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ρ_*^∞-Orthogonality and its Geometrical Properties

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Abstract In this paper, we define the concept of a ρ_*^n-functional and we use norm derivatives to introduce ρ_*^∞-orthogonality and discuss its geometrical properties. We describe relationships between ρ_*^∞-orthogonality and Birkhoff-James orthogonality on one hand and semi-orthogonality on the other. In particular, we provide illustrations showing that ρ_*^∞-orthogonality cannot be compared to other well-known orthogonalities. Finally, we characterize the inner product spaces using the properties of ρ_*^∞-orthogonality.

1 Introduction

The notion of orthogonality is one of the fascinating ideas in studying the geometry of normed spaces. The orthogonality relations derived from the norm derivatives provide a useful framework for studying the geometric structure of a normed space. One of the most intriguing orthogonality concepts is the Birkhoff-James orthogonality, which was first proposed by Birkhoff [3] and then modified by James [9]. Isosceles and Pythagorean orthogonalities were first introduced to normed space by James [8] in 1945.

In 1986, Amir [1] introduced the functional using norm derivative, which is helpful in analyzing the geometry of normed spaces. The concept of the norm derivatives naturally arises from the Gateaux derivative of the norm's two-sided limiting feature, making them suitable generalizations of the latter. Furthermore, Milićić [13] provided a new functional and orthogonality, a combination of norm derivatives. Later, Zamani and Moslehian [15] extended the norm orthogonality as a convex combination of norm derivatives. Recently, Enderami et al. [6] introduced a new functional and on the basis of norm derivatives, an orthogonality relation in complex normed spaces is established.

Inspired by these results, we introduce a functional and ρ_*^∞-orthogonality relation in the framework of a normed space with a complex field. Also, we investigate its interesting geometric properties in complex normed space. We also obtain an association between the orthogonality defined in the paper with Birkhoff-James orthogonality and semi-orthogonality. Consequently, we give some examples that illustrate that the ρ_*^∞-orthogonality is incomparable with other renowned orthogonalities. In the last, we characterize the inner product space using the properties of ρ_*^∞.

2 Preliminaries

In this section, we provide some fundamental definitions, notations and results which will be used in the sequel.

Throughout the paper, we consider a complex normed space $(\mathcal{X}, \|.\|)$.

According to Birkhoff-James (see [3, 9]), a vector $p \in \mathcal{X}$ is considered to be orthogonal to a vector $q \in \mathcal{X}$, denoted by $p \perp_B q$, if

$$\|p + rq\| \geq \|p\|, \text{ for all } r \in \mathbb{R}.$$

In 1986, Amir [1] defined the norm derivatives as, for all $p, q \in \mathcal{X}$,

$$\rho_{\pm}(p,q) = \lim_{\eta \to 0^{\pm}} \frac{\|p + \eta q\|^2 - \|p\|^2}{2\eta} = \|p\| \lim_{\eta \to 0^{\pm}} \frac{\|p + \eta q\| - \|p\|}{\eta}.$$

Following are the well-known properties of norm derivatives (see [2]) in $\mathcal{X}$, for all $p, q \in \mathcal{X}$ and for every $\gamma = |\gamma| e^{i\theta}, \delta = |\delta| e^{i\omega}$ in $\mathbb{C}$, we obtain

(i) $\rho_-(p,q) \leq \rho_+(p,q)$;

(ii) $\rho_{\pm}(p, \gamma p) = Re(\gamma)\|p\|^2$,

(iii) $\rho_{\pm}(p, \gamma p + q) = Re(\gamma)\|p\|^2 + \rho_{\pm}(p,q)$,

(iv) $\rho_{\pm}(\gamma p, \delta q) = |\gamma||\delta| \rho_{\pm}(p, e^{i(\omega - \theta)} q)$,

(v) $|\rho_{\pm}(\gamma p, \delta q)| \leq \|\gamma p\| \|\delta q\|$,

(vi) $\rho_{\pm}(p, \gamma q) = Re\langle p, \gamma q \rangle$, for inner product space.

(vii) $p \perp_{\rho_{\pm}} q$ if and only if $\rho_{\pm}(p,q) = 0$.

