operators*, Nonlinear Anal. 73 (2010), 1481-1490.

Author information

Abdelhak El Haddouchi, LMPA Laboratory, M2A Group, Department of Mathematics, FST, Errachidia, University Moulay Ismail of Meknes, B.P: 509
Boutalamine, 52000, Errachidia, Morocco.
E-mail: abdelhak.elhaddouchi@gmail.com

Abdellatif Sadrati, LMPA Laboratory, MASD Group, Department of Mathematics, FST, Errachidia, University Moulay Ismail of Meknes, B.P: 509
Boutalamine, 52000, Errachidia, Morocco.
E-mail: abdo2sadrati@gmail.com

Received: 2024-03-22
Accepted: 2024-05-14