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Quasi-Conformal Curvature Tensor of $\mathtt{NC}_{10}$-Manifold

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Abstract This study aims to investigate the quasi-conformal curvature tensor of $\mathtt{NC}_{10}$-manifold. The components of this tensor were determined using the adjoined G-structure space. Three quasi-conformal invariants were identified in relation to the vanishing quasi-conformal curvature tensor. Subsequently, three types of $\mathtt{NC}_{10}$-manifold were established. Furthermore, the necessary conditions for these classes to be an η-Einstein manifold were established.

1 Introduction

A generalisation of the cosymplectic manifold, the $\mathtt{C}_{10}$-manifold is a class of almost contact metric manifolds introduced by Chiena and Gonzales in [10]. In [19], Rustanov considered a generalisation of the $\mathtt{C}_{10}$-manifold and a closely related cosymplectic manifold, called the $\mathtt{NC}_{10}$-manifold. He derived the complete set of structure equations and computed the components of the Riemannian curvature tensor, Ricci tensor and ΦHS-curvature tensor. Rustanov also demonstrated that the normal $\mathtt{NC}_{10}$-manifold and integrable manifold are cosymplectic. In [20], Rustanov et al. established that the $\mathtt{NC}_{10}$-manifold belongs to the class $\mathbf{CR}_3$ and that the local structure of the $\mathtt{NC}_{10}$-manifold is a manifold of classes $\mathbf{CR}_1$ and $\mathbf{CR}_2$.

In our previous works, we have examined various types of curvature tensors that have the Riemannian curvature tensor in their structures. For instance, see references [1], [2], [3] and [4]. For related studies, we refer to the citations [5], [21] and [22].

In this paper, we focus on the quasi-conformal curvature tensor ($\mathbf{QC}$-tensor) of the $\mathtt{NC}_{10}$-manifold. Specifically, we elucidate the geometric significance of the vanishing of this tensor. Numerous authors have studied this tensor; notably, De and Sakar [11] investigated the $\mathbf{QC}$-tensor of an (K,μ)-contact metric manifold. In [13] Hasseb, Siddiqi and Shahid examined the $\mathbf{QC}$-tensor on a Kenmotsu manifold. De and Hazra [9] explored the $\mathbf{QC}$-tensor in the context of space-time and $f(R,G)$-gravity.

2 Preliminaries

This section briefly summarizes some of the basic facts and concepts which have a relationship with the present work.

Definition 2.1. *[6]* If $\mathbf{M}$ is a $2\mathfrak{n}+1$ dimensional smooth manifold, an almost contact metric structure ($\mathbf{ACO}_n$-structure) is quadrilateral $\Upsilon = (\eta,\zeta,\mathbf{g},\Phi)$ of tensor fields, where η is a *contact* 1*-form*; ζ is *a characteristic vector*; $\mathbf{g} = \langle .,. \rangle$ is a Riemannian metric, Φ is a structure tensor of sort $(1;1)$ called an *endomorphism*, furthermore the subsequent conditions verified:

$$(1)\ \Phi^2 = -id + \eta\otimes\zeta; \quad (2)\ \Phi(\zeta) = 0\ ; \quad (3)\ \eta\circ\Phi = 0; \quad (4)\ \eta(\zeta) = 1\ .$$

$$\mathbf{g}(\Phi Q, \Phi O) = \mathbf{g}(Q,O) - \eta(Q)\eta(O),\ Q,O \in \mathfrak{X}(\mathbf{M})$$

In this occurrence, the manifold $\mathbf{M}$ accompaied by the quadrilateral Υ is called an $\mathbf{ACO}_n$-manifold.

More details for the constract of the associated-$\mathbf{G}$-structure space ($\mathbf{ASG}$-space), we recommend reviewing the citations [15] and [17].

Lemma 2.1. *[16]* The components of Φ_l and $\mathfrak{g}_l$ in the **ASG**-space, exemplified by the beneath matrices respectively:

$$(\Phi^l_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \ (g_{lj}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

Here I_n refere to the identity matrix of $n \times n$ order.