Milićić [13] defined the mapping $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by

$$\rho(p,q) = \frac{\rho_-(p,q) + \rho_+(p,q)}{2},$$

for all $p, q \in \mathcal{X}$ and introduced the corresponding ρ-orthogonality in terms of norm derivatives :

$$p \perp_{\rho} q \text{ if and only if } \rho(p,q) = \frac{\rho_-(p,q) + \rho_+(p,q)}{2} = 0.$$

It's interesting to note that the relations $\perp_{\rho_+}, \perp_{\rho_-}$ and $\perp_{\rho}$ are generally incomparable in a non-smooth normed space but are equivalent in an inner product space.
In 2015, Chen and Lu [5] introduced the notion of ρ_*-orthogonality which is defined by:

$$p \perp_{\rho_*} q \text{ if and only if } \rho_*(p,q) = \rho_-(p,q)\rho_+(p,q) = 0,$$

where $p, q \in \mathcal{X}$.
Furthermore, Zamani and Moslehian [15] introduced ρ_λ-orthogonality as a generalisation of orthogonality relations dependent on norm derivatives,

$$p \perp_{\rho_\lambda} q \text{ if and only if } \rho_\lambda(p,q) = \lambda \rho_-(p,q) + (1 - \lambda)\rho_+(p,q) = 0,$$

for each $p, q \in X$ and $\lambda \in [0,1]$. In addition, they provided a ρ_λ based characterization of inner product spaces.

Due to Lumer [12] and Giles [7], there exists a mapping $[.,.] : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ for every normed space $(\mathcal{X}, \|.\|)$, known as a semi-inner product (s.i.p.), satisfying the following properties, for all $p, q, w \in \mathcal{X}$ and $\gamma, \delta \in \mathbb{C}$,

(i) $[\gamma p + \delta q, w] = \gamma[p, w] + \delta[q, w]$,

(ii) $[p, \gamma q] = \bar{\gamma}[p, q]$,

(iii) $[p, p] = \|p\|^2$,

(iv) $|[p, q]| \leq \|p\| \|q\|$.

In an arbitrary normed space, the idea of orthogonality can be presented in a variety of ways. A semi-orthogonality of the components p and q in a semi inner product $[.,.]$ is defined by

$$p \perp_s q \text{ if and only if } [q, p] = 0.$$

The structure of the paper is as follows.
In Section 3, first we define a functional $\rho_*^n : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ defined by

$$\rho_*^n(p,q) = \frac{2}{n}\sum_{\ell=1}^{n} \rho_*(p, c_\ell q),$$

where the scalars c_ℓ, $\ell = 1, 2, \ldots, n$ are the n-th roots of unity in $\mathbb{C}$.

Thereafter, we generalize the functional $\rho_*^n : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by defining a $\rho_*^\infty : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ as

$$\rho_*^\infty(p,q) = \frac{1}{\pi}\int_0^{2\pi} \rho_*(p, e^{i\theta} q)d\theta.$$

Also, we discuss their basic geometrical properties using norm derivatives.

On the basis of the mapping ρ_*^∞, we define an orthogonality relation in Section 4 as follows

$$p \perp_{\rho_*^\infty} q \text{ if } \rho_*^\infty(p,q) = 0.$$

In addition, we demonstrate certain comparable relations using s.i.p. $[.,.]$ on $\mathcal{X}$. Also, we show that $\perp_{\rho_*^\infty} \subseteq \perp_B$ if $|\rho_*^\infty(p,q)| \leq \|p\|\|q\|$ for all $p, q \in \mathcal{X}$.

3 A new functional form

Now, we introduce a new functional using ρ_* functional in the setting of complex normed space.

Definition 3.1. For the n-th root of unity $c_1, c_2, ...c_n$ we define a functional ρ_*^n on $\mathcal{X}$ as

$$\rho_*^n(p,q) = \frac{2}{n}\sum_{\ell=1}^{n} \rho_*(p, c_\ell q),$$

for all $p, q \in \mathcal{X}$.

Remark 3.2. (i) From the above definition, for $n = 1, c_1 = 1$, then we get $\rho_*^1 = \frac{2}{1}\{\rho_*(p, 1q)\} = 2\rho_*(p,q)$.

(ii) For $n = 2$, $c_1 = 1$ and $c_2 = -1$, we have $\rho_*^2 = \frac{2}{2}\{\rho_*(p, 1q) + \rho_*(p, -1q)\} = 2\rho_*(p,q)$.

(iii) For $n = 1$ and $n = 2$ we now have $\sum_{i=1}^{n} c_n^2 = n$, but when we take $n > 2$, this fact is not true. So, It's fascinating to think about the nth roots of unity, where n is greater than 2.

In this part, we begin with some properties of the functional ρ_* in $\mathcal{X}$.

