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On the Convergence of Random Fourier–Jacobi Series in $L^{p,(\eta,\tau)}_{[-1,1]}$ space

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Abstract Liu and Liu introduced the random Fourier transform, which is a random Fourier series in Hermite functions, and applied it to image encryption and decryption. They expected its applications in optics and information technology, which motivated us to investigate random Fourier series in orthogonal polynomials. In this article, the different modes of convergence of random Fourier–Jacobi series $\sum_{n=0}^{\infty} d_n r_n(\omega)\varphi_n(y)$ is discussed, where $r_n(\omega)$ are random variables and d_n are scalars. The $r_n(\omega)$ are associated with continuous stochastic processes, such as symmetric stable process and Wiener process. The $\varphi_n(y)$ are viewed as the Jacobi polynomials or variants of these polynomials depending upon the selection of random variables associated with the stochastic processes. The scalars d_n are the Fourier–Jacobi coefficients of a function f in $L^{p,(\eta,\tau)}_{[-1,1]}$, $p \geq 1$ space. It is observed that the mode of convergence of the random series depends on the choice of the scalars d_n and the stochastic processes. Further, the continuity properties such as continuity in quadratic mean and almost sure continuity of the sum functions are studied.

1 Introduction

The Fourier series was invented in the early 1800s to solve the problem of heat diffusion in a continuous medium. It was extended to the Fourier series in orthogonal polynomials, which has a role in mathematical physics [1, 6]. Later, it was extended to random Fourier series, which is an inherent part of signal processing and image processing. In 2007, Liu and Liu [3, 4] attempted to define a random Fourier transform in orthogonal Hermite functions, which is a Fourier–Hermite series with random coefficients. The random Fourier transform they introduced is

$$\mathcal{R}[f(y)] := \sum_{n=0}^{\infty} d_n \mathcal{R}(\lambda_n)\varphi_n(y),$$

where $\varphi_n(y)$ are orthogonal Hermite functions, $\mathcal{R}(\lambda_n) := \exp[i\pi \text{ Random(n)}]$ are randomly chosen values from the unit circle in $\mathbb{C}$ and d_n are the Fourier–Hermite coefficients of the function f in $L^2(\mathbb{R})$, i.e.,

$$d_n := \int_{-\infty}^{\infty} f(y)\varphi_n(y)dy.$$

They applied it to image encryption and decryption. They also expected its application in optics and information technology. It motivated us to look into random Fourier series in orthogonal polynomials. Jacobi polynomials are a family of orthogonal polynomials that play a significant role in mathematical physics, particularly in problems involving orthogonal functions, spectral analysis, and special functions. Out of a curiosity rooted in mathematics, we started exploring random Fourier series involving Jacobi polynomials. Recently [10], we studied the convergence of random Fourier series

$$\sum_{n=0}^{\infty} d_n r_n(\omega)\varphi_n(y) \tag{1.1}$$

in orthogonal Jacobi polynomials $\varphi_n(y)$. The scalars d_n are the Fourier–Jacobi coefficients of a function f in some class of continuous functions and the random variables $r_n(\omega)$ are the Fourier–Jacobi coefficients of a stochastic process.

It is known that, if $X(t,\omega),\ t \in \mathbb{R}$ is a continuous stochastic process with independent increments and f is a continuous function in $[a,b]$, then the stochastic integral

$$\int_a^b f(t)dX(t,\omega) \tag{1.2}$$

is defined in the sense of probability and is a random variable (Lukacs [5, p. 148]). Further, if $X(t,\omega),\ t \in \mathbb{R}$ is a symmetric stable process of index $\alpha \in [1,2]$ and $f \in L^p_{[a,b]}$, $p \geq 1$, then the stochastic integral (1.2) is defined in the sense of probability for $p \geq \alpha$ (c.f. [9]).

Consider the space $L^{p,(\eta,\tau)}_{[-1,1]}$ of all measurable functions f on the segment $[-1,1]$ with the weight function $\rho^{(\eta,\tau)}(y) := (1-y)^\eta(1+y)^\tau,\ \ \eta,\tau > -1$, such that

$$\int_{-1}^{1} |f(y)\rho^{(\eta,\tau)}(y)|^p dy < \infty.$$

This space is equipped with the norm

$$||f||_{L^{p,(\eta,\tau)}_{[-1,1]}} = \left\{ \int_{-1}^{1} |f(y)\rho^{(\eta,\tau)}(y)|^p dy \right\}^{\frac{1}{p}},\ \text{for } p \geq 1.$$

If $f \in L^{p,(\eta,\tau)}_{[-1,1]},\ p \geq 1$, i.e., $f\rho^{(\eta,\tau)} \in L^p_{[-1,1]},\ \eta,\tau > -1$, then the stochastic integral

$$\int_{-1}^{1} f(t)\rho^{(\eta,\tau)}(t)dX(t,\omega)$$

will exist in probability for $p \geq \alpha \geq 1$.