Definition 2.2. *[19]* An $\mathtt{NC}_{10}$-structure is $\mathbf{ACO}_n$-structure accompanied with the condition below

$$\nabla_O(\Phi)U + \nabla_U(\Phi)O = \xi\nabla_O(\eta)\Phi U + \xi\nabla_U(\eta)\Phi O + \eta(O)\nabla_{\Phi U}\xi + \eta(U)\nabla_{\Phi O}\xi; \quad U, O \in \mathfrak{X}(\mathfrak{M})$$

A manifold $\mathbf{M}$ accompanied with the $\mathtt{NC}_{10}$-structure is called a $\mathtt{NC}_{10}$-manifold.

Lemma 2.2. *[19]* Due to the **ASG**-space, the second structure equations of the aforementioned manifold in the last Definition have the following forms

(i) $dw^c_b + w^a_c \wedge w^c_b = (A^{ad}_{bc} - 2C^b + C^{adh}C_{hbc} - F^{ad}F_{bc})w^c \wedge w_d;$

(ii) $dF_{ab} - F_{cb}w^c_a - F_{ac}w^c_b = 0;$

(iii) $dF^{ab} + F^{cb}w^a_c + F^{ac}w^b_c = 0;$

(iv) $dC^{abc} + C^{dbc}w^a_d + C^{adc}w^b_d + C^{abd}w^d_c = C^{abcd}w_d;$

(v) $dC_{abc} - C_{dbc}w^d_a - C_{adc}w^d_b - C_{abd}w^d_c = C_{abcd}w^d,$

where

(i) $F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}}; F_{ab} = -\sqrt{-1}\Phi^0_{a,b}; F_{ab} + F_{ba} = 0;$

(ii) $C^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}; C_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c} \text{and} C_{[abc]} = C_{abc}; C^{[abc]} = C^{abc};$

(iii) $F_{ad}C^{dbc} = F^{ad}C_{dbc} = 0; A^{ad}_{[bc]} = A^{[ad]}_{bc} = 0;$

Lemma 2.3. *[19]* The $\mathtt{NC}_{10}$-manifold is called a manifold of class $\mathtt{C}_{10}$ iff, $C^{abc} = C_{abc} = 0$

Regarding to the the **ASG**-space, the non-flat components of the Riemannian tensor of the class $\mathtt{NC}_{10}$ immersed in the next Lemma.

Lemma 2.4. *[20]* The components of the aforementioned tensor for the manifold of class $\mathtt{NC}_{10}$ is identified below

(i) $\mathbf{R}^a_{bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc};$

(ii) $\mathbf{R}^{\hat{a}}_{bcd} = C^{acdb} - F_{ab}F_{cd};$

(iii) $\mathbf{R}^b_{00a} = F_{ac}F^{cb};$

(iv) $\mathbf{R}^a_{\hat{b}cd} = 2C^{abh}C_{hcd}.$

Definition 2.3. *[8]* A $(2,0)$-tensor which is specified by $\mathbf{r}_{ik} = -\mathbf{R}^t_{ikt}$ is called a Ricci tensor.

Lemma 2.5. Due to the **ASG**-space, the components of the Ricci tensor for $\mathtt{NC}_{10}$-manifold are identified below.

(i) $\mathbf{r}_{oo} = -2F_{ab}F^{ba};$

(ii) $\mathbf{r}_{a\hat{b}} = \mathbf{r}_{\hat{b}a} = A^{bc}_{ac} - 3C^{bcd}C_{dca} - F_{ac}F^{cb}.$

the residual components are zero. Furthermor, the scalar curvature is specified by $\kappa = -2F_{ab}F^{ba} - 6C^{abc}C_{cba} + 2A^{ab}_{ab}$, here $\kappa = \mathbf{g}^{ik}\mathbf{r}_{ik}$.