Proposition 3.3. *For all $p, q \in \mathcal{X}$ and for every $\gamma = |\gamma|e^{i\theta}, \delta = |\delta|e^{i\omega}$ in $\mathbb{C}$, we have*

(i) $\rho_*(p, \gamma p) = \{Re(\gamma)\}^2\|p\|^4$,

(ii) $\rho_*(p, \gamma p + q) = \{Re(\gamma)\}^2\|p\|^4 + 2Re(\gamma)\|p\|^2\rho(p,q) + \rho_*(p,q)$,

(iii) $\rho_*(\gamma p, \delta q) = |\gamma|^2|\delta|^2\rho_*(p, e^{i(\omega-\theta)}q)$,

(iv) $|\rho_*(\gamma p, \delta q)| \leq \|\gamma p\|^2\|\delta q\|^2$,

(v) $\rho_*(p, \gamma q) = \{Re\langle p, \gamma q\rangle\}^2$, *for inner product space.*

Proof. (i) We know that, $\rho_\pm(p, \gamma p) = Re(\gamma)\|p\|^2$.
Therefore,

$$\rho_*(p, \gamma p) = \rho_-(p, \gamma p)\rho_+(p, \gamma p) = \{Re(\gamma)\}^2\|p\|^4.$$

(ii) Consider,

$$\rho_+(p, \gamma p + q) = \rho_+(p,q) + Re(\gamma)\|p\|^2.$$

and

$$\rho_-(p, \gamma p + q) = \rho_-(p, q) + Re(\gamma)\|p\|^2.$$

Therefore,

$$\begin{aligned}\rho_*(p, \gamma p + q) &= \rho_-(p, \gamma p + q)\rho_+(p, \gamma p + q) \\ &= \{Re(\gamma)\|p\|^2 + \rho_-(p, q)\}\{Re(\gamma)\|p\|^2 + \rho_+(p, q)\} \\ &= \{Re(\gamma)\}^2\|p\|^4 + 2Re(\gamma)\|p\|^2\rho(p, q) + \rho_*(p, q).\end{aligned}$$

(iii) Consider

$$\rho_\pm(\gamma p, \delta q) \quad = \quad |\gamma||\delta|\rho_\pm(p, e^{i(\omega-\theta)q}).$$

Therefore,

$$\rho_*(\gamma p, \delta q) = |\gamma|^2|\delta|^2\rho_*(p, e^{i(\omega-\theta)}).$$

(iv) Using the definition of ρ_*, we have

$$\begin{aligned}|\rho_*(\gamma p, \delta q)| &= |\rho_-(\gamma p, \delta q)\rho_+(\gamma p, \delta q)| \\ &= |\rho_-(\gamma p, \delta q)||\rho_+(\gamma p, \delta q)| \\ &\leq \|\gamma p\|\|\delta q\|\|\gamma p\|\|\delta q\| \\ &= \|\gamma p\|^2\|\delta q\|^2.\end{aligned}$$

(v) Consider,

$$\rho_\pm(p, \gamma q) \quad = \quad Re\langle p, \gamma q\rangle.$$

So, $\rho_*(p, \gamma q) = \rho_-(p, \gamma q)\rho_+(p, \gamma q) = \{Re\langle p, \gamma q\rangle\}^2$. □

The next consequence is obvious (see [6]).

Lemma 3.4. *Suppose that $c_1, c_2, \ldots, c_n$ are the nth roots of unity. Then $\sum_{\ell=1}^{n} c_\ell^2 = 0 = \sum_{\ell=1}^{n} \overline{c_\ell}^2$, for $n > 2$.*

We discuss some basic properties of functional ρ_*^n.

Proposition 3.5. *(i) $\rho_*^n(p, p) = \|p\|^4$, for all $p \in \mathcal{X}$.*

(ii) $|\rho_*^n(p, q)| \leq 2\|p\|^2\|q\|^2$, for every p, q in $\mathcal{X}$.

(iii) If the norm of $\mathcal{X}$ derives from an inner product $\langle ., .\rangle$, then $\rho_*^n(p, q) = \langle p, q\rangle\overline{\langle p, q\rangle}$ for every $p, q \in \mathcal{X}$.