In particular, if $f(t)$ are the orthonormal Jacobi polynomials $p_n^{(\gamma,\delta)}(t)$ with respect to Jacobi weight $\rho^{(\gamma,\delta)}(t) := (1-t)^\gamma(1+t)^\delta$ on $[-1,1]$, then the stochastic integrals

$$A_n(\omega) := \int_{-1}^{1} p_n^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)dX(t,\omega) \tag{1.3}$$

exist, for $\gamma,\delta > -1,\ \ \eta,\tau \geq 0$ and are random variables. These $A_n(\omega)$ are called the Fourier–Jacobi coefficients of the symmetric stable process $X(t,\omega)$ and are not independent (see Theorem 2.1). However, they are independent if these are associated with the Wiener process (see Theorem 3.1). In our work, the random coefficients $r_n(\omega)$ in the series (1.1) are considered to be these $A_n(\omega)$, which are dependent or independent depending on the choice of the stochastic process such as the symmetric stable process $X(t,\omega)$ or the Wiener process $W(t,\omega)$, respectively. The scalars d_n are chosen to be the Fourier–Jacobi coefficients a_n of a function f in $L^{p,(\eta,\tau)}_{[-1,1]}$ space defined as

$$a_n := \int_{-1}^{1} f(t)p_n^{(\gamma,\delta)}(t)\rho^{(\gamma,\delta)}(t)dt,\ \gamma,\delta > -1. \tag{1.4}$$

A key insight of this paper is to study the convergence of the random series (1.1) in Jacobi polynomials associated with stochastic processes like the symmetric stable process and the Wiener process. We also find the conditions on η,τ,γ,δ as well as investigate the weighted space

$L^{p,(\eta,\tau)}_{[-1,1]}$, so that the Fourier–Jacobi coefficients a_n of its functions make the random series (1.1) to converge. Further, the continuity properties of the sum functions are also discussed.

This article is structured as follows:
Section 2 establishes the convergence of random Fourier–Jacobi series (1.1) associated with symmetric stable process $X(t,\omega)$ of index $\alpha \in (1,2]$ as well as for $\alpha = 1$. Section 3 is devoted to study the random Fourier–Jacobi series (1.1) associated with the Wiener process. In this case, the orthogonal polynomials $\varphi_n(y)$ are considered to be the modified Jacobi polynomials $q_n^{(\gamma,\delta)}(y)$ (see equation (3.1)) in the segment $[0,1]$. It is shown that the random Fourier–Jacobi series (1.1) converges in quadratic mean. Moreover, under a strong condition on the scalars d_n, the convergence is upgraded to almost sure convergence. The sum functions of the random Fourier–Jacobi series (1.1) associated with stochastic processes are found to be stochastic integrals. The continuity properties of the sum functions are proved in Section 4. It is obtained that the sum functions of the random series (1.1) associated with the symmetric stable process and the Wiener process are weakly continuous in probability and continuous in quadratic mean, respectively. Further, the almost sure continuity of the sum function associated with the Wiener process is established.

2 Random Fourier–Jacobi series associated with symmetric stable process

This section deals with the random series

$$\sum_{n=0}^{\infty} a_n A_n p_n^{(\gamma,\delta)}(y) \tag{2.1}$$

in orthonormal Jacobi polynomials $p_n^{(\gamma,\delta)}(y)$, $\gamma,\delta > -1$, where the random coefficients $A_n(\omega)$ are associated with the symmetric stable process $X(t,\omega)$ of index $\alpha \in [1,2]$ defined as in (1.3). The following lemma establishes the random coefficients $A_n(\omega)$ are not independent.

Theorem 2.1. *If $X(t,\omega), t \in \mathbb{R}$, is a symmetric stable process of index $\alpha \in [1,2]$, then the random variables $A_n(\omega)$ associated with $X(t,\omega)$ are not independent for $\gamma,\delta > -1$ and $\eta,\tau \geq 0$.*

Proof. To prove $A_n(\omega)$ are not independent, it is sufficient to show that the characteristic function of $\Big(A_n(\omega) + A_m(\omega)\Big)$ is not the same as the product of the characteristic functions of $A_n(\omega)$ and $A_m(\omega)$. It is known that the characteristic function of the stochastic integral $\int_{-1}^{1} f(t)dX(t,\omega)$ is

$$\exp\Big(-C|x|^\alpha \int_{-1}^{1} |f(t)|^\alpha dt\Big), \tag{2.2}$$

where C is a constant. So, the characteristic function of $A_n(\omega)$ is

$$\exp\Big(-C|x|^\alpha \int_{-1}^{1} \Big|p_n^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)\Big|^\alpha dt\Big).$$

Now the characteristic function of sum of the random variables $(A_n(\omega) + A_m(\omega))$, i.e., the characteristic function of

$$\int_{-1}^{1} \Big\{p_n^{(\gamma,\delta)}(t) + p_m^{(\gamma,\delta)}(t)\Big\}\rho^{(\eta,\tau)}(t)dX(t,\omega)$$

is

$$\exp\Big(-C|x|^\alpha \int_{-1}^{1} \Big|\Big(p_n^{(\gamma,\delta)}(t) + p_m^{(\gamma,\delta)}(t)\Big)\rho^{(\eta,\tau)}(t)\Big|^\alpha dt\Big).$$