Definition 2.4. *[23]* On the $2n+1$-dimensional $\mathbf{ACO}_n$-manifold Υ. A quasi-conformal curvature $(4,0)$-tensor ($\mathbf{QC}$-tensor) $\check{\mathbf{C}}$ is defined via the specified

$$\check{\mathbf{C}}_{ijkl} = \mathcal{A}\mathbf{R}_{ijkl} + \mathcal{B}(\mathbf{r}_{jl}\mathbf{g}_{ik} + \mathbf{r}_{ik}\mathbf{g}_{jl} - \mathbf{r}_{il}\mathbf{g}_{jk} - \mathbf{r}_{jk}\mathbf{g}_{il}) - \frac{\kappa}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})[\mathbf{g}_{ik}\mathbf{g}_{jl} - \mathbf{g}_{il}\mathbf{g}_{jk}],$$

where $\check{\mathbf{C}}_{ijkl} = -\check{\mathbf{C}}_{jikl} = -\check{\mathbf{C}}_{ijlk} = \check{\mathbf{C}}_{klij}$. and $\mathcal{B}, \mathcal{A}$ are the constants which are not jointly zero.

Definition 2.5. *[7]* The Ricci tensor $\mathbf{r}$ of an $\mathbf{ACO}_n$-manifold Υ that attain the relevance

$$\mathbf{r} = \alpha\mathbf{g} + \beta\eta\otimes\eta;$$

called an η-Einstein manifold. In the case where β equal to zero, thereupon $\mathbf{M}$ will be called an Einstein manifold, here the functions α and β are smooth.

3 The Fundamental Classes of Quasi-conformal $\mathtt{NC}_{10}$-manifold

Theorem 3.1. The components for the quasi-conformal tensor of the $\mathtt{NC}_{10}$-manifold are identified below:

(i) $\check{\mathbf{C}}_{abcd} = \mathcal{A}(C_{acdb} - F_{ab}F_{cd})$;

(ii) $\check{\mathbf{C}}_{\hat{a}\hat{b}cd} = 2\mathcal{A}C^{abe}C_{ecd} + 4\mathcal{B}(\mathbf{r}^{[a}_{[c}\delta^{b]}_{d]}) - \frac{\delta^{ab}_{cd}}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea})$;

(iii) $\check{\mathbf{C}}_{\hat{a}bc\hat{d}} = \mathcal{A}(A^{ad}_{bc} - C^{ade}C_{ebc}) + \mathcal{B}(\mathbf{r}^d_b\delta^a_c + \mathbf{r}^a_c\delta^d_b) - \frac{\delta^a_c\delta^d_b}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea})$.

(iv) $\check{\mathbf{C}}_{\hat{a}00d} = \mathcal{A}F_{dc}F^{ca} - \mathcal{B}(\mathbf{r}_{00}\delta^a_d + \mathbf{r}^a_d) + \frac{\delta^a_d}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea})$

Proof. By using the the Definition 2.4, Lemmas 2.4 and 2.5, we can find the components. □

In the following we highlight on the vanishing the components of the quasi-conformal tensor and their geometric meaning.

Let $\check{\mathbf{C}}_{\hat{a}00d} = 0$, then applying the same procedure as in [16] and [17] on the relations $\check{\mathbf{C}}_{\hat{a}00d} = 0, \check{\mathbf{C}}_{a00d} = 0, \check{\mathbf{C}}_{000d} = 0$, means $\check{\mathbf{C}}_{i00d} = 0$, then

$$\check{C}(U,\zeta)\zeta = 0, \quad \forall U \in \mathfrak{X}(\mathbf{M}) \tag{3.1}$$

The opposite is also holds, if (3.1) is true then the relation $\check{\mathbf{C}}_{\hat{a}00d} = 0$ holds. Thus the relations are equivalent in the the $\mathbf{ASG}$-space.

Definition 3.1. An $\mathtt{NC}_{10}$-manifold whose quasi-conformal tensor fulfilles the identity (3.1) is called a manifold of class $\check{\mathbf{C}}_1$.

Theorem 3.2. If the $\mathtt{NC}_{10}$-manifold is a manifold of class $\check{\mathbf{C}}_1$, then it is an η-Einstein manifold, where $\alpha = \frac{-1}{2n+1}(\frac{\mathcal{A}}{n\mathcal{B}} + (2n+5))F_{ae}F^{ea} + \frac{1}{2n+1}(\frac{\mathcal{A}}{2n\mathcal{B}} + 2)(2A^{ae}_{ae} - 6C^{aeh}C_{eha})$, $\beta = \frac{1}{2n+1}(\frac{\mathcal{A}}{n\mathcal{B}} - (2n-3))F_{ae}F^{ea} - \frac{1}{2n+1}(\frac{\mathcal{A}}{2n\mathcal{B}} + 2)(2A^{ae}_{ae} - 6C^{aeh}C_{eha})$.