Proof. (i) For $p \in \mathcal{X}$, we have

$$\begin{aligned}\rho_*^n(p, p) &= \frac{2}{n}\sum_{\ell=1}^{n}\rho_*(p, c_\ell p) \\ &= \frac{2}{n}\sum_{\ell=1}^{n}\{Re(c_\ell)\}^2\|p\|^4 \; [since\ \rho_*(p, \gamma p) = Re(\gamma)\|p\|^2] \\ &= \frac{\|p\|^4}{2n}\sum_{\ell=1}^{n}(c_\ell + \bar{c}_\ell)^2 \\ &= \frac{\|p\|^4}{2n}\sum_{\ell=1}^{n}({c_\ell}^2 + 2|c_\ell|^2 + {\bar{c}_\ell}^2) \\ &= \frac{\|p\|^4}{2n}(\sum_{\ell=1}^{n}{c_\ell}^2 + 2\sum_{\ell=1}^{n}|c_\ell|^2 + \sum_{\ell=1}^{n}{\bar{c}_\ell}^2) \\ &= \frac{\|p\|^4}{2n}(0 + 2n + 0) \;\; \text{(using Lemma 3.4)} \\ &= \|p\|^4.\end{aligned}$$

(ii) For $p, q \in \mathcal{X}$, we have

$$\begin{aligned} |\rho_*^n(p,q)| &= \frac{2}{n}|\sum_{\ell=1}^{n}\rho_*(p,c_\ell p)| \\ &\leq \frac{2}{n}\sum_{\ell=1}^{n}|\rho_*(p,c_\ell q)| \\ &\leq \frac{2}{n}\sum_{\ell=1}^{n}\|p\|^2\|c_\ell q\|^2 \\ &= \frac{2\|p\|^2\|q\|^2}{n}\sum_{\ell=1}^{n}|c_\ell|^2 \\ &= 2\|p\|^2\|q\|^2. \end{aligned}$$

(iii) Suppose that the norm of $\mathcal{X}$ derives from an inner product $\langle .,. \rangle$. Then for all $p, q \in \mathcal{X}$,

$$\begin{aligned} \rho_*^n(p,q) &= \frac{2}{n}\sum_{\ell=1}^{n}c_\ell\rho_*(p,c_\ell q) \\ &= \frac{2}{n}\sum_{\ell=1}^{n}\{Re\langle p,c_\ell q\rangle\}^2 \ \ \text{(using Proposition 3.3(v))} \\ &= \frac{1}{2n}\sum_{\ell=1}^{n}(\langle p,c_\ell q\rangle + \overline{\langle p,c_\ell q\rangle})^2 \\ &= \frac{1}{2n}\sum_{\ell=1}^{n}(\bar{c_\ell}\langle p,q\rangle + c_\ell\overline{\langle p,q\rangle})^2 \\ &= \frac{\langle p,q\rangle^2}{2n}\sum_{\ell=1}^{n}\bar{c_\ell}^2 + \frac{\overline{\langle p,q\rangle}^2}{2n}\sum_{\ell=1}^{n}c_\ell^2 + \frac{2\langle p,q\rangle\overline{\langle p,q\rangle}}{2n}\sum_{\ell=1}^{n}|c_\ell|^2 \\ &= \langle p,q\rangle\overline{\langle p,q\rangle}. \end{aligned}$$

□

Remark 3.6. We know that if $g : [0, 1] \to \mathbb{C}$ is a continuous function, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\ell=1}^{n}g(\frac{k}{n}) = \int_0^1 g(t)dt. \tag{3.1}$$

Now, replacing c_ℓ with $e^{2k\pi i/n}$ we get

$$\rho_*^n(p,q) = \frac{2}{n}\sum_{\ell=1}^{n}\rho_*(p,e^{2k\pi i/n}q). \tag{3.2}$$

Letting $n \to \infty$ in (3.2), we get

$$\lim_{n\to\infty}\frac{2}{n}\sum_{\ell=1}^{n}\rho_*(p,e^{2k\pi i/n}q) = 2\int_0^1 \rho_*(p,e^{i2\pi t}q)dt.$$

Taking $\theta = 2\pi t$, we have

$$\lim_{n\to\infty}\rho_*^n(p,q) = \frac{1}{\pi}\int_0^{2\pi}\rho_*(p,e^{i\theta}q)d\theta. \tag{3.3}$$

Therefore,

$$\rho_*^\infty(p,q) = \lim_{n\to\infty}\rho_*^n(p,q) = \frac{1}{\pi}\int_0^{2\pi}\rho_*(p,e^{i\theta}q)d\theta. \tag{3.4}$$