The product of characteristic functions of random variables $A_n(\omega)$ and $A_m(\omega)$ is

$$\exp\Big(-C|x|^\alpha \int_{-1}^{1} \Big|p_n^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)\Big|^\alpha dt\Big).\exp\Big(-C|x|^\alpha \int_{-1}^{1} \Big|p_m^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)\Big|^\alpha dt\Big)$$
$$= \exp\Big(-C|x|^\alpha \int_{-1}^{1} \Big(\Big|p_n^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)\Big|^\alpha + \Big|p_m^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)\Big|^\alpha\Big)dt\Big),$$

which is not equal to the characteristic function of the sum $\Big(A_n(\omega)+A_m(\omega)\Big)$. Hence, $A_n(\omega)$ are not independent random variables for $\gamma,\delta>-1$ and $\eta,\tau\geq 0$. $\square$

Theorem 2.2 below is on the convergence of the random Fourier–Jacobi series (2.1). The convergence of the series (2.1) is established in the sense of probability.

In fact, a sequence of random variables X_n is said to converge in probability to a random variable X if

$$\lim_{n\to\infty} P(|X_n-X|>\epsilon)=0,\ \text{for } \epsilon>0.$$

Theorem 2.2. *Let $X(t,\omega), t\in\mathbb{R}$, be a symmetric stable process of index $\alpha\in(1,2]$ and $A_n(\omega)$ be defined as in (1.3). If $\gamma,\delta>-1,\eta,\tau\geq 0$ satisfy the following conditions*

$$\Big|\eta-\frac{\gamma}{2}-\frac{1}{2}+\frac{1}{p}\Big|<\min\Big(\frac{1}{4},\frac{1}{2}+\frac{1}{2}\gamma\Big), \tag{2.3}$$

$$\Big|\tau-\frac{\delta}{2}-\frac{1}{2}+\frac{1}{p}\Big|<\min\Big(\frac{1}{4},\frac{1}{2}+\frac{1}{2}\delta\Big), \tag{2.4}$$

and a_n are the Fourier–Jacobi coefficients of $f\in L^{p,(\eta,\tau)}_{[-1,1]}$ defined as in (1.4), then the random Fourier–Jacobi series (2.1) converges in probability to the stochastic integral

$$\int_{-1}^{1} f(y,t)\rho^{(\eta,\tau)}(t)dX(t,\omega), \tag{2.5}$$

for $p\geq\alpha>1$.

The proof of this theorem requires the following result:

Lemma 2.3. *[9] Let $f(t)$ be any function in $L^p_{[a,b]}$ and $X(t,\omega)$ be a symmetric stable process of index α, for $1\leq\alpha\leq 2$. Then for all $\epsilon>0$,*

$$P\left(\left|\int_a^b f(t)dX(t,\omega)\right|>\epsilon\right)\leq\frac{C2^{\alpha+1}}{(\alpha+1)\epsilon'^{\alpha}}\int_a^b|f(t)|^{\alpha}dt,$$

where $\epsilon'<\epsilon$ and C is a positive constant, if $p\geq\alpha>1$.

Proof of Theorem 2.2.
Let

$$\mathbf{S}_n^{(\gamma,\delta)}(f,y,\omega):=\sum_{k=0}^{n}a_kA_k(\omega)p_k^{(\gamma,\delta)}(y) \tag{2.6}$$

be the nth partial sum of the random Fourier–Jacobi series (2.1). The integral form of (2.6) is

$$\begin{aligned}\mathbf{S}_n^{(\gamma,\delta)}(f,y,\omega) &:= \sum_{k=0}^{n}a_k\left(\int_{-1}^{1}p_k^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)dX(t,\omega)\right)p_k^{(\gamma,\delta)}(y)\\ &= \int_{-1}^{1}\sum_{k=0}^{n}a_kp_k^{(\gamma,\delta)}(t)p_k^{(\gamma,\delta)}(y)\rho^{(\eta,\tau)}(t)dX(t,\omega)\\ &= \int_{-1}^{1}\mathbf{s}_n^{(\gamma,\delta)}(f,y,t)\rho^{(\eta,\tau)}(t)dX(t,\omega),\end{aligned}$$

where

$$\mathbf{s}_n^{(\gamma,\delta)}(f,y,t):=\sum_{k=0}^{n}a_kp_k^{(\gamma,\delta)}(y)p_k^{(\gamma,\delta)}(t).$$