Proof. Assume that $\mathbf{M}$ is $\mathtt{NC}_{10}$-manifold of class $\check{\mathbf{C}}_1$, the relation (3.1) holds, it follows that $\check{\mathbf{C}}_{\hat{a}00b} = 0$

$$\mathcal{A}F_{be}F^{ea} - \mathcal{B}(\mathbf{r}_{00}\delta^a_b + \mathbf{r}^a_b) + \frac{\delta^a_b}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea}) = 0. \tag{3.2}$$

Taking into account the Lemma 2.5 and symmetrising and then antisymmetrising relevance (3.2) utilizing indices (e, b) we get

$$\mathcal{B}\mathbf{r}^a_b = \mathcal{B}F_{ae}F^{ea}\delta^a_b + \frac{\delta^a_b}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea})$$

Means that

$$\mathbf{r}^a_b \ =[\frac{-1}{2n+1}(\frac{\mathcal{A}}{n\mathcal{B}} + (2n+5))F_{ae}F^{ea} + \frac{1}{2n+1}(\frac{\mathcal{A}}{2n\mathcal{B}} + 2)(2A^{ae}_{ae} - 6C^{aeh}C_{eha})]\delta^a_b;$$

$$\mathbf{r}^a_b \ =\alpha\delta^a_b, \quad \text{where} \quad \alpha = \frac{-1}{2n+1}(\frac{\mathcal{A}}{n\mathcal{B}} + (2n+5))F_{ae}F^{ea} + \frac{1}{2n+1}(\frac{\mathcal{A}}{2n\mathcal{B}} + 2)(2A^{ae}_{ae} - 6C^{aeh}C_{eha}).$$

Furthermore,

$$\mathbf{r}_{00} = \alpha + \beta, \quad \text{whereas} \quad \beta = \frac{1}{2n+1}(\frac{\mathcal{A}}{n\mathcal{B}} - (2n-3))F_{ae}F^{ea} - \frac{1}{2n+1}(\frac{\mathcal{A}}{2n\mathcal{B}} + 2)(2A^{ae}_{ae} - 6C^{aeh}C_{eha}). \tag{3.3}$$

Thus, $\mathbf{M}$ is η-Einstein manifold. □

Assume that $\check{\mathbf{C}}_{\hat{a}bc\hat{d}} = 0$. Using the same technique as in [16] and [17] on the relations $\check{\mathbf{C}}_{\hat{a}bc\hat{d}} = 0$, $\check{\mathbf{C}}_{abc\hat{d}} = 0$, $\check{\mathbf{C}}_{0bc\hat{d}} = 0$, which means that the relations $\check{\mathbf{C}}_{ibc\hat{d}} = 0$, so we get

$$\check{\mathbf{C}}(\Phi^2U, \Phi^2O)\Phi^2Z + \check{\mathbf{C}}(\Phi^2U, \Phi O)\Phi Z - \check{\mathbf{C}}(\Phi U, \Phi^2O)\Phi Z + \check{\mathbf{C}}(\Phi U, \Phi O)\Phi^2Z = 0, \quad \forall U, O, Z \in \mathfrak{X}(\mathbf{M}). \tag{3.4}$$

Conversely, if the relevance 3.4 satisfied, then it can be formulated as

$$\check{\mathbf{C}}^i_{jkl}\Phi^j_h\Phi^h_r\Phi^k_m\Phi^m_p\Phi^l_s\Phi^s_q + \check{\mathbf{C}}^i_{jkl}\Phi^j_r\Phi^k_m\Phi^m_p\Phi^l_q - \check{\mathbf{C}}^i_{jkl}\Phi^j_r\Phi^k_P\Phi^l_s\Phi^s_q + \check{\mathbf{C}}^i_{jkl}\Phi^j_h\Phi^h_r\Phi^k_p\Phi^l_q = 0.$$

Taking into consideration the **ASG**-space, the Lemma (2.1), the relation above turn into:

$$4\check{\mathbf{C}}_{\hat{a}bc\hat{d}} + 4\check{\mathbf{C}}_{a\hat{b}\hat{c}d} = 0, \quad i.e. \quad \check{\mathbf{C}}_{\hat{a}bc\hat{d}} = 0, \check{\mathbf{C}}_{a\hat{b}\hat{c}d} = 0$$

.