Following, we examine the basic geometrical properties of ρ_*^∞. As the integral functions under study are 2π-periodic, we'll frequently use the following equality:

$$\int_\phi^{2\pi+\phi} k(\theta)d\theta = \int_0^{2\pi} k(\theta)d\theta \ \ (\phi \in \mathbb{R}).$$

Proposition 3.7. *For all $p, q \in \mathcal{X}$, $\gamma, \delta \in \mathbb{C}$, we get*

(i) $\rho_*^\infty(\gamma p, \delta q) = |\gamma|^2|\delta|^2\rho_*^\infty(p, q)$

(ii) $\rho_*^\infty(p, \gamma p + q) = \gamma\bar{\gamma}\|p\|^4 + \rho_*^\infty(p, q) + \frac{2\|p\|^2}{\pi}\int_0^{2\pi} Re(e^{i\theta}\gamma)\rho(p, e^{i\theta}q)d\theta,$

(iii) $|\rho_*^\infty(p, q)| \leq 2\|p\|^2\|q\|^2.$

Proof. Choose $\gamma = |\gamma|e^{i\phi}$ and $\delta = |\delta|e^{i\psi}$, for some $\psi, \phi \in [0, 2\pi)$. Therefore by using definition of ρ_*^∞, we get

$$\begin{aligned}
\rho_*^\infty(\gamma p, \delta q) &= \frac{1}{\pi}\int_0^{2\pi} \rho_*(|\gamma|e^{i\phi}p, |\delta|e^{i\psi}e^{i\theta}q)d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \rho_*(|\gamma|e^{i\phi}p, |\delta|e^{i(\theta+\psi)}q)d\theta \\
&= \frac{|\gamma|^2|\delta|^2}{\pi}\int_0^{2\pi} \rho_*(p, e^{i(\theta+\psi-\phi)}q)d\theta \ [\textit{Using } (nd3)] \\
&= \frac{|\gamma|^2|\delta|^2}{\pi}\int_{\psi-\phi}^{2\pi+\psi-\phi} \rho_*(p, e^{it}q)dt \ [taking\ t = \theta + \psi - \phi] \\
&= \frac{|\gamma|^2|\delta|^2}{\pi}\int_0^{2\pi} \rho_*(p, e^{it}q)dt \\
&= |\gamma|^2|\delta|^2\rho_*^\infty(p, q).
\end{aligned}$$

(ii) First, we note that

$$\begin{aligned}
\int_0^{2\pi}\{Re(e^{i\theta}\gamma)\}^2 d\theta &= \frac{1}{2}\int_0^{2\pi}(e^{i\theta}\gamma + e^{-i\theta}\bar{\gamma})^2 d\theta \\
&= \frac{1}{4}\int_0^{2\pi}(e^{2i\theta}\gamma^2 + \bar{\gamma}^2e^{-2i\theta} + 2\gamma\bar{\gamma})d\theta \\
&= \frac{\gamma^2}{4}\int_0^{2\pi} e^{2i\theta}d\theta + \frac{\bar{\gamma^2}}{4}\int_0^{2\pi} e^{-2i\theta}d\theta + \frac{2\gamma\bar{\gamma}}{4}\int_0^{2\pi} d\theta \\
&= 0 + 0 + \frac{2\gamma\bar{\gamma}}{4}2\pi \\
&= \gamma\bar{\gamma}\pi.
\end{aligned}$$

Further,

$$\begin{aligned}
\rho_*^\infty(p,\gamma p+q) &= \frac{1}{\pi}\int_0^{2\pi} \rho_*(p,e^{i\theta}(\gamma p+q))d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \rho_*(p,e^{i\theta}\gamma p+e^{i\theta}q))d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \rho_*(p,e^{i\theta}\gamma p)d\theta + \frac{1}{\pi}\int_0^{2\pi} 2Re(e^{i\theta}\gamma)\|p\|^2\rho(p,e^{i\theta}q)d\theta \\
&+ \frac{1}{\pi}\int_0^{2\pi} \rho_*(p,e^{i\theta}q))d\theta \\
&= \frac{1}{\pi}\|p\|^4\int_0^{2\pi}\{Re(e^{i\theta}\gamma)\}^2 d\theta + \frac{2\|p\|^2}{\pi}\int_0^{2\pi} Re(e^{i\theta}\gamma)\rho(p,e^{i\theta}q)d\theta \\
&+ \frac{1}{\pi}\int_0^{2\pi} \rho_*(p,e^{i\theta}q))d\theta \\
&= \frac{1}{\pi}\|p\|^4\gamma\bar{\gamma}\pi + \frac{2\|p\|^2}{\pi}\int_0^{2\pi} Re(e^{i\theta}\gamma)\rho(p,e^{i\theta}q)d\theta + \rho_*^\infty(p,q) \\
&= \gamma\bar{\gamma}\|p\|^4 + \rho_*^\infty(p,q) + \frac{2\|p\|^2}{\pi}\int_0^{2\pi} Re(e^{i\theta}\gamma)\rho(p,e^{i\theta}q)d\theta.
\end{aligned}$$