With the help of Lemma 2.3,

$$P\left(\left|\int_{-1}^{1} f(y,t)\rho^{(\eta,\tau)}(t)dX(t,\omega) - \mathbf{S}_n^{(\gamma,\delta)}(f,y,\omega)\right| > \epsilon\right)$$
$$= P\left(\left|\int_{-1}^{1} f(y,t)\rho^{(\eta,\tau)}(t)dX(t,\omega) - \int_{-1}^{1} \mathbf{s}_n^{(\gamma,\delta)}(f,y,t)\rho^{(\eta,\tau)}(t)dX(t,\omega)\right| > \epsilon\right)$$
$$\le \frac{C2^{\alpha+1}}{(\alpha+1)\epsilon'^{\alpha}} \int_{-1}^{1} \left|\Big(f(y,t) - \mathbf{s}_n^{(\gamma,\delta)}(f,y,t)\Big)\rho^{(\eta,\tau)}(t)\right|^{\alpha} dt, \text{ for } \epsilon' < \epsilon.$$

For $f \in L^{p,(\eta,\tau)}_{[-1,1]}$, $p > 1$, if the weights $\gamma, \delta, \eta, \tau$ satisfy the conditions (2.3), (2.4), then the nth partial sum $\mathbf{s}_n^{(\gamma,\delta)}(f,t)$ converges to $f(t)$ (by Theorem 1 in [8]). Hence, for $p \ge \alpha > 1$,

$$\lim_{n\to\infty} \int_{-1}^{1} \left|\Big(f(y,t) - \mathbf{s}_n^{(\gamma,\delta)}(f,y,t)\Big)\rho^{(\eta,\tau)}(t)\right|^{\alpha} dt = 0.$$

This implies the convergence of random Fourier–Jacobi series (2.1) to the stochastic integral (2.5) in probability for $p \ge \alpha > 1$.

The following theorem demonstrates the convergence of random Fourier–Jacobi series (2.1) associated with the symmetric stable process of index $\alpha = 1$.

Theorem 2.4. *Let $X(t,\omega)$ be a symmetric stable process of index $\alpha = 1$. If a_n and $A_n(\omega)$ defined as in (1.4) and (1.3) are the Fourier–Jacobi coefficients of a function f in $L^{1,(\eta,\tau)}_{[-1,1]}$, $\eta, \tau \ge 0$ and $X(t,\omega)$, respectively, then the random series (2.1) converges in probability to the stochastic integral (2.5) provided*

$$\gamma - \eta \ge 0 \text{ and } \delta - \tau \ge 0. \tag{2.7}$$

Proof. The proof follows the steps of Theorem 2.2 under the conditions (2.7) for $\gamma, \delta, \eta, \tau$ and uses the result of [7]. □

3 Random Fourier–Jacobi series associated with Wiener process

In this section, we consider the stochastic process $X(t,\omega)$ to be the Wiener process $W(t,\omega)$, $t \ge 0$ and the nth degree polynomials

$$q_n^{(\gamma,\delta)}(t) := p_n^{(\gamma,\delta)}(2t-1),\ n \in \mathbb{N} \cup 0, \gamma, \delta > -1, \tag{3.1}$$

as the orthogonal polynomials instead of the polynomials $\varphi_n(t)$ in the random series (1.1). These $q_n^{(\gamma,\delta)}(t)$ are orthogonal in the interval $[0,1]$ and form a complete orthonormal set in the interval $[0,1]$ with respect to the weight

$$\sigma^{(\gamma,\delta)}(t) := (1-t)^{\gamma} t^{\delta},\ \gamma, \delta > -1.$$

We know that the stochastic integral

$$\int_a^b f(t)dW(t,\omega) \tag{3.2}$$

exists in quadratic mean for $f \in L^2_{[a,b]}$ [9].

The random sequence $\{X_n\}_{n=0}^{\infty}$ is said to converge in quadratic mean to a random variable X if

$$\lim_{n\to\infty} E\Big(|X_n - X|^2\Big) = 0.$$

The stochastic integral (3.2) is normally distributed random variable with mean zero and finite variance $\int\limits_a^b |f(t)|^2 dt$, if $f(t)$ is a function in $L^2_{[a,b]}$ (c.f. Lukacs [5, p. 148]). The $q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)$ remains continuous for $\eta, \tau \geq 0$ and hence the stochastic integrals

$$B_n(\omega) := \int\limits_0^1 q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)dW(t,\omega) \tag{3.3}$$

with weight function $\sigma^{(\eta,\tau)}(t), \eta, \tau \geq 0$ exist in quadratic mean. These $B_n(\omega)$ are random variables with mean zero and finite variance. The following lemma proves the independence of random variables $B_n(\omega)$.