Hence, on the **ASG**-space, the relevance (3.4) and $\check{\mathbf{C}}_{\hat{a}bc\hat{d}} = 0$ are equivalent.

Definition 3.2. An $\texttt{NC}_{10}$-manifold whose quasi-conformal tensor fulfilles the equality (3.4)is called a manifold of class $\check{\mathbf{C}}_2$.

Theorem 3.3. If the manifold $\texttt{NC}_{10}$ is a manifold of class $\check{\mathbf{C}}_2$, then it is an η-Einstein manifold, where $\alpha = \frac{1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}} + 1)(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea})$, $\beta = \frac{-1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}} + 1)(2A^{ae}_{ae} - 6C^{aeh}C_{eha}) - \frac{-1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}} + (4n+3))F_{ae}F^{ea}$.

Proof. Assume that $\mathbf{M}$ is $\texttt{NC}_{10}$-manifold of class $\check{\mathbf{C}}_2$, then $\check{\mathbf{C}}_{\hat{a}bc\hat{d}} = 0$, that is

$$\mathcal{A}(A^{ad}_{bc} - C^{ade}C_{ebc}) + \mathcal{B}(\mathbf{r}^d_b\delta^a_c + \mathbf{r}^a_c\delta^d_b) - \frac{\delta^a_c\delta^d_b}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea}) = 0 \tag{3.5}$$

Contracting the relevance (3.5) utilizing indices (c, d), we derive

$$\mathcal{A}(A^{ac}_{bc} - C^{ace}C_{ebc}) + \mathcal{B}(\mathbf{r}^a_b + \nabla^a_b) - \frac{\delta^a_b}{2n+1}(\frac{\mathcal{A}}{2n} + 2\mathcal{B})(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea}) = 0 \tag{3.6}$$

Symmetrising and later antisymmetrising the relevance (3.6) utilizing indices (a, c), we infer

$$\mathbf{r}^a_b = \frac{1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}} + 1)(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea})\delta^a_b;$$

$$\mathbf{r}^a_b = \alpha\delta^a_b, \quad \text{whereas} \quad \alpha = \frac{1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}} + 1)(2A^{ae}_{ae} - 6C^{aeh}C_{eha} - 2F_{ae}F^{ea}).$$

Also,
$\mathbf{r}_{00}=\alpha+\beta$, whereas $\beta=\frac{-1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}}+1)(2A^{ae}_{ae}-6C^{aeh}C_{eha})-\frac{-1}{2n+1}(\frac{\mathcal{A}}{4n\mathcal{B}}+(4n+3))F_{ae}F^{ea}$ Hence, **M** is η-Einstein manifold. □

Consider the equalities $\check{\mathbf{C}}_{\hat{a}\hat{b}cd}=0$,$\check{\mathbf{C}}_{a\hat{b}cd}=0$, $\check{\mathbf{C}}_{0\hat{b}cd}=0$ which means that $\check{\mathbf{C}}_{i\hat{b}cd}=0$, it follows that

$$\check{\mathbf{C}}(\Phi^2U,\Phi^2O)\Phi^2Z+\check{\mathbf{C}}(\Phi^2U,\Phi O)\Phi Z-\check{\mathbf{C}}(\Phi U,\Phi^2O)\Phi Z+\check{\mathbf{C}}(\Phi U,\Phi O)\Phi^2Z=0,\quad \forall U,O,Z\in\mathfrak{X}(\mathbf{M}). \tag{3.7}$$

As mentioned above, on the **ASG**-space, the identity (3.7) is equivalent to the relations $\check{\mathbf{C}}_{\hat{a}\hat{b}cd}=0$.

Definition 3.3. An $\mathtt{NC}_{10}$-manifold whose quasi-conformal tensor fulfilles the equality (3.7)is called a manifold of class $\check{\mathbf{C}}_3$.