(iii) For $p,q\in\mathcal{X}$, we obtain

$$\begin{aligned}
|\rho_*^\infty(p,q)| &= |\frac{1}{\pi}\int_0^{2\pi} \rho_*(p,e^{i\theta}q)d\theta| \\
&\leq \frac{1}{\pi}\int_0^{2\pi} |\rho_*(p,e^{i\theta}q)d\theta| \\
&= \frac{1}{\pi}\int_0^{2\pi} |\rho_*(p,e^{i\theta}q)|d\theta \\
&\leq \frac{1}{\pi}\int_0^{2\pi} \|p\|^2\|e^{i\theta}q\|^2 d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \|p\|^2\|q\|^2 d\theta \\
&= \frac{1}{\pi}\|p\|^2\|q\|^2.2\pi \\
&= 2\|p\|^2\|q\|^2.
\end{aligned}$$

□

Theorem 3.8. *Let a complex vector space $\mathcal{X}$ be equipped with two norms denoted by $\|.\|_1$ and $\|.\|_2$ over the field $\mathbb{C}$. $\|.\|_1$ and $\|.\|_2$ are norm equivalent if and only if there exists a positive constant k such that,*

$$|\rho_{*,1}^\infty(p,q)-\rho_{*,2}^\infty(p,q)| \leq k\min\{\|p\|_1^2\|q\|_1^2, \|q\|_2^2\|q\|_2^2\}$$

for all $p,q\in\mathcal{X}$ and where $\rho_{*,c}^\infty$ is a functional ρ_*^∞ with respect to $\|.\|_c$, for $c=1,2$.

Proof. Suppose that $\|.\|_1$ and $\|.\|_2$ are norm equivalent. So, $m\|.\|_1 \leq \|.\|_2 \leq M\|.\|_1$ for some positive numbers m, M. Using Proposition 3.7 (iii), we have

$$|\rho_{*,1}^\infty(p,q)| \leq 2\|p\|_1^2\|q\|_1^2,$$

and

$$|\rho_{*,2}^\infty(p,q)| \leq 2\|p\|_2^2\|q\|_2^2.$$

Therefore,

$$\begin{aligned}|\rho^{\infty}_{*,1}(p,q)-\rho^{\infty}_{*,2}(p,q)| &\leq 2(\|p\|_1^2\|q\|_1^2+\|p\|_2^2\|q\|_2^2)\\ &\leq 2(\|p\|_1\|q\|_1+M^2\|p\|_1^2\|q\|_1^2)\\ &= 2(1+M^2)\|p\|_1^2\|q\|_1^2.\end{aligned}$$

On the similar lines, we get

$$|\rho^{\infty}_{*,1}(p,q)-\rho^{\infty}_{*,2}(p,q)| \leq 2(1+\frac{1}{m^2})\|p\|_2^2\|q\|_2^2.$$

Taking $k=\max\{2(1+M^2),2(1+\frac{1}{m^2})\}$, we have,

$$|\rho^{\infty}_{*,1}(p,q)-\rho^{\infty}_{*,2}(p,q)| \leq k\min\{\|p\|_1^2\|q\|_1^2,\|p\|_2^2\|q\|_2^2\}.$$

Conversely, suppose that for every $p\in\mathcal{X}$ and $k>0$, we have

$$|\rho^{\infty}_{*,1}(p,p)-\rho^{\infty}_{*,2}(p,p)| \leq k\min\{\|p\|_1^4,\|p\|_2^4\}.$$

If $\min\{\|p\|_1^4,\|p\|_2^4\}=\|p\|_1^4$, then

$$|\|p\|_1^4-\|p\|_2^4| \leq k\|p\|_1^4 \leq k\|p\|_2^4.$$

Therefore, we get

$$\|p\|_2 \leq \sqrt[4]{1+k}\|p\|_1, \quad \|p\|_1 \leq \sqrt[4]{1+k}\|p\|_2,$$

and

$$\frac{1}{\sqrt[4]{1+k}}\|p\|_1 \leq \|p\|_2 \leq \sqrt[4]{1+k}\|p\|_1.$$

Similarly, we get the result, if $\min\{\|p\|_1^4,\|p\|_2^4\}=\|p\|_2^4$. □

4 ρ^{∞}_{*}-orthogonality

Here, we use the functional (3.4) to define ρ^{∞}_{*}-orthogonality in a complex normed space.