Theorem 3.1. *If $X(t,\omega)$ is the Wiener process $W(t,\omega), t \geq 0$, then the random variables $B_n(\omega)$ associated with $W(t,\omega)$ are independent.*

Proof. The Wiener process $W(t,\omega)$ has orthogonal increments and if $f, g \in L^2_{[a,b]}$, then by Doob [2, p. 427],

$$E\Big(\int_a^b f(t)dW(t,\omega)\overline{\int_a^b g(t)dW(t,\omega)}\Big) = \int_a^b f(t)\overline{g(t)}dt,$$

where $\overline{g(t)}$ is the complex conjugate of $g(t)$.
Thus, for $t \in [0,1]$,

$$\begin{aligned} E\Big(B_n(\omega)\overline{B_m(\omega)}\Big) &= E\Bigg(\int_0^1 q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)\overline{\int_0^1 q_m^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)}\Bigg) \\ &= \int_0^1 q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)\overline{q_m^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)}dt \\ &= \int_0^1 q_n^{(\gamma,\delta)}(t)q_m^{(\gamma,\delta)}(t)\{\sigma^{(\eta,\tau)}(t)\}^2 dt. \end{aligned}$$

Since $\sigma^{(\eta,\tau)}(t)$ is bounded by C in $[0,1]$, where C is a positive constant, Hence

$$\begin{aligned} E\Big(B_n(\omega)\overline{B_m(\omega)}\Big) &= \int_0^1 q_n^{(\gamma,\delta)}(t)q_m^{(\gamma,\delta)}(t)\{\rho^{(\eta,\tau)}(t)\}^2 dt \\ &\leq C\int_0^1 q_n^{(\gamma,\delta)}(t)q_m^{(\gamma,\delta)}(t)\rho^{(\eta,\tau)}(t)dt = 0. \end{aligned}$$

This proves the fact that $\{B_n(\omega)\}_{n=0}^{\infty}$ is a sequence of independent random variables for $t \in [0,1]$ and $\gamma, \delta, \eta, \tau \geq 0$. □

Now consider the random series

$$\sum_{n=0}^{\infty} b_n B_n(\omega) q_n^{(\gamma,\delta)}(y), \tag{3.4}$$

where $B_n(\omega)$ are defined as in (3.3) and b_n are scalars defined by

$$b_n := \int_0^1 f(t)q_n^{(\gamma,\delta)}(t)\sigma^{(\gamma,\delta)}(t)dt. \tag{3.5}$$

The $B_n(\omega)$ and b_n are called the modified Fourier–Jacobi coefficients of $W(t,\omega)$ and the function f, respectively. If b_n as defined in (3.5) are the Fourier–Jacobi coefficients of function $f \in L^{2,(\eta,\tau)}_{[0,1]}, \eta, \tau \geq 0$ with respect to the Jacobi polynomials $q_n^{(\gamma,\delta)}(t)$, $\gamma, \delta > -1$, then it can be shown that the random series (3.4) converges in quadratic mean to the stochastic integral

$$\int\limits_0^1 f(y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega). \tag{3.6}$$

The proof of this result needs the following theorem, which is a modified form of Theorem 1 in [8].

Theorem 3.2. *If $f \in L^{2,(\eta,\tau)}_{[0,1]}$ and the following conditions are satisfied by $\eta, \tau \geq 0,\ \gamma,\ \delta > -1$,*

$$\begin{aligned} \Big|\eta - \frac{1}{2}\gamma\Big| &< \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\gamma\Big), \\ \Big|\tau - \frac{1}{2}\delta\Big| &< \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\delta\Big), \end{aligned}$$

then

$$\lim_{n\to\infty} \int_0^1 \Big|\Big\{\mathbf{v}_n^{(\gamma,\delta)}(f,y) - f(y)\Big\}\sigma^{(\eta,\tau)}(y)\Big|^2 dy = 0,\ \textit{for } y \in [0,1],$$

where $\mathbf{v}_n^{(\gamma,\delta)}(f,y)$ is the nth partial sum of the Fourier–Jacobi series $\sum_{n=0}^{\infty} b_n q_n^{(\gamma,\delta)}(y)$.

The following theorem establishes the convergence of the random Fourier–Jacobi series (3.4) in modified Jacobi polynomials $q_n^{(\gamma,\delta)}(t)$ associated with the Wiener process $W(t,\omega)$.

Theorem 3.3. *Let $W(t,\omega), t \geq 0$ be the Wiener process and $B_n(\omega)$ be defined as in (3.3). If the weights $\gamma, \delta > -1,\ \eta, \tau \geq 0$ satisfy the conditions*

$$\Big|\eta - \frac{1}{2}\gamma\Big| < \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\gamma\Big),\ \Big|\tau - \frac{1}{2}\delta\Big| < \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\delta\Big), \tag{3.7}$$

and b_n are the Fourier–Jacobi coefficients of $f \in L^{2,(\eta,\tau)}_{[0,1]}$, then the random Fourier–Jacobi series (3.4) converges to the integral (3.6) in quadratic mean.