Theorem 3.4. If the manifold $\mathtt{NC}_{10}$ is a manifold of class $\check{\mathbf{C}}_3$, then the first tensor identical to zero .

Proof. Assume that **M** is $\mathtt{NC}_{10}$-manifold of class $\check{\mathbf{C}}_3$, we have

$$2\mathcal{A}C^{abe}C_{ecd}+4\mathcal{B}(\mathbf{r}^{[a}_{[c}\delta^{b]}_{d]})-\frac{\delta^{ab}_{cd}}{2n+1}(\frac{\mathcal{A}}{2n}+2\mathcal{B})(2A^{ae}_{ae}-6C^{aeh}C_{eha}-2F_{ae}F^{ea})=0 \tag{3.8}$$

Symmetrising and later antisymmetrising (3.8) using indices (e,b) and then contracting using indices (b,c) and (a,d), it follows that

$$C^{abe}C_{eba}=0.$$

Therefore,

$$\sum_{a,b,e}|C_{abe}|^2=0\Leftrightarrow C_{abe}=0 \tag{3.9}$$

□

According to the Lemma 2.3, the Theorem 3.4 can be formulated as the following:

Theorem 3.5. If the manifold $\mathtt{NC}_{10}$ is a manifold of class $\check{\mathbf{C}}_3$, then it is a manifold of class $\mathtt{C}_{10}$.

Theorem 3.6. If the manifold $\mathtt{NC}_{10}$ is a manifold of class $\check{\mathbf{C}}_3$, then it is an η-Einstein manifold, where $\alpha=\frac{-3n\mathcal{B}+\mathcal{A}(n-1)}{n(n-2)(2n+1)\mathcal{B}}(A^{ae}_{ae}-F_{ae}F^{ea})$, $\beta=\frac{1}{n(n-2)(2n+1)\mathcal{B}}[(-3n\mathcal{B}+\mathcal{A}(n-1))A^{ae}_{ae}+(-n\mathcal{B}(4n^2-6n-7)+\mathcal{A}(n-1))F_{ae}F^{ea}$.

Proof. Assume that **M** is $\mathtt{NC}_{10}$-manifold of class $\check{\mathbf{C}}_3$, accordingly $\check{\mathbf{C}}_{\hat{a}\hat{b}cd}=0$ and consider the Theorem 3.4, we deduce

$$\mathcal{B}(\mathbf{r}^b_d\delta^a_c+\mathbf{r}^a_c\delta^b_d-\mathbf{r}^a_d\delta^b_c-\mathbf{r}^b_c\delta^a_d)-\frac{\delta^a_c\delta^b_d-\delta^a_d\delta^b_c}{2n+1}(\frac{\mathcal{A}}{2n}+2\mathcal{B})(2A^{ae}_{ae}-2F_{ae}F^{ea})=0 \tag{3.10}$$

Contracting (3.10) using indices (a,c), we conclude

$$(n-2)\mathcal{B}\mathbf{r}^b_d+\mathcal{B}\mathbf{r}^a_a\delta^b_d-\frac{(n-1)\delta^b_d}{2n+1}(\frac{\mathcal{A}}{2n}+2\mathcal{B})(2A^{ae}_{ae}-2F_{ae}F^{ea})=0$$

Then,

$$\mathbf{r}^b_d=\frac{-3n\mathcal{B}+\mathcal{A}(n-1)}{n(n-2)(2n+1)\mathcal{B}}(A^{ae}_{ae}-F_{ae}F^{ea})\delta^b_d$$

$$\mathbf{r}^b_d=\alpha\delta^b_d,\quad \text{where}\quad \alpha=\frac{-3n\mathcal{B}+\mathcal{A}(n-1)}{n(n-2)(2n+1)\mathcal{B}}(A^{ae}_{ae}-F_{ae}F^{ea}).$$

Furthermore, we have
$\mathbf{r}_{00}=\alpha+\beta$,

whereas,$\beta=\frac{1}{n(n-2)(2n+1)\mathcal{B}}[(-3n\mathcal{B}+\mathcal{A}(n-1))A^{ae}_{ae}+(-n\mathcal{B}(4n^2-6n-7)+\mathcal{A}(n-1))F_{ae}F^{ea}$.

Therefore, **M** is η-Einstein manifold.. □

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