Definition 4.1. Define a ρ^{∞}_{*}-orthogonality as

$$p\perp_{\rho^{\infty}_{*}} q \text{ if and only if } \rho^{\infty}_{*}(p,q)=0,$$

for every $p,q\in\mathcal{X}$.

It is always worth examining relationships between various types of orthogonality.

Theorem 4.2. *$\perp_{\rho^{\infty}_{*}}=\perp_s$ if $\rho^{\infty}_{*}(p,q)=\overline{[q,p]}$, for every $p,q\in\mathcal{X}$, where $[.,.]$ is a s.i.p. on $\mathcal{X}$.*

Proof. Let $\rho^{\infty}_{*}(p,q)=\overline{[q,p]}$, for all $p,q\in\mathcal{X}$.
If $\rho^{\infty}_{*}(p,q)=0$ then we have $\overline{[q,p]}=0$. So $\perp_{\rho^{\infty}_{*}}\subseteq\perp_s$.
Also if $\overline{[q,p]}=0$, then we get $\rho^{\infty}_{*}(p,q)=0$. So $\perp_s\subseteq\perp_{\rho^{\infty}_{*}}$.
Hence from the above condition we obtain $\perp_{\rho^{\infty}_{*}}=\perp_s$. □

The example below demonstrates that in general $\perp_B\not\subset\perp_{\rho^{\infty}_{*}}$.

Example 4.3. Let $\mathcal{X}=l^1$ be the space of summable sequences over $\mathbb{C}$ equipped with its standard norm. Let $p=(0,0,1,0,0,\ldots)$ and $q=(3,0,1,0,0,\ldots)$. Here, for any $\xi\in\mathbb{C}$, we have

$\|p+\xi q\| = 3|\xi| + |1+\xi| \geq 1 = \|p\|$. So, $p \perp_B q$.
Consider,

$$\begin{aligned}\frac{\|p+te^{i\theta}q\| - \|p\|^2}{t} &= \frac{\|(3te^{i\theta},0,1+te^{i\theta},\ldots)\| - \|(0,0,1,0,0,\ldots)\|}{t}\\ &= \frac{|3te^{i\theta}| + |1+te^{i\theta}| - 1}{t}\\ &= \frac{3|t|}{t} + \frac{|1+te^{i\theta}|-1}{t}.\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_+(p,e^{i\theta}q) &= \lim_{t\to 0^+}\frac{3|t|}{t} + \lim_{t\to 0^+}\frac{|1+te^{i\theta}|-1}{t}\\ &= 3 + \lim_{t\to 0^+}\frac{|1+te^{i\theta}|^2-1}{t(|1+te^{i\theta}|+1)}\\ &= 3 + \lim_{t\to 0^+}\frac{1+2tRe(e^{i\theta})+t^2-1}{t(|1+te^{i\theta}|+1)}\\ &= 3 + 2Re(e^{i\theta}).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho_-(p,e^{i\theta}q) &= \lim_{t\to 0^-}\frac{3|t|}{t} + \lim_{t\to 0^-}\frac{|1+te^{i\theta}|-1}{t}\\ &= -3 + 2Re(e^{i\theta}).\end{aligned}$$

Further,

$$\begin{aligned}\rho_*(p,e^{i\theta}q) &= \rho_-(p,e^{i\theta}q)\rho_+(p,e^{i\theta}q)\\ &= \{-3+2Re(e^{i\theta})\}\{3+2Re(e^{i\theta})\}\\ &= \{-3+2\cos\theta)\}\{3+2\cos\theta)\}\\ &= 4\cos^2\theta - 9\end{aligned}$$

Hence

$$\begin{aligned}\rho_*^\infty(p,q) &= \frac{1}{\pi}\int_0^{2\pi}\rho_*(p,e^{i\theta}y)d\theta\\ &= \frac{1}{\pi}\int_0^{2\pi}(4\cos^2\theta - 9)d\theta\\ &= -14\pi \neq 0.\end{aligned}$$

Thus $p \not\perp_{\rho_*^\infty} y$. Hence $\perp_B \not\subset \perp_{\rho_*^\infty}$.