Proof. Let

$$T_n^{(\gamma,\delta)}(f,y,\omega) := \sum_{k=0}^{n} b_k B_k(\omega) q_k^{(\gamma,\delta)}(y),\ \gamma,\delta > -1 \tag{3.8}$$

be the nth partial sum of the random Fourier–Jacobi series (3.4). The integral form of (3.8) is

$$\int_0^1 \mathbf{v}_n^{(\gamma,\delta)}(f,y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega),\ \eta,\tau \geq 0,$$

with

$$\mathbf{v}_n^{(\gamma,\delta)}(f,y,t) := \sum_{k=0}^{n} b_k q_k^{(\gamma,\delta)}(y) q_k^{(\gamma,\delta)}(t),$$

for $t \in [0,1]$. We know that (Lukacs [5, p. 147]) for $g \in L^2_{[a,b]}$, if $W(t,\omega)$ is the Wiener process, then

$$E\Big|\int_a^b g(t)dW(t,\omega)\Big|^2 = \beta^2 \int_a^b |g(t)|^2 dt, \tag{3.9}$$

where β is a constant associated with the normal law of increment of the process $W(t,\omega)$, for $t \in [a,b]$. Hence, the equality

$$\begin{aligned} &E\Big(\Big|\int_0^1 f(y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega) - T_n^{(\gamma,\delta)}(f,y,\omega)\Big|^2\Big) \\ &= E\Big(\Big|\int_0^1 f(y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega) - \int_0^1 \mathbf{v}_n^{(\gamma,\delta)}(f,y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)\Big|^2\Big) \\ &= \beta^2 \int_0^1 \Big|\Big(f(y,t) - \mathbf{v}_n^{(\gamma,\delta)}(f,y,t)\Big)\sigma^{(\eta,\tau)}(t)\Big|^2 dt. \end{aligned}$$

If $f \in L^{2,(\eta,\tau)}_{[0,1]}$ and $\gamma, \delta, \eta, \tau$ satisfy the conditions in (3.7), then by Theorem 3.2,

$$\lim_{n\to\infty} \int_0^1 \Big|\Big(f(t) - \mathbf{v}_n^{(\gamma,\delta)}(f,t)\Big)\sigma^{(\eta,\tau)}(t)\Big|^2 dt = 0.$$

Hence

$$\lim_{n\to\infty}\int_0^1 \left|\left(f(y,t)-\mathbf{v}_n^{(\gamma,\delta)}(f,y,t)\right)\sigma^{(\eta,\tau)}(t)\right|^2 dt = 0,$$

which implies convergence of the random series (3.4) in quadratic mean to the integral (3.6). □

The almost sure convergence of the random Fourier–Jacobi series (3.4) is derived in Theorem 3.5. The Kolmogorov Theorem stated below is required to prove it.

Theorem 3.4. *(Kolmogorov Theorem)*
Let $(X_n)_{n=1}^{\infty}$ be independent random variables with expected values $E[X_n]=\mu_n$ and variances $Var(X_n)=\sigma_n^2$, such that $\sum_{n=1}^{\infty}\mu_n$ converges in $\mathbb{R}$ and $\sum_{n=1}^{\infty}\sigma_n^2$ converges in $\mathbb{R}$. Then $\sum_{n=1}^{\infty}X_n$ converges in $\mathbb{R}$ almost surely.

The following facts about $p_n^{(\gamma,\delta)}(y)$ will be useful to establish the almost sure convergence of the series (3.4). We know that the Jacobi polynomials satisfy the following inequality

$$|p_n^{(\gamma,\delta)}(y)| \le C' n^{-\frac{1}{2}}(1-y+n^{-2})^{-\frac{1}{2}\gamma-\frac{1}{4}},\ 0\le y\le 1,\ \gamma,\delta>-1, \tag{3.10}$$

where C' is a constant independent of y and n [11, p. 167].

The equality

$$p_n^{(\gamma,\delta)}(y)=(-1)^n p_n^{(\gamma,\delta)}(-y) \text{ (see [11, p. 71])}$$

extends the inequality (3.10), to hold for all $y\in[-1,1]$. This can be applied for the modified orthonormal Jacobi polynomials $q_n^{(\gamma,\delta)}(y)$ in the interval $[0,1]$, and we obtain

$$\begin{aligned}|q_n^{(\gamma,\delta)}(y)| &\le C' n^{-\frac{1}{2}}(1-y+n^{-2})^{-\frac{1}{2}\gamma-\frac{1}{4}}\\ &= \frac{C' n^{\gamma}}{[(1-y)n^2+1]^{\gamma/2+1/4}},\end{aligned}$$

where $1/[(1-y)n^2+1]$ is bounded by $1/2$ in $[0,1]$.
Hence

$$|q_n^{(\gamma,\delta)}(y)| \le \frac{C' n^{\gamma}}{2^{\gamma/2+1/4}},$$

i.e.,

$$|q_n^{(\gamma,\delta)}(y)| \le C n^{\gamma}, \tag{3.11}$$

where C is a constant independent of y and n.

Theorem 3.5. *Let $W(t,\omega),\ t\ge 0$, be the Wiener process and $B_n(\omega)$ be as defined in (3.3). If the scalars b_n are Fourier–Jacobi coefficients of function $f\in L^{2,(\eta,\tau)}_{[0,1]}$ and $\gamma,\delta>-1,\ \eta,\tau\ge 0$ satisfy the conditions (3.7) in Theorem 3.3 and*

$$\sum_{n=0}^{\infty}\{n^{(2\gamma)}|b_n|\}^2<\infty, \tag{3.12}$$

then the series (3.4) converges almost surely to the stochastic integral (3.6).