Theorem 4.4. *$\perp_{\rho_*^\infty} \subseteq \perp_B$ if $|\rho_*^\infty(p,q)| \leq \|p\|^2\|q\|^2$, for every p,q in $\mathcal{X}$ over $\mathbb{C}$.*

Proof. First we suppose that $|\rho_*^\infty(p,q)| \leq \|p\|^2\|q\|^2$ holds for every $p,q \in \mathcal{X}$. Consider $p \perp_{\rho_*^\infty} q$, we have

$$\begin{aligned}\|p\|^4 &= \rho_*^\infty(p,p+tq)\\ &\leq \|p\|^2\|p+tq\|^2.\end{aligned}$$

Assume that $p \neq 0$. Then we now have, $\|p+tq\| \geq \|p\|$, that is $p \perp_B q$. Hence $\perp_{\rho_*^\infty} \subseteq \perp_B$. □

The example below demonstrates that the relationships between $\perp_{\rho_+}, \perp_{\rho_-}, \perp_{\rho_*}$ and $\perp_{\rho_*^\infty}$ are not generally comparable.

Example 4.5. Take $\mathcal{X} = l^1$ with $\|.\|_1$. Let $p = (0,1,0,0,\ldots)$ and $q = (0,i,0,0,\ldots)$. On calculating, we have $\rho_\pm(p,q) = 0$, $\rho_*(p,q) = 0$ and $\rho_*^\infty(p,q) = 1$. This implies that $\perp_{\rho_\pm} \not\subset \perp_{\rho_*^\infty}$ and $\perp_{\rho_*} \not\subset \perp_{\rho_*^\infty}$.
Again let $p = (1,0,1,0,0,\ldots)$ and $q = (1,0,-1,0,0,\ldots)$. On calculating, we have $\rho_\pm(p,q) = 1$, $\rho_*(p,q) = 1$ and $\rho_*^\infty(p,q) = 0$. This implies that $\perp_{\rho_*^\infty} \not\subset \perp_{\rho_*}$.

5 Characterization

In this section, we give a result on the characterization of inner product spaces using the properties of ρ_*^∞-functional.

Theorem 5.1. *Let $(\mathcal{X}, \|.\|)$ be a normed space. If the norm in $\mathcal{X}$ comes from an inner product then $\rho_*^\infty(p,q) = \rho_*^\infty(q,p)$.*

Proof. Let $(\mathcal{X}, \|.\|)$ be a normed space, and the norm in $\mathcal{X}$ comes from an inner product. Consider,

$$\begin{aligned}
\rho_*^\infty(p,q) &= \frac{1}{\pi}\int_0^{2\pi} \rho_*(p, e^{i\theta}q)d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi} \{Re\langle p, e^{i\theta}q\rangle\}^2 d\theta \quad \text{(using Proposition 3.3}(v)) \\
&= \frac{1}{4\pi}\int_0^{2\pi} \left(\langle p, e^{i\theta}q\rangle + \overline{\langle p, e^{i\theta}q\rangle}\right)^2 d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \left(e^{-i\theta}\langle p,q\rangle + e^{i\theta}\overline{\langle p,q\rangle}\right)^2 d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \left(e^{-2i\theta}\langle p,q\rangle^2 + e^{2i\theta}\overline{\langle p,q\rangle}^2 + 2\langle p,q\rangle\overline{\langle p,q\rangle}\right) d\theta \\
&= \frac{1}{4\pi}\left(0 + 0 + 4\pi\langle p,q\rangle\overline{\langle p,q\rangle}\right) \\
&= \langle p,q\rangle\overline{\langle p,q\rangle} = \langle p,q\rangle\langle q,p\rangle.
\end{aligned}$$

Similarly, one can prove that $\rho_*^\infty(q,p) = \langle q,p\rangle\overline{\langle q,p\rangle} = \langle q,p\rangle\langle p,q\rangle$. Hence, $\rho_*^\infty(p,q) = \rho_*^\infty(q,p)$. □

Corollary 5.2. *Let $(\mathcal{X}, \|.\|)$ be a normed space, and the norm in $\mathcal{X}$ comes from an inner product. Then $p \perp q$ (that is $\langle p,q\rangle = 0$) if and only if $p \perp_{\rho_*^\infty} q$.*

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