Proof. We know that the independent random variables $B_n(\omega)$ are normally distributed with mean zero and finite variance. Hence, each $b_n B_n(\omega) q_n^{(\gamma,\delta)}(y)$, for $n=1,2,\ldots$ are normally distributed, independent random variables with mean zero and finite variance. Now, using the identity (3.9), the sum of the variance of these random variables is

$$\begin{aligned}\sum_{n=0}^{\infty}E\left|b_nB_n(\omega)q_n^{(\gamma,\delta)}(y)\right|^2 &= \sum_{n=0}^{\infty}E\left|b_n\int_0^1 q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)q_n^{(\gamma,\delta)}(y)\right|^2\\ &= \sum_{n=0}^{\infty}E\left|\int_0^1 b_nq_n^{(\gamma,\delta)}(y)q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)\right|^2\\ &= \sum_{n=0}^{\infty}\int_0^1\left|b_nq_n^{(\gamma,\delta)}(y)q_n^{(\gamma,\delta)}(t)\sigma^{(\eta,\tau)}(t)\right|^2dt.\end{aligned}$$

Since $q_n^{(\gamma,\delta)}(t)$ are bounded by Cn^γ (inequality (3.11)) and $\sigma^{(\eta,\tau)}(t)$ is bounded for $\eta,\tau \geq 0$, we have the inequality

$$\sum_{n=0}^{\infty} E\left|b_n B_n(\omega) q_n^{(\gamma,\delta)}(y)\right|^2 \quad \leq \quad K\sum_{n=0}^{\infty}\left(|b_n| n^{2\gamma}\right)^2,$$

which will be finite for $f \in L^{2,(\eta,\tau)}_{[0,1]}$, if $\sum\limits_{n=0}^{\infty}\left(|b_n|n^{2\gamma}\right)^2$ is finite. Now by Theorem 3.4, the series (3.4) converges to the integral $\int\limits_0^1 f(y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)$ almost surely in $\mathbb{R}$, if γ,δ,η,τ satisfy the conditions (3.7) in Theorem 3.3. □

4 Continuity property of the sum functions

4.1 Sum function associated with symmetric stable process

The sum function of the random Fourier–Jacobi series (2.1) associated with the symmetric stable process $X(t,\omega)$ is shown to be weakly continuous in probability.

We know that a function $f(t,\omega)$ is said to be weakly continuous in probability at $t = t_0$ if for all $\epsilon > 0$,

$$\lim_{h\to 0} P(|f(t_0+h,\omega) - f(t_0,\omega)| > \epsilon) = 0.$$

If a function $f(t,\omega)$ is weakly continuous at every $t_0 \in [a,b]$, then the function $f(t,\omega)$ is said to be weakly continuous in probability in the closed interval $[a,b]$.

The proof of this result requires the following lemma.

Lemma 4.1. *[12, p. 37] If f is periodic or in $L^p_{[a,b]}$, $1 \leq p < \infty$ or continuous function, then the integral*

$$\left\{\int_a^b \left|f(x+t) - f(x)\right|^p dx\right\}^{1/p}$$

tends to 0 *as* t *tends to* 0.

Theorem 4.2. *In Theorem 2.2 and Theorem 2.4, the sum function (2.5) of the random Fourier–Jacobi series (2.1) associated with the symmetric stable process according to the respective conditions of γ,δ,η,τ are weakly continuous in probability.*

Proof. With the help of Lemma 2.3,

$$P\left(\left|\int_{-1}^1 f(x,t)\rho^{(\eta,\tau)}(t)dX(t,\omega)) - \int_{-1}^1 f(y,t)\rho^{(\eta,\tau)}(t)dX(t,\omega)\right| > \epsilon\right)$$

$$\leq \frac{C2^{\alpha+1}}{(\alpha+1)\epsilon'^{\alpha}}\int_{-1}^1 \left|\Big(f(x,t) - f(y,t)\Big)\rho^{(\eta,\tau)}(t)\right|^\alpha dt,$$

where $0 < \epsilon' < \epsilon$ and $p \geq \alpha \in [1,2]$. Since the weight $\rho^{(\eta,\tau)}(t)$ is bounded, the integral

$$\int_{-1}^1 \left|\Big(f(x,t) - f(y,t)\Big)\right|^\alpha dt$$

tends to 0 as $y \to x$, by Lemma 4.1. This confirms that the sum function (2.5) is weakly continuous in probability. □

4.2 Sum function associated with Wiener process

Theorem 4.3. *The sum function (3.6) of the random Fourier–Jacobi series (3.4) under the condition on γ,δ and η,τ in Theorem 3.3 is continuous in quadratic mean.*

Proof. By the use of Equation (3.9),

$$
\begin{aligned}
&E\Big(\Big|\int_0^1 f(y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega) - f(x,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)\Big|^2\Big)\\
&= E\Big(\Big|\int_0^1 \Big(f(y,t) - f(x,t)\Big)\sigma^{(\eta,\tau)}(t)dW(t,\omega)\Big|^2\Big)\\
&= \beta^2\int_0^1 \Big|\Big(f(y,t) - f(x,t)\Big)\sigma^{(\eta,\tau)}(t)\Big|^2 dt.
\end{aligned}
$$

For $\eta,\tau \geq 0$, the Jacobi weight $\sigma^{(\eta,\tau)}(t)$ is bounded by $C > 0$. Then

$$
\begin{aligned}
&E\Big(\Big|\int_0^1 f(y,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega) - f(x,t)\sigma^{(\eta,\tau)}(t)dW(t,\omega)\Big|^2\Big)\\
&\leq \beta^2 C^2\int_0^1 \Big|\Big(f(y,t) - f(x,t)\Big)\Big|^2 dt.
\end{aligned}
$$

Hence, by Lemma 4.1, the right–hand side tends to zero as $y \to x$. This proves that the sum function (3.6) in Theorem 3.3 is continuous in quadratic mean. □

The following theorem establishes the improvement of the continuity property of the sum function (3.6) from quadratic mean to almost surely.

Theorem 4.4. *The sum function (3.6) of the random Fourier–Jacobi series (3.4) is almost surely continuous if*

$$\sum_{n=0}^{\infty}(n^{\gamma}|b_n|) < \infty, \tag{4.1}$$

in addition to the conditions on $\gamma, \delta, \eta, \tau$ stated in Theorem 3.3.

Proof. By Weistrass M–test, the series (3.4) converges uniformly to a continuous function almost surely for almost all $y \in [0,1]$, if

$$\sum_{n=0}^{\infty}\Big|b_n B_n(\omega) q_n^{(\gamma,\delta)}(y)\Big| < \infty.$$

It is sufficient to show that

$$\sum_{n=0}^{\infty}E\Big|b_n B_n(\omega) q_n^{(\gamma,\delta)}(y)\Big| < \infty.$$

Now

$$\sum_{n=0}^{\infty}E\Big|b_n B_n(\omega) q_n^{(\gamma,\delta)}(y)\Big| \quad\leq\quad K\sum_{n=0}^{\infty}(|b_n|n^{\gamma}), \text{ where } K \text{ is a constant,}$$

as $q_n^{(\gamma,\delta)}(y)$ are bounded by Cn^{γ} and $B_n(\omega)$ are bounded. If the sum in the right–hand side series is finite, then the almost sure continuity of the sum function (3.6) is established for almost all $y \in [0,1]$. □

5 Remark

In all our results, the weights associated with the Jacobi polynomials are considered to be $\gamma,\delta > -1$ and $\eta,\tau \geq 0$. The results that we obtained can be summarized as follows:

(i) If $X(t,\omega), t \in \mathbb{R}$, is a symmetric stable process of the index $\alpha = 1$ and weights γ,δ,η,τ satisfy

$$\gamma - \eta \geq 0 \text{ and } \delta - \tau \geq 0,$$

then the random Fourier–Jacobi series (2.1) converges in probability to the integral (2.5).

(ii) If the index $\alpha \in (1,2]$ and

$$\Big(\Big|\eta - \frac{\gamma}{2} - \frac{1}{2} + \frac{1}{p}\Big| < \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\gamma\Big),$$
$$\Big(\Big|\tau - \frac{\delta}{2} - \frac{1}{2} + \frac{1}{p}\Big| < \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\delta\Big),$$

are satisfied by the weights $\gamma, \delta, \eta, \tau$, then the random Fourier–Jacobi series (2.1) converges in probability to the integral (2.5).

(iii) The sum function (2.5) is weakly continuous in probability, if $X(t,\omega)$ is the symmetric stable process of index $\alpha \in [1,2]$.

(iv) If $X(t,\omega)$ is the Wiener process $W(t,\omega), t \geq 0$, and the conditions

$$\Big|\eta - \frac{\gamma}{2}\Big| < \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\gamma\Big),$$
$$\Big|\tau - \frac{\delta}{2}\Big| < \min\Big(\frac{1}{4}, \frac{1}{2} + \frac{1}{2}\delta\Big),$$

are satisfied by $\gamma, \delta, \eta, \tau$, then the random Fourier–Jacobi series (3.4) converges in quadratic mean to the integral (3.6).

(v) The sum function (3.6) associated with the Wiener process is continuous in quadratic mean.

(vi) In addition to the conditions as in (iv) on $\gamma, \delta, \eta, \tau$, if the Fourier–Jacobi coefficients b_n of the function $f \in L^{2,(\gamma,\delta)}_{[0,1]}$ satisfy the strong condition

$$\sum_{n=0}^{\infty} \left(|b_n| n^{2\gamma}\right)^2 < \infty,$$

then the random Fourier–Jacobi series (3.4) converges almost surely to the stochastic integral (3.6).

(vii) The sum function (3.6) associated with the Wiener process is almost surely continuous if

$$\sum_{n=0}^{\infty} |b_n| n^{\gamma} < \infty.$$